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Institute of Actuaries

ITUTE OF ACTUARIES'

TEXT BOOK

OF THE

PRINCIPLES OF INTEREST, LIFE ANNUITIES, AND ASSURANCES,

AND THEIR PRACTICAL APPLICATION.

PART II.

LIFE CONTINGENCIES

(*Including Life Annuities and Assurances*).

SECOND EDITION.

BY GEORGE KING,

FELLOW OF THE INSTITUTE OF ACTUARIES AND FELLOW OF THE FACULTY OF ACTUARIES.

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PREFACE.

In issuing the Second Part of the Text-Book (the preparation of which was entrusted to Mr. George King, in succession to Mr. W. Sutton who was unable to continue the work for reasons explained in the Preface to the First Part,) the Council of the Institute of Actuaries desire to express their great satisfaction at the able manner in which the whole work has been performed by these gentlemen. At the same time they wish it to be understood that their object in publishing this work is to place in the hands of Students and others, a Treatise on the *Principles* of Interest (including Annuities-Certain) and the *Theory* of Life Annuities and Assurances, and that they do not hold themselves responsible for any opinions expressed therein by the respective Authors. They also point out that the numerical examples are to be regarded simply as Illustrations of the Mathematical Theory, and not as Rules or Directions for applying the theory to actual practice.

PREFACE TO THE SECOND EDITION

The custom that volumes published by the Institute of Actuaries should be furnished with a short Preface must not be broken, but neither this Second Edition nor its Author needs any introduction to the many learners who have already profited by his labours.

When the First Edition (which has been also published abroad in French) was nearly sold out, the Council requested Mr. King to make any revision he thought desirable, though, as it appeared to them, and they are glad to observe that he concurs, but little alteration was necessary. What was required, however, has now been made, the principal changes being set out in the Introduction; and the book is again sent forth on its way to aid those who are preparing themselves for examination, and to refresh from time to time the memories of students of an older growth.

That it may be as successful as its forerunner is the earnest hope of the Council, who desire once more to thank the Author for this renewed example of his skill. That much advantage may accrue to those who will take the trouble to master its pages will be the reward to Council and Author of their enterprise and his endeavour.

C. D. H.

5*th February*, 1902.

INTRODUCTION BY THE AUTHOR.

In the preparation of Part II (Life Contingencies) of the Text-Book, an effort has been made to render each chapter a complete, if brief, monograph in itself; but, from exigencies of space, it has not been possible to mention everything previously written on the subject. A selection had to be made, and many of those formulas and methods which are now practically obsolete have been omitted.

The volume, from the nature of the case, includes but little that is actually new in the way of investigation. Nevertheless, it has not been thought expedient to give many references to other writings, principally because, in the majority of instances, there would be difficulty in determining who is the original authority. The initials *J.I.A.*, when they occur, refer to the Journal of the Institute of Actuaries, which, for the past forty years, has been the chief repository of the results of original research.

In reading the book for the first time the student may, without being thereby inconvenienced, pass over those paragraphs which are numbered in parentheses, thus, [ ].

They contain the more difficult demonstrations, requiring, for the most part, an elementary knowledge of the higher mathematics; and they are so arranged that the sequence of the other portions of the book will not be interfered with by their omission.

Three chapters have been incorporated, on *Finite Differences, Interpolation,* and *Summation,* respectively. These subjects are not within the scope of the TEXT-BOOK, as originally planned; but it was found, as the work progressed, that many formulas had to be used which could hardly be taken for granted; and it became apparent that the most convenient course would be to gather all such matters into chapters by themselves. While excellent books exist on the *Calculus of Finite Differences,* there is not any treatise which exactly meets the requirements of the actuarial student; and it is therefore hoped that these three chapters will not be deemed superfluous.

Advantage has been taken of the remarkable applicability to the Institute of Actuaries' Healthy Males Mortality Table of Makeham's formula for the law of mortality, in order to provide, for the first time, complete tables for finding the values of joint-life annuities up to four lives inclusive. These are given, with many other tables, at six different rates of interest; so that the TEXT-BOOK may be not only a manual for students, but also a useful book of reference to the practising actuary.

In conclusion, the Author tenders his best thanks to Mr. G. F. HARDY for the assistance he rendered in the preparation of those portions of Chap. xxiv which treat of approximate formulas for the evaluation of integrals; also to Mr. H. J. ROTHERY, and Messrs. J. HOWARD BARNES, R. CROSS, D. J. McG. McKENZIE, and E. A. RUSHER, for the skill and labour they have devoted to the computation of the majority of the Numerical Tables. It is believed that, on account of the great care which they have bestowed on this important section of the work, any errors which may hereafter be found therein will prove to be few and insignificant.

G. K.

LONDON, *October*, 1887.

INTRODUCTION TO THE SECOND EDITION.

BY THE AUTHOR.

In preparing the Second Edition of Part II of the Text-Book, it has not been found necessary to make many great changes. Effect has been given to the published list of errata in the First Edition, and the few other errors discovered have also been corrected. The following are the principal alterations introduced:—

Students had found difficult the investigation of the values of ${}_{n}p_{\overset{r}{\overline{xyz\ldots(m)}}}$ and ${}_{n}p_{\overset{[r]}{\overline{xyz\ldots(m)}}}$, and therefore Arts. 26 to 32 of Chap. ii have been re-written and very much simplified.

In the First Edition, De Morgan's proof was given that, if Simpson's Rule for finding the value of an annuity on three joint lives prevail, then Gompertz's Law of Mortality must hold; but that proof is intricate, and perhaps not quite satisfactory. Therefore Arts. 17 to 31 of Chap. xii have been entirely recast. De Morgan's proof has been abandoned, and the demonstration substituted which was first put forward by Mr. R. Henderson, and subsequently amplified by Mr. A. Levine. This has the advantage of also covering the much more general proposition set out on p. 207 of the present edition.

Additional formulas, derived by means of the Differential Calculus, have been given in Chap. x, Art. 17, for the value of $\overline{A}_x$, and in Chap. xiii, Art. 32, for that of $\overline{A}^1_{xy}$. Also, an alternative solution has been added of the curious problem discussed in Chap. xvi, Art. 66. In Chap. viii the applicability of Conversion Tables to Temporary Assurances has been shown, and the construction of such Tables for continuous benefits explained.

It was discovered that in Chap. v, Arts. 14 and 15, a discrepancy existed between the statement of the question and the solution. These Articles have therefore been re-written, and the discrepancy removed.

Notwithstanding the foregoing changes, and some others of less importance, it has been found possible so to plan that each chapter should begin and end on the same pages as in the First Edition, an arrangement not without its conveniences.

Opportunity has been taken to include in the volume a set of Tables at 2½ per-cent interest, prepared by Mr. H. J. Baker, and this has necessitated the renumbering of the Tables at higher rates. Also, to facilitate the calculation of the values of benefits on joint lives, Tables Nos. xli and xlii have been introduced, showing the Uniform Seniority for two and for three lives respectively, with an explanation, on page 563, of their application. Moreover, Messrs. A. S. Hume and W. Stott having prepared (*J.I.A.* xxxvi, 393) Tables, at 3 per-cent interest, of Annual Premiums for Endowment Assurances on two Joint Lives of equal ages, these likewise have been incorporated in Table No. xliii. It is hoped that this new tabular matter will prove useful to the practising actuary.

G. K.

London, *February*, 1902.

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ANALYTICAL GUIDE

TO THE

TABLES.

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	Table.	Page.	Table.	Page.	Table.	Page.	Table.	Page.	Table.	Page.	Table.	Page.	Table.	Page.	Table.	Page.
Constants	...	493	...	503	...	511	...	519	...	527	...	535	...	543	...	551
LIFE TABLE.																
l_x, d_x, μ_x, $\Delta\mu_x$, p_x, p_x^{-1}, q_x, L_x, N'_x, T_x, $\mathring{e}_x$	i	494														
log l_x, log p_x, Δ log p_x, Colog l_x, Colog p_x, Δ Colog p_x, log μ_x, log L_x, log d_x, Δ log d_x, $-$ Δ log d_x, log $\frac{1}{2}(p_x^{-1}+1)$, Δ log $\frac{1}{2}(p_x^{-1}+1)$, log $(p_x^{-1}-1)$, Δ log $(p_x^{-1}-1)$,	ii	498														
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a_x	...	...	vi	507	xi	515	xvi	523	xxi	531	xxvi	539	xxxi	547	xxxvi	555
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$P_{xx:\overline{n}\|}$	...	...	...	...	xliii	566	...	...	...	...	...	...	...	...	...	...
CONVERSION TABLES.																
Single Premiums	...	...	xxxix	560	xxxix	560	xxxix	560	xxxix	560	xxxix	560	xxxix	560	xxxix	560
Annual ,, ...	...	...	xl	561	xl	561	xl	561	xl	561	xl	561	xl	561	xl	561

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KEY TO THE NOTATION.

The System of Notation used in this volume is, with a few additions, that which was settled by a special Committee of the Council of the Institute of Actuaries, and published in 1872 as an appendix to the *Institute of Actuaries' Life Tables*. It is for the most part graphic in its character, and when the principles on which it is based are understood, the symbols convey their own interpretation. Hence it is not necessary to give here a complete tabular statement. A few explanatory sentences will suffice.

A letter enclosed in brackets, thus (x), denotes "a person aged x."

For each class of functions a principal letter is used, and its meaning is modified by suffixes, &c., as circumstances may require.

We have

l = Number living.
d = Number dying.
L = Population.
p = Probability of life.
$\left.\begin{matrix} q \\ \mathrm{Q} \end{matrix}\right\}$ = Probability of death.*
μ = Force of mortality.
m = Central death-rate.
e = Expectation of life.
E = Endowment.
$\left.\begin{matrix} a \\ \mathbf{a} \end{matrix}\right\}$ = Annuity, or Life Interest.
A = Assurance, or Single Premium, or Reversion.
$\left.\begin{matrix} \mathrm{P} \\ \pi \end{matrix}\right\}$ = Annual Premium; where P refers generally to ordinary net premiums, and π to special premiums.

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* When the Universal Notation was adopted by the Second International Actuarial Congress, held in London in 1898, it was resolved to limit the symbol q to probabilities involving only one year, and to employ the symbol Q for longer terms. It has not, however, been found possible to adhere strictly to this resolution in the Second Edition of the Text Book, because to have done so would have involved a great many alterations in the stereotyped plates.

z = Rate of sickness.
s = Value of sickness benefit.
c = Contribution for sickness benefit.
(NP) = Next Presentation.
F = Fines.
$\left.\begin{matrix}\mathrm{V}\\ \mathrm{U}\end{matrix}\right\}$ = Policy-value.
W = Free policy.

The ages of the lives to which the principal symbol refers are denoted by letters placed as suffixes at the *lower right* corner of the principal symbol; and if the letters in the suffix are not distinguished by any special mark, then joint lives are intended. Thus,

a_{xyz} = Value of an annuity, first payment at the end of a year, to continue during the joint lives of (x), (y), and (z).
a_{xyz} = Value of a similar annuity, first payment, however, to be made at once.

If one of the letters in the suffix is enclosed in a right angle, then a term-certain is meant, and not the age of a life. Thus,

$a_{x\overline{n}|}$ = Value of an annuity to continue during the joint duration of a life aged x and a term of n years certain; that is, a temporary annuity on the life of (x).

If a perpendicular bar separates the letters in the suffix, then the status after the bar is supposed to follow the status before the bar. Thus,

$a_{y|x}$ = Value of an annuity to a person aged x, to begin on the death of a person aged y.
$\mathrm{A}_{z|xy}$ = Value of an assurance payable on the failure of the joint lives (x) and (y), provided these joint lives survive the life (z).

If a horizontal bar appears above the letters in the suffix, then survivors of the lives, and not joint lives, are intended. The number of survivors is denoted by a letter over the extreme end of the bar. If that letter, say r, is not distinguished by any mark, then the meaning is, *at least r* survivors; but if it is enclosed in square brackets, $[r]$, then the meaning is, *exactly* r survivors. If no letter appears over the bar, then unity is supposed, and the meaning is, at least *one* survivor. Thus,

$e_{\overline{wxyz\ldots(m)}}^{\;r}$ = Expectation of life of the last *r* survivors of *m* lives.

$p_{\overline{wxyz\ldots(m)}}^{\;[r]}$ = Probability that exactly *r* lives out of *m* lives will survive a year.

$a_{\overline{xyz}}$ = Value of an annuity on the last survivor of (x), (y), and (z).

If the letters in the suffix, besides having a bar over them, are each enclosed in brackets, ( ), it shows that the lives are to be nominated in succession, each on the death of the one before it, and are not all nominated at the outset. Numerals are often used to denote the *order* of succession when it is not necessary to distinguish the ages of the lives. The letter relating to the life in possession is not placed in brackets. Thus,

$$\left.\begin{array}{r} A_{\overline{x(y)(z)\&c.}} \\ \text{or}\quad A_{\overline{0(1)(2)\ldots(n)}} \end{array}\right\} = \left\{\begin{array}{l} \text{Value of an assurance to be paid on the death of the} \\ n\text{th life to be nominated in succession after the life} \\ \text{at present in possession.} \end{array}\right.$$

When numerals are placed above or below the letters of the suffix, they designate the order in which the lives are to fail. The numeral placed *over* the suffix points out the life whose failure will finally determine the event; and the numerals placed under the suffix indicate the order in which other lives involved are to fail. Thus,

$$\left.\begin{array}{r} Q^1_{xyz} \\ Q^2_{xyz} \\ Q^3_{xyz} \end{array}\right\} = \text{Total probability that } (x) \text{ will die} \left\{\begin{array}{llll} \text{first} & \text{of the} & \text{three} & \text{lives.} \\ \text{second} & ,, & ,, & ,, \\ \text{third} & ,, & ,, & ,, \end{array}\right.$$

$A^{\overset{4}{}}_{\underset{321}{wxyz}}$ = Value of assurance payable on the death of (w) if he die last of the four lives, the other lives failing in the order, (z) first, (y) second, and (x) third.

$$\left.\begin{array}{r} a_{\underset{1}{\overline{yz}}|x} \\ \text{or}\quad a^{2}_{yz|x} \end{array}\right\} = \left\{\begin{array}{l} \text{Value of annuity to } (x) \text{ after the failure of the} \\ \text{survivor of } (y) \text{ and } (z)\text{, provided } (z) \text{ fail before } (y). \end{array}\right.$$

$A_{\overline{xy}:\underset{2}{z}}$ = Value of assurance payable on the death of the survivor of (x) and (y), if he die before (z).

When, for the sake of distinctness in the symbol, it is desirable tc separate the letters in the suffix, a colon is placed between them. A colon is used instead of a point, so as not to conflict with the decimal point when numbers take the place of letters. Thus we write $a_{x+n:y+n}$, and $A_{35:40}$, &c.

The various marks affecting the suffix may be used in combination. Thus,

$Q^1_{x:\overline{y(\overline{t|})}}$ = Total probability that (x) will die before (y), or within t years after the death of (y).

Here we write $\overline{t|}$ to show that a term-certain is meant. Brackets are added, $(\overline{t|})$, to show that the term-certain is not to commence until the failure of the life aged y. A bar appears, $\overline{y(\overline{t|})}$, to point out that the survivor of y and $\overline{t|}$ is in question; and the numeral, 1, is placed over x to show that the life aged x must fail before the status represented by $\overline{y(\overline{t|})}$.

A letter at the *lower left* corner of the principal symbol relates to the number of years covered by the probability or benefit in question. Thus,

$_np_x$ = Probability that (x) will live n years.

$_nP_x$ = Annual premium, limited to n payments, for an assurance on (x).

$_nV_x$ = Value of policy on (x), which has been in force n years.

paid by company to

$_nE_x$ = Value of endowment on (x), payable if he survive n years.

If the letter comes before a perpendicular bar it shows that a deferred period is meant; while if the letter comes after a perpendicular bar it shows that a temporary period is meant. Thus,

$_{|n}q_x$ = Probability that (x) will die within n years.

$_{n|}q_x$ = Probability that (x) will die in a year, deferred n years, that is, that he will die in the $(n+1)$th year.

$_{n|}a_x$ = Value of an annuity on (x) deferred n years, that is, the first payment to be made at the end of $n+1$ years.

$_{|n}a_x$ = Value of a temporary annuity on (x) for n years.

A letter at the *upper left* corner of the principal symbol means that the ages of *all* the lives mentioned in the suffix are to be increased. Thus,

$^nl_{xyz} = l_{x+n:y+n:z+n}$

$^na_{y|x} = a_{y+n|x+n}$

A letter in brackets at the *upper right* corner of the principal symbol shows the number of intervals into which the year is to be divided. Thus,

$a^{(m)}$ = Value of an annuity payable by m instalments of $\frac{1}{m}$ each throughout the year.

$P^{(m)}$ = Premium per annum payable at intervals m times throughout the year.

$A^{(m)}$ = Value of an assurance payable at the end of the interval of $\frac{1}{m}$ in which (x) dies. Therefore $A^{(m)}$ represents the value of an assurance payable $\frac{1}{2m}$ of a year after death. Hence:

$A^{\left(\frac{t}{2}\right)}$ = Value of an assurance payable $\frac{1}{t}$ of a year after death.

If the year be divided into an infinite number of infinitesimal parts, that is, if m become indefinitely great, instead of writing (∞), a bar is placed over the principal symbol. Thus,

$\bar{a}$ = Value of a continuous or momently annuity.

$\bar{A}$ = Value of an assurance payable at the moment of death.

In the case of premiums and policy-values the notation requires attention, as it departs a little from the foregoing rules. We have,

$P^{(m)}$ = as above, the premium per annum payable at intervals m times throughout the year.

$V^{(m)}$ = Value of a policy secured by the premium $P^{(m)}$.

${}^{(m)}P$ = Annual premium for the assurance $A^{(m)}$.

${}^{(m)}V$ = Policy-value in connection with the assurance $A^{(m)}$.

${}^{(\infty)}P$ = Annual premium for the assurance $\bar{A}$.

${}^{(\infty)}V$ = Policy-value in connection with the assurance $\bar{A}$.

In the case of church livings also, the foregoing rules of notation are not strictly adhered to. We have,

(NP) = Value of next presentation.

$(NP)^{(n)}$ = Value of next n presentations.

$(NP)^{\infty}$ = Value of advowson.

A small circle placed over the principal symbol shows that the benefit is to be complete. Thus,

$\mathring{e}$ = Complete expectation of life.
$\mathring{a}$ = Value of complete annuity.

In the case of reversionary annuities, distinction has to be made between those where the times of year at which payments are to take place are determined at the outset, and those where the times depend on the moment of failure of the preceding status. We therefore write,

$a_{y|x}$ = Value of annuity to (x) first payment at the end of the year of the death of (y), or, on the average, six months after his death.
$\hat{a}_{y|x}$ = Value of annuity to (x), first payment one year after death of (y).

Compound symbols are sometimes used. Thus,

$$\left.\begin{array}{lll}(\mathrm{I}a) & = & \text{Value of annuity} \\ (\mathrm{IA}) & = & \text{,, assurance}\end{array}\right\}\text{commencing at 1 and increasing 1 per annum.}$$

$(\mathrm{v}a)$ = Value of varying annuity.
(vA) = ,, ,, assurance.

If the whole benefit is to be temporary the symbol of limitation is placed outside the brackets. Thus,

$(\mathrm{I}a)_{x\overline{n}|}$ = Value of temporary increasing annuity.
$(\mathrm{IA})^{1}_{x\overline{n}|}$ = ,, ,, ,, assurance.
$(\mathrm{v}a)_{\overline{xn}|}$ = ,, ,, varying annuity.
$(\mathrm{vA})^{1}_{\overline{xn}|}$ = ,, ,, ,, assurance

If only the increase or variation is to be temporary, but the benefit to be for the whole of life, then the symbol of limitation is placed immediately after the symbol I or v. Thus,

$$\left.\begin{array}{lll}(\mathrm{I}_{\overline{n}|}a)_x & = & \text{Value of whole-life annuity} \\ (\mathrm{I}_{\overline{n}|}\mathrm{A})_x & = & \text{,, ,, assurance}\end{array}\right\}\text{increasing for } n \text{ years.}$$

$$\left.\begin{array}{lll}(\mathrm{v}_{\overline{n}|}a)_x & = & \text{,, ,, annuity} \\ (\mathrm{v}_{\overline{n}|}\mathrm{A})_x & = & \text{,, ,, assurance}\end{array}\right\}\text{varying for } n \text{ years.}$$

In connection with annual premiums, sometimes it is not convenient to place all the explanatory suffixes, &c., round the letter P. In such cases the letter P is merely placed before the symbol for the benefit. Thus,

$\mathrm{P}a_{y|x}$ = Annual premium for a reversionary annuity.

As regards *Interest*, the following notation is employed.

i = Effective rate of interest, that is, the interest on 1 actually realized in 1 year. When more rates than one are involved in a question, i denotes the principal rate, the other rates being distinguished by an accent, thus i', or by using other letters such as I or J.

$j_{(m)}$ = $m\{(1+i)^{\frac{1}{m}}-1\}$=nominal rate of interest, convertible m times a year, when the effective rate is i.

δ = $j_{(\infty)}=\log_e(1+i)$=force of interest=force of discount.

v = $(1+i)^{-1}$.

d = $1-v=vi$.

Commutation Columns.

For the Notation used in connection with Commutation Columns, see Chap. xvi, Art. 3; Chap. vii, Arts. 53 and 54; and Chap. xiii, Art. 17.

INSTITUTE OF ACTUARIES' TEXT-BOOK.

PART II.

LIFE CONTINGENCIES.

CHAPTER I.

THE MORTALITY TABLE.

ELEMENTARY ILLUSTRATIONS.

1. THE Mortality Table is the instrument by means of which are measured the probabilities of life and the probabilities of death.

2. Could we find 100,000 children all born at the same moment, and could we follow them throughout life, and enter in a column the numbers who remain at the end of each successive year until all have passed away, we should form the column *living* (*see* Table I), headed with the symbol l_x; where l_x represents the number who attain the *precise age* x.

3. If, instead of recording the survivors on each anniversary of birth, we were to count those of the 100,000 simultaneously born who die before reaching the first anniversary, and place that number opposite age 0 in the table; and if we were also to count those who die between the first and second anniversaries, the second and third, and so on, and place the resulting numbers opposite the ages 1, 2, &c., of the table; we should form the column headed d_x; where d_x represents the

number out of the l_x persons attaining the precise age x, who die before reaching the age $x+1$. That is,

$$d_x = l_x - l_{x+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

4. It is evident from the principles on which these columns are constructed, that the sum of the numbers in column d, from age x to the oldest age in the table, both inclusive, is equal to l_x; a relation which we may symbolize by the equation

$$\Sigma d = l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where the suffixes are omitted, because the equation is perfectly general.*

5. We have so far supposed all the persons observed to have been born at the same moment of time; but that assumption is not necessary. The table could be constructed with equal facility, were we able to trace throughout life, as above described, any 100,000 children, selected no matter how various might be the dates of their births. We should only have to register, as before, the numbers completing the first, second, third, &c., years of age; or the numbers dying in these respective years.

6. Further, we could, under suitable circumstances, construct the table without tracing the history of individual children. If there were a community which had been undisturbed by emigration or immigration for a whole generation, say 100 years; and if in that community there had been exactly 100,000 births every year, all the births always taking place on the same day of the year, say 1 January; and if, throughout the whole period, the law of mortality had never varied; it is evident that, by taking on 1 January in any year a complete census of the population according to age, we should form the column l_x. We should have the 100,000 just born; and we should have the survivors of the 100,000 who were born exactly 1, 2, 3, &c., years ago; and the total population on that day would be the sum of the column l_x. Or if, instead of taking a census of the population, we were to extract from the burial register for any one year the numbers dying in the first, second, third, &c., years of age; we should form the column d_x. For in any year of age, say for example between the ages 59 and 60, those entered in the burial register would be those dying out of the 100,000 births which took place on 1 January 59 years ago.

7. It is, however, not necessary to assume that the births all happen

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* It should be noted that this use of the symbol Σ, though convenient, is not strictly in accordance with the notation of the Calculus of Finite Differences. *See* Chap. xxiv, Art. 3.

simultaneously. It is more in accordance with nature to suppose that they occur at equal intervals throughout the year; in which case, at a census at any given date, there would be found persons living at all the intermediate fractional ages in each year of life. Under such circumstances the total population would not be exactly the sum of the column l_x. In each year of age a certain number of persons would be found to exist; but that number would not be exactly represented by any value in the column l. As we have seen, l_x is the number who, out of a given number born, attain the precise age x; but in the population situated as described, persons would be found of every intermediate fractional age between x and $x+1$. The mortality table, however, by a very convenient and natural assumption, may be made to represent such a population. If we suppose the numbers dying in any year of age, to die at equal intervals throughout the year—that is, if we assume the deaths to be equally distributed throughout the year of age—the numbers who attain the middle of the year of age will be the arithmetical mean of those commencing the year and those completing it; that is

$$l_{x+\frac{1}{2}}=\tfrac{1}{2}(l_x+l_{x+1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, on the same assumption as to deaths in the community now under consideration, the population between the ages x and $x+1$ would also be $l_{x+\frac{1}{2}}$. While, therefore, as before, the enumeration of the deaths in any given calendar year would produce the column d_x; a census would not give the column l_x: it would give a table of *population*, which however would correspond to a column formed of the values of $l_{x+\frac{1}{2}}$. The total population at all ages would therefore be $l_{\frac{1}{2}}+l_{1+\frac{1}{2}}+l_{2+\frac{1}{2}}+$&c. which, on the assumption of uniform distribution of deaths, is equal to $\frac{1}{2}(l_0+l_1)+\frac{1}{2}(l_1+l_2)+$&c., or $\frac{1}{2}l_0+l_1+l_2+$&c.

8. We may here state once for all that, in dealing algebraically with questions in life contingencies, the assumption is usually made that the deaths in each year of age are uniformly distributed throughout the year. If the numbers in the column d_x were equal at all ages, the assumption would be in strict accordance with fact; but in every table based on actual statistics, the value of d_x varies from age to age; and if, for example, the deaths in the $(n+1)$th year be more numerous than those in the nth, then the deaths in any portion of the nth year towards its close will no doubt be more numerous than the deaths in any equal portion towards its commencement. The assumption, then, of a uniform distribution of deaths, introduces an error; but we shall hereafter see that the error, except perhaps for the two years immediately succeeding

birth, is infinitesimal; and in all practical calculations may be disregarded. When the differential calculus is made use of, and in some cases when the calculus of finite differences is employed, it is not necessary to presuppose a uniform distribution of deaths.

9. By definition of the symbol, $l_{x+\frac{1}{2}}$ represents the number, out of l_0 born, who attain the precise age $x+\frac{1}{2}$. We have also seen that, in a stationary population, supported by l_0 annual births, and not affected by migrations, $l_{x+\frac{1}{2}}$ represents the number living in that population between the ages x and $x+1$. It will be convenient to distinguish these two ideas by difference of symbol, although the values of the quantities may be the same; and we shall therefore use the symbol L_x to represent the *population*, in a stationary community, between the ages x and $x+1$; while we shall retain the symbol $l_{x+\frac{1}{2}}$ to represent the number, out of l_0 born, who attain the precise age $x+\frac{1}{2}$. When births and deaths are uniformly distributed over the year, $L_x=l_{x+\frac{1}{2}}$; but under other circumstances the equality does not necessarily exist.

10. A population cannot be found of the nature supposed above, for no population in the world is stationary. Many causes of disturbance occur; and therefore it is not possible, by a simple census, or by the mere enumeration of the deaths, to construct a true mortality table; but yet by an inverse process we can, from the principles above suggested, attain the desired result. As already stated, the mortality table affords us the means of measuring the probabilities of life and also of death; and now, conversely, we may add that, having given these probabilities, we can form the mortality table.

11. If an event may happen in a ways, and fail in β ways, all these ways being equally likely; the probability of the happening of the event is $\frac{a}{a+\beta}$, and the probability of the failure of the event is $\frac{\beta}{a+\beta}$; Writing $\frac{a}{a+\beta}=p$, and $\frac{\beta}{a+\beta}=q$, we have $p+q=1$; and $p=1-q$; and $q=1-p$. Now, in dealing with a group of unknown persons, all similarly circumstanced and all of the same age, we must assume that they all enjoy an equal chance of life, and run an equal risk of death. The number l_x represents such a group. Of the l_x, there survive a year l_{x+1}; and the probability of any one of them living a year is therefore $\frac{l_{x+1}}{l_x}$. There die within a year d_x; and the probability of any one of them dying within a year is therefore $\frac{d_x}{l_x}$. If now we write $a=l_{x+1}$, and $\beta=d_x$, and consequently $\alpha+\beta=l_x$; and if to p and q we add the suffix x, to indicate the age with which we are at present concerned, we have

$$p_x=\frac{l_{x+1}}{l_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$q_x=\frac{d_x}{l_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where p_x is the probability that a person of the precise age x will survive one full year; and q_x is the probability that that person will die within one year. In the same way $p_{x+1}=\frac{l_{x+2}}{l_{x+1}}$; $p_{x+2}=\frac{l_{x+3}}{l_{x+2}}$; and so on.

From the relation between p and l established in the last paragraph, it follows that $l_{x+1}=l_x\times p_x$; $l_{x+2}=l_x\times p_x\times p_{x+1}$; and, generally,

$$l_{x+n}=l_x\times p_x\times p_{x+1}\times p_{x+2}\times \ldots \times p_{x+n-1} \quad . \quad . \quad . \quad (6)$$

and it would therefore appear that, having l_x, and the values of p_x, p_{x+1}, &c., to the oldest age, we can, by continued multiplication, calculate the values of l for all the higher ages; and if $x=0$, that is, if we know the number born, and the probability at each age of living a year, we can construct the entire mortality table. But since it is not required that the numbers against the several ages in the mortality table shall represent *absolute* numbers living, but only *relative* numbers living, we can give to l_0 any value we think suitable; and then, from a complete column of p_x, construct a mortality table. The arbitrary value of l_0—or of l_x where x is the youngest tabulated age—is called the *Radix* of the table.

12. If now we take any community of sufficient size to supply trustworthy averages; and if we make a census, at the middle of any calendar year, of the population in each year of age; and if we also keep a record of the deaths, in each year of age, during that calendar year; we can, by dividing the latter by the former, get the ratio of *deaths* to *population* for each year of age; which ratio, for the year from x to $x+1$, we shall denote by the symbol m_x.

13. A very simple relation exists between m_x and p_x. We have seen that, in the mortality table, the population between the ages x and $x+1$ is represented by $l_{x+\frac{1}{2}}$, which is equal to $\frac{1}{2}(l_x+l_{x+1})$, or, what is the same thing, $l_x-\frac{1}{2}d_x$, or $l_{x+1}+\frac{1}{2}d_x$; while d_x represents the deaths corresponding to that population; whence

$$m_x=\frac{d_x}{l_{x+\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Also

$$p_x = \frac{l_x - d_x}{l_x} = \frac{l_x - \frac{1}{2}d_x - \frac{1}{2}d_x}{l_x - \frac{1}{2}d_x + \frac{1}{2}d_x}$$

$$= \frac{l_{x+\frac{1}{2}} - \frac{1}{2}d_x}{l_{x+\frac{1}{2}} + \frac{1}{2}d_x} = \frac{1 - \frac{1}{2}m_x}{1 + \frac{1}{2}m_x} = \frac{2l_{x+\frac{1}{2}} - d_x}{2l_{x+\frac{1}{2}} + d_x} \quad ✓$$

$$= \frac{2 - m_x}{2 + m_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

14. Having then, from the census returns and the burial register, as explained in Art. 12, obtained the ratio m_x for all values of x, we can, by means of formula 8, pass to p_x; and from p_x, by continued multiplication, we can construct the mortality table.

15. We have said, Art. 10, that, in consequence of the constant fluctuations in every population, it is not possible by a census alone, or by a mere enumeration of the deaths, to construct a true mortality table; and it may be useful to illustrate this remark. The mortality table shows how many persons complete each year of age out of a given number born. Now, suppose a population of such a character that it always doubles itself by natural increase in the course of sixty years. If at present there are 100,000 annual births, then sixty years ago there were but 50,000. If we only count the deaths, it is evident that the mortality among infants is doubled, relatively to the mortality among persons sixty years of age; for the deaths among children under one year of age, are the deaths occurring out of 100,000 annual births; while those among persons aged 60, are due to only 50,000 annual births; and we must therefore double the latter, to bring it up to the same proportion as the former. If d'_x represent the enumerated deaths in the year of age x to $x+1$; and, as before, d_x the deaths according to the mortality table; we have $d_{60} = 2d'_{60}$; and $d_0 = d'_0$. From these two equations we can find the correction to apply to the recorded deaths, in order to form the column d_x. If r be the annual rate of increase of the population, we have $(1+r)^{60} = 2$; whence $(1+r) = \sqrt[60]{2}$, and

$$d_0 = d'_0$$

$$d_1 = (1+r)d'_1$$

$$d_2 = (1+r)^2 d'_2$$

$$\&c. = \&c.$$

$$d_x = (1+r)^x d'_x$$

$$\&c. = \&c.$$

16. It is hardly necessary to say that, as the increase of a population is affected by migrations, and as the natural rate of increase varies at different periods, the correction of the last paragraph is not practically applicable; and, to construct a true mortality table, recourse must be had to an actual comparison of the living with the dying.

The Northampton Table may be adduced as an example of a table constructed upon false principles from the deaths alone; and the Carlisle Table as an example of one correctly constructed.

17. In Arts. 12 and 13 we spoke of m_x, the ratio of *deaths* to *population*. This ratio has been called by Farr the *rate of mortality*; but almost all other writers reserve the name *rate of mortality* for the probability of dying in a year,—the q_x of Art. 11; and in this respect we shall follow the example of the majority. The convenient name *central death rate* has been given to m_x, a name which we shall retain for the function. From equation 8 we can easily deduce the relation

$$q_x = \frac{2m_x}{2+m_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

18. The records of a life assurance office afford a very important source from which to derive the materials for the construction of a mortality table; but a different method must be followed from that sketched out in this chapter.

CHAPTER II.

PROBABILITIES OF LIFE.

1. It is not intended here to expound the Theory of Probability; but, on the contrary, in the following pages a knowledge of the elements of that theory on the part of the reader is assumed. In commencing this chapter it will, however, be convenient to recapitulate, without demonstration, a few of the more simple propositions; and we shall then show how a mortality table is used to calculate the probabilities of life.

2. If an event may happen in a ways, and fail in β ways, all these ways being equally likely, the probability of the happening of the event is $\frac{a}{a+\beta}$; and the probability of the failure of the event is $\frac{\beta}{a+\beta}$. Writing $\frac{a}{a+\beta}=p$, and $\frac{\beta}{a+\beta}=q$; the probability of the happening of the event is p, or $1-q$; and the probability of the failure of the event is q, or $1-p$.

3. If there be several independent events, the probabilities of the happening of which are respectively p_1, p_2, p_3, &c., then the probability that all the events will happen is $p_1 \times p_2 \times p_3 \times$ &c.;—that all will fail, $(1-p_1)(1-p_2)(1-p_3)$ &c.; that all will not happen, that is, that at least one will fail, $1-p_1 \times p_2 \times p_3 \times$ &c.;—that all will not fail, that is, that at least one will happen, $1-(1-p_1)(1-p_2)(1-p_3)$ &c.;—that the first only will happen, and all the rest fail, $p_1(1-p_2)(1-p_3)$&c.;—that exactly one will happen, and all the rest fail, $p_1(1-p_2)(1-p_3)$ &c. $+p_2(1-p_1)(1-p_3)$&c. $+p_3(1-p_1)(1-p_2)$&c. + &c.

4. If there be two or more events which cannot concur, then the probability that one or other of them will happen is the sum of their separate probabilities.

5. If there be two events dependent on each other in such a way that before the second can happen the first must have happened, then

the probability of the concurrence of the two events is, the product of the probability of the first, into the probability that when that has happened the second will follow.

6. Passing now to the Mortality Table; if there be l_x persons living at age x; and if of these, l_{x+n} survive to age $x+n$; then the probability that a person aged x will survive n years is $\frac{l_{x+n}}{l_x}$. To this probability we shall assign the symbol ${}_np_x$, or, when $n=1$, p_x, so that

$$ {}_np_x=\frac{l_{x+n}}{l_x} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1) $$

In the same way, the probability that a person aged y will survive n years is $\frac{l_{y+n}}{l_y}$, or ${}_np_y$. Thus (see Table I.) the probability that a person aged 60 will survive 10 years is $\frac{37977}{58842}=\cdot 64541$, and the probability that a person aged 25 will survive 10 years is $\frac{86137}{93044}=\cdot 92576$.

7. The probability that a life aged x, and a life aged y, will both survive n years, is ${}_np_x \times {}_np_y$; which we shall write ${}_np_{xy}$. Thus the probability that a father aged 60, and his son aged 25, will both be alive at the end of 10 years, is $\cdot 64541 \times \cdot 92576 = \cdot 59749$.

The probability ${}_np_{xy}$ we can derive directly from the mortality table, as follows. There are l_x persons aged x, and l_y persons aged y; and there are therefore $l_x \times l_y$ pairs of persons, one of whom is aged x, and the other aged y, because each of the persons of the l_x group may be paired off with each of the persons of the l_y group. Of the l_x persons, l_{x+n} will survive n years; and of the l_y persons, l_{y+n} will survive n years. Therefore, at the end of n years, $l_{x+n} \times l_{y+n}$ pairs will remain unbroken, of the $l_x \times l_y$ pairs that entered on the n years. Therefore the probability that any particular pair will remain unbroken is

$$ \frac{l_{x+n} \times l_{y+n}}{l_x \times l_y} = {}_np_x \times {}_np_y = {}_np_{xy} \quad . \; . \; . \; . \quad (2) $$

8. It will be convenient if we denote the product $l_x \times l_y$ by the symbol l_{xy}. Then ${}_np_{xy}=\frac{l_{x+n:y+n}}{l_{xy}}$. Also, we shall often find it useful to have a more concise symbol for such a function as $l_{x+n:y+n}$; because otherwise, if many lives are involved, the suffix will become awkwardly lengthy. We shall therefore write ${}^nl_{xy}$ for $l_{x+n:y+n}$, whenever such a contraction is desirable; the letter or number placed at the upper left-hand corner of the symbol, denoting that all the lives included in the suffix are

to be supposed to be to that extent equally increased in age. Thus $^nl_{xyz}=l_{x+n:y+n:z+n}$; and $^{10}l_{60:25}=l_{70:35}$. A similar system of notation will be followed for all other symbols, as well as for l_x. Thus $_n^tp_{xyz}$ represents the probability that three lives, aged respectively $x+t$, $y+t$, and $z+t$, will jointly survive n years.

9. For brevity we shall also frequently write (x) for "a person aged x". Thus, when we shall speak of the probability that (x) will survive n years, we shall mean the probability that a person aged x will survive n years.

10. The probability that (x) will not live n years, or that (x) will die within n years, is $1-{}_np_x$, which is equal to $\dfrac{l_x-l_{x+n}}{l_x}$. This probability we shall denote by $_{|n}q_x$; or simply by q_x, when $n=1$.[*] Therefore

$$_{|n}q_x=1-{}_np_x=\frac{l_x-l_{x+n}}{l_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

11. The probability that neither of the two lives (x) and (y) will survive n years, or that both of them will die within n years, is $(1-{}_np_x)\,(1-{}_np_y)={}_{|n}q_x\times{}_{|n}q_y$; and this probability may be denoted by $_{|n}q_{\overline{xy}}$, the bar over the suffix informing us that the survivor of the lives is in question, and not the joint lives. Thus

$$_{|n}q_{\overline{xy}}={}_{|n}q_x\times{}_{|n}q_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

12. The probability that at least one of the lives (x) and (y) will survive n years, is the probability that both lives will not die within the period; that is, $1-{}_{|n}q_{\overline{xy}}$. Writing this in the form $1-(1-{}_np_x)\,(1-{}_np_y)$, and multiplying out, we have the probability that the survivor of the lives will live n years,

$$_np_{\overline{xy}}={}_np_x+{}_np_y-{}_np_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

13. The probability that (x) will survive n years, and (y) die within the n years, is

$$_np_x(1-{}_np_y)={}_np_x-{}_np_{xy}$$

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[*] It should be noted here that when a probability is limited to a fixed number of years, we express the fact of limitation by placing a perpendicular bar to the left of the symbol representing the probability, and writing on the *right* of the bar the number of years of limitation. Thus $_{|n}q_x$ represents the probability that (x) will die within the next n years. If the probability be deferred, we write the number of years on the *left* of the bar. Thus $_{t|n}q_x$ represents the probability that (x) will die in the n years succeeding the first t years; that is, that he will survive t years, and then die within n years.

14. The probability that one of the lives, (x) and (y), will survive n years, and the other fail within the n years, is composed of the sum of two probabilities: first that (x) will survive and (y) die, which is (Art. 13) ${}_np_x - {}_np_{xy}$: and secondly, that (y) will survive and (x) die, which is ${}_np_y - {}_np_{xy}$. The probability we seek is therefore ${}_np_x + {}_np_y - 2\,{}_np_{xy}$. This is the probability that *exactly* one of the lives will survive: and it must be carefully distinguished from the probability that *at least* one of the lives will survive.

15. The probability that at least one of the lives (x) and (y) will fail within n years, that is, that the joint lives will fail, is $1 - {}_np_x \times {}_np_y$, or $1 - {}_np_{xy}$. This probability we shall denote by ${}_{\lvert n}q_{xy}$. Thus

$${}_{\lvert n}q_{xy} = 1 - {}_np_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

16. The probability that (x) will die in the nth year from the present time, is the probability that he will survive $n-1$ years, multiplied by the probability that, when he has so survived, he will then die within one year. The second of these factors is evidently q_{x+n-1}, and the whole probability is therefore ${}_{n-1}p_x \times q_{x+n-1}$. This probability, which is denoted by ${}_{n-1|}q_x$, may be derived directly from the table as follows. Of the l_x persons alive at the age x, d_{x+n-1} will die in the nth year; and the probability of dying in the nth year is therefore $\dfrac{d_{x+n-1}}{l_x}$. This may be written $\dfrac{l_{x+n-1}}{l_x} \times \dfrac{d_{x+n-1}}{l_{x+n-1}}$, which is seen to be the same as ${}_{n-1}p_x \times q_{x+n-1}$, already given.

l_{x+n-1} start nth year

The same probability may be arrived at in another, and perhaps more useful, way. The probability in question is evidently the difference between the probability that (x) will survive $n-1$ years, and the probability that he will survive n years; that is

$${}_{n-1|}q_x = {}_{n-1}p_x - {}_np_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This last result can easily be shown to be identical with the result already obtained, for

$$\begin{aligned} {}_{n-1|}q_x = {}_{n-1}p_x - {}_np_x &= \frac{l_{x+n-1}}{l_x} - \frac{l_{x+n}}{l_x} \\ &= \frac{d_{x+n-1}}{l_x}, \text{ as before} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8) \end{aligned}$$

With regard to the symbol ${}_{n-1|}q_x$ for the probability that (x) will die in a year deferred $n-1$ years, we may remark that if it were written in full it would stand ${}_{n-1|1}q_x$. No confusion will, however, be caused

$n-1|q_{\overline{xy}} - {}_np_{\overline{xy}}$

t we omit the symbol for unity, and merely write ${}_{n-1|}q_x$, in the same way that we write p_x for ${}_1p_x$. In fact, the symbol for unity may be omitted whenever the omission will not lead to misunderstanding.

17. We have, by definition, $d_x = l_x - l_{x+1}$. By analogy then, $d_{xy} = l_{xy} - l_{x+1:y+1}$, or $l_x \times l_y - l_{x+1} \times l_{y+1}$. Thus we see that, although l_{xy} is equal to $l_x \times l_y$, yet d_{xy} is *not* equal to $d_x \times d_y$; because $d_x \times d_y = (l_x - l_{x+1})(l_y - l_{y+1}) = l_{xy} - l_{x:y+1} - l_{x+1:y} + l_{x+1:y+1}$. The student must make a particular note of this fact, or he will be apt to fall into error.

18. The probability that *both* (x) and (y) will die in the nth year from the present time is ${}_{n-1|}q_x \times {}_{n-1|}q_y = ({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y - {}_np_y) = \frac{d_{x+n-1}}{l_x} \cdot \frac{d_{y+n-1}}{l_y}$.

19. The probability that the *first* death among the two lives will happen in the nth year from the present time,—that is, that the joint lives will fail in the nth year,—is the probability that both the lives will survive for $n-1$ years, less the probability that they will both survive for n years. In symbols,

$$ {}_{n-1|}q_{xy} = {}_{n-1}p_{xy} - {}_np_{xy} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

20. The probability that the *second* death among the two lives will happen in the nth year, is the probability that at least one of them will survive $n-1$ years, less the probability that at least one of them will survive n years. In symbols,

$$ {}_{n-1|}q_{\overline{xy}} = {}_{n-1}p_{\overline{xy}} - {}_np_{\overline{xy}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This may also be written

$$ {}_{n-1|}q_{\overline{xy}} = {}_{n-1|}q_x + {}_{n-1|}q_y - {}_{n-1|}q_{xy} \quad . \quad . \quad . \quad . \quad (11)$$

21. The probability that *one only* of the two lives will fail in the nth year, is the sum of the two probabilities; first that (x) will die in the nth year, and that (y) will not so die, namely ${}_{n-1|}q_x(1 - {}_{n-1|}q_y)$; and secondly, that (y) will die in the nth year, and that (x) will not so die, namely ${}_{n-1|}q_y(1 - {}_{n-1|}q_x)$. The probability we seek is therefore ${}_{n-1|}q_x(1 - {}_{n-1|}q_y) + {}_{n-1|}q_y(1 - {}_{n-1|}q_x) = {}_{n-1|}q_x + {}_{n-1|}q_y - 2\,{}_{n-1|}q_x \times {}_{n-1|}q_y$.

22. The probability that *neither* of the two lives will fail in the nth year, is the probability that (x) will not so die, multiplied by the probability that (y) will not so die; that is $(1 - {}_{n-1|}q_x)(1 - {}_{n-1|}q_y) = 1 - {}_{n-1|}q_x - {}_{n-1|}q_y + {}_{n-1|}q_x \times {}_{n-1|}q_y$.

23. The probability that *one at least* of the two lives (x) and (y) will fail in the nth year, is evidently the arithmetical complement of the probability found in the last article; that is ${}_{n-1|}q_x + {}_{n-1|}q_y - {}_{n-1|}q_x \times {}_{n-1|}q_y$.

that x will die before 2d yr and y survive [illegible]

$q_{\overline{xyz}} = 1 - (1 - {}_{10}p_x)(1 - {}_{10}p_y)(1 - {}_{10}p_z)$

24. The probability that (x) will survive n years, and that (y) will survive $n-1$ years is ${}_np_x \times {}_{n-1}p_y$. In its present form we cannot express this probability as a joint-life probability, because the two factors composing it are not homogeneous; these factors being the probability that (x) will live n years, and that (y) will live $n-1$ years. In our future investigations compound probabilities of this kind will frequently appear; and we shall find that the analysis will be much assisted if we can resolve them into joint-life probabilities. By means of a simple algebraical artifice such resolution is easy. Because ${}_np_{y-1} = p_{y-1} \times {}_{n-1}p_y$, therefore ${}_{n-1}p_y = \dfrac{{}_np_{y-1}}{p_{y-1}}$. Substituting this value in the compound probability we have

$$ {}_np_x \times {}_{n-1}p_y = \frac{{}_np_x \times {}_np_{y-1}}{p_{y-1}} = \frac{{}_np_{x:y-1}}{p_{y-1}} \quad . \quad . \quad . \quad (12)$$

In the numerator we have now an ordinary joint-life probability, and in the denominator we have a probability independent of the value of n. In Table I we have given the values of p_x^{-1} so as to facilitate the calculation of such probabilities, by changing the process of division into multiplication.

25. In complicated questions in life contingencies, the chief difficulty often is to unravel the probabilities. For instance, we have seen that, in the case of a temporary probability, ${}_{|n}q_x \times {}_{|n}q_y = {}_{|n}q_{\overline{xy}}$; and we might hastily assume that a similar equation holds for deferred probabilities, and that ${}_{n-1|}q_x \times {}_{n-1|}q_y = {}_{n-1|}q_{\overline{xy}}$; but this last equation is quite wrong. By Art. 20 the correct expression for ${}_{n-1|}q_{\overline{xy}}$ is ${}_{n-1|}q_x + {}_{n-1|}q_y - {}_{n-1|}q_{xy}$. In fact, the probability ${}_{n-1|}q_{\overline{xy}}$ includes a case omitted from the probability ${}_{n-1|}q_x \times {}_{n-1|}q_y$. The probability ${}_{n-1|}q_x \times {}_{n-1|}q_y$ supposes both the lives to fail in the nth year. In the probability ${}_{n-1|}q_{\overline{xy}}$ both lives may fail in the nth year, but they need not necessarily do so. The conditions are satisfied if the survivor die in the nth year; and the other life may fail either in that year or in any previous year.

Only thought and practice can make the student expert and accurate in unravelling probabilities; and to assist him on the threshold of the subject we have given the foregoing illustrations at some length. He can set himself many more examples of a similar character; and he will find it very instructive practically to make use of the mortality table, and, taking lives of specified ages, to calculate the numerical values of the probabilities he deduces. He must remember that the probabilities are not mere abstract conceptions, but actual quantities, having definite numerical values, depending upon the ages assumed for (x) and (y), and upon the mortality table employed.

26. The principles which we have so far investigated have been illustrated with only two lives; but it is evident that we may include as many lives as we please. We may have, for instance, $_np_{wxyz}$, and $_np_{\overline{wxyz}}$. We now proceed to the examination of these more general cases.

27. Let $_np_{\overline{wxyz\ldots(m)}}^{[r]}$ represent the probability that *exactly* r lives out of m lives will survive n years,—the letter (m), in brackets, relating to the number of lives involved, and not to the age of a life. The investigation for the present being perfectly general, and not depending on the value of n, we may in the meantime write the probability in question $p_{\overline{wxyz\ldots(m)}}^{[r]}$.

To find $p_{\overline{wxyz\ldots(m)}}^{[r]}$.

The required probability is the sum of the probabilities that r designated lives will survive, and $m-r$ lives fail, for all the combinations of r lives that can be made out of m lives, and is

$$\text{A.}\quad p_{\overline{wxyz\ldots(m)}}^{[r]}=\left\{\begin{array}{l} \quad p_{wx\ldots(r)}(1-p_y)(1-p_z)\ldots(m-r\text{ factors})\\ +\,p_{wy\ldots(r)}(1-p_x)(1-p_z)\ldots(\text{ do. do. })\\ +\,p_{wz\ldots(r)}(1-p_x)(1-p_y)\ldots(\text{ do. do. })\\ +\quad\&\text{c.,}\qquad\&\text{c.,}\qquad\&\text{c.}\end{array}\right\}$$

If by Z^r we represent the sum of the probabilities of r joint lives surviving, for all the combinations of r lives which can be made out of m lives; so that, for example, when there are four lives,

$$Z^2=(p_{wx}+p_{wy}+p_{wz}+p_{xy}+p_{xz}+p_{yz});$$

and if by (m,r) we denote the number of combinations of m things taken r together; so that $(m,r)=\dfrac{\underline{|m}}{\underline{|r}\,\underline{|m-r}}$, where the symbol $\underline{|m}$ denotes the product of all the natural numbers, from 1 to m inclusive; then the following equation B, where the law of the terms is sufficiently manifest, we shall now prove to be identical with Scheme A:

$$\text{B}\quad p_{\overline{wxyz\ldots(m)}}^{[r]}=Z^r-Z^{r+1}\times\frac{(m-r,1)(m,r)}{(m,r+1)}+Z^{r+2}\times\frac{(m-r,2)(m,r)}{(m,r+2)}$$
$$-Z^{r+3}\times\frac{(m-r,3)(m,r)}{(m,r+3)}+\&\text{c.}$$

It will be seen that Z^r occurs in Scheme A only once.

The scheme consists of (m,r) lines; and the number of probabilities in each line, each probability including $r+1$ lives, is the same as the number of bracket factors, that is $(m-r,1)$. Therefore, in the scheme, the total number of probabilities, each of which includes $r+1$ lives, is $(m,r)(m-r,1)$. Among these, every possible probability which includes $r+1$ lives is to be found, but each of them occurs more than once; and, as all of them are symmetrically involved, they are all repeated the same number of times. To find how often each distinct probability is repeated, we must therefore divide the total number of probabilities just found, $(m,r)(m-r,1)$, by the number of distinct probabilities, $(m,r+1)$, and we have $\dfrac{(m,r)(m-r,1)}{(m,r+1)}$, which, taken negatively, is the coefficient of Z^{r+1} arising from Scheme A, and which we have given in equation B.

A precisely similar argument holds for the coefficients of Z^{r+2}, Z^{r+3}, &c., and therefore equation B is established.

The coefficients admit of great reduction. The coefficient of Z^{r+t} is, irrespective of sign, $\dfrac{(m-r,t)(m,r)}{(m,r+t)}$, which is equal to

$$\frac{\underline{\left|m-r\right.}}{\underline{\left|t\right.}\,\underline{\left|m-r-t\right.}}\times\frac{\underline{\left|m\right.}}{\underline{\left|r\right.}\,\underline{\left|m-r\right.}}\times\frac{\underline{\left|r+t\right.}\,\underline{\left|m-r-t\right.}}{\underline{\left|m\right.}}=\frac{\underline{\left|r+t\right.}}{\underline{\left|t\right.}\,\underline{\left|r\right.}}=\frac{(r+1)(r+2)\ldots(r+t)}{\underline{\left|t\right.}}.$$

Therefore, giving to t successively the values 1, 2, 3, &c., and noticing that the terms are of alternate sign, we have

$$\text{C.}\quad p_{\overline{wxyz\ldots(m)}}^{[r]}=Z^r-(r+1)Z^{r+1}+\frac{(r+1)(r+2)}{\underline{\left|2\right.}}Z^{r+2}-\&c. \quad . \quad (13)$$

The expression on the right of equation C follows the law of the binomial theorem, and we can write symbolically

$$\text{D.}\quad p_{\overline{wxyz\ldots(m)}}^{[r]}=Z^r(1+Z)^{-(r+1)}=\frac{Z^r}{(1+Z)^{r+1}} \quad . \; . \; . \; . \; . \; . \quad (14)$$

a very compact expression, easily remembered, and consequently very useful. It should be observed that all terms must be omitted after that containing Z^{r+t}, where $r+t=m$, because a higher index than m for Z is, by the nature of the case impossible.

28. As an example, let it be required to determine, when $m=4$, the values of ${}_np_{\overline{wxyz}}^{[1]}$, ${}_np_{\overline{wxyz}}^{[2]}$, ${}_np_{\overline{wxyz}}^{[3]}$, and ${}_np_{\overline{wxyz}}^{[4]}$.

By C, ${}_np_{\overline{wxyz}}^{[1]}=Z-2Z^2+3Z^3-4Z^4$

$$={}_np_w+{}_np_x+{}_np_y+{}_np_z$$
$$-2({}_np_{wx}+{}_np_{wy}+{}_np_{wz}+{}_np_{xy}+{}_np_{xz}+{}_np_{yz})$$
$$+3({}_np_{wxy}+{}_np_{wxz}+{}_np_{wyz}+{}_np_{xyz})-4{}_np_{wxyz}.$$

Again, $\quad {}_np_{\overline{wxyz}}^{[2]} = Z^2 - 3Z^3 + 6Z^4$

$$= {}_np_{wx} + {}_np_{wy} + {}_np_{wz} + {}_np_{xy} + {}_np_{xz} + {}_np_{yz}$$
$$-3({}_np_{wxy} + {}_np_{wxz} + {}_np_{wyz} + {}_np_{xyz}) + 6\,{}_np_{wxyz}.$$

Again, $\quad {}_np_{\overline{wxyz}}^{[3]} = Z^3 - 4Z^4$

$$= {}_np_{wxy} + {}_np_{wxz} + {}_np_{wyz} + {}_np_{xyz} - 4\,{}_np_{wxyz}.$$

Finally, $\quad {}_np_{\overline{wxyz}}^{[4]} = Z^4 = {}_np_{wxyz}.$

29. It will be noticed that, when we have reached an expression involving only probabilities of joint lives, we cannot make any further simplifications. The expression, in fact, is in its "lowest terms"; and it only remains to calculate the numerical values of the joint-life probabilities by means of the mortality table. Compound probabilities, such as ${}_np_{\overline{wxyz\ldots(m)}}^{[r]}$, and ${}_{|n}q_{\overline{xy}}$, must be reduced to simple probabilities which involve only single or joint lives.

30. Let ${}_np_{\overline{wxyz\ldots(m)}}^{\,r}$ represent the probability that *at least* r lives out of m lives will survive n years. To find ${}_np_{\overline{wxyz\ldots(m)}}^{\,r}$.

This probability is the sum of the probabilities that exactly r lives will survive, exactly $r+1$, exactly $r+2$, &c., up to exactly m. We must therefore take the sum of the following expressions, derived from equation C by putting r successively equal to r, $r+1$, $r+2$, &c.

E. Exactly r lives,

$$Z^r - (r+1)Z^{r+1} + \frac{(r+1)(r+2)}{\underline{|2}}Z^{r+2} - \frac{(r+1)(r+2)(r+3)}{\underline{|3}}Z^{r+3} + \&c.$$

Exactly $r+1$ lives, $\quad Z^{r+1} - (r+2)Z^{r+2} + \frac{(r+2)(r+3)}{\underline{|2}}Z^{r+3} - \&c.$

Exactly $r+2$ lives, $\quad Z^{r+2} - (r+3)Z^{r+3} + \&c.$

Exactly $r+3$ lives, $\quad Z^{r+3} - \&c.$

&c. &c. &c.

Taking now the sums of the various columns in Statement E, we have the coefficient of Z^r equal to unity, and the coefficient of Z^{r+1} equal to $-r$. The coefficient of Z^{r+2} is

$$\frac{(r+1)(r+2)}{\underline{|2}} - (r+2) + 1 = \frac{(r+1)(r+2) - 2(r+1)}{\underline{|2}} = \frac{r(r+1)}{\underline{|2}}.$$

In finding the coefficient of Z^{r+3}, it will be noticed that the last three lines in the column for Z^{r+3} are of precisely the same form as the three lines in the column for Z^{r+2} if only we write $r+1$ for r. Therefore, the sum of the coefficients in these lines is $\frac{(r+1)(r+2)}{\underline{|2}}$ and the total coefficient of Z^{r+3}, taking in the first line, is

$$-\frac{(r+1)(r+2)(r+3)}{\underline{|3}}+\frac{(r+1)(r+2)}{\underline{|2}}$$

$$=-\frac{(r+1)(r+2)}{\underline{|3}}\{(r+3)-3\}=-\frac{r(r+1)(r+2)}{\underline{|3}}.$$

Observing the law of the terms, we see that, irrespective of sign, the coefficient of Z^{r+t} must be

$$\frac{r(r+1)(r+2)\ldots(r+t-1)}{\underline{|t}}.$$

If the law holds for the coefficient of Z^{r+t}, it also holds for that of Z^{r+t+1}. In the column for Z^{r+t+1} the sum of the coefficients in all the lines except the first is of the same form, with opposite sign, as the sum of all the coefficients in the column for Z^{r+t}, which, by hypothesis is

$$\frac{r(r+1)(r+2)\ldots(r+t-1)}{\underline{|t}},$$

the only difference being that we must write $r+1$ for r, by doing which we obtain

$$\frac{(r+1)(r+2)(r+3)\ldots(r+t)}{\underline{|t}}.$$

The coefficient of Z^{r+t+1} in the first line is

$$\frac{(r+1)(r+2)\ldots(r+t+1)}{\underline{|t+1}},$$

and the total coefficient of Z^{r+t+1} is the difference between these last two expressions, that is

$$\frac{(r+1)(r+2)\ldots(r+t+1)}{\underline{|t+1}}-\frac{(r+1)(r+2)\ldots(r+t)}{\underline{|t}}$$

$$=\frac{(r+1)(r+2)\ldots(r+t)}{\underline{|t+1}}\{(r+t+1)-(t+1)\}$$

$$=\frac{r(r+1)(r+2)\ldots(r+t)}{\underline{|t+1}},$$

which is of the same form as the coefficient of Z^{r+t}. But we have seen that the form holds when $t=3$, and it must therefore hold universally. Therefore

$$\text{F.}\quad {}_np_{\overset{r}{\overline{wxyz\ldots(m)}}}=\text{Z}^r-r\text{Z}^{r+1}+\frac{r(r+1)}{\underline{|2}}\text{Z}^{r+2}-\&\text{c.}$$

$$+(-1)^t\frac{r(r+1)\ldots(r+t-1)}{\underline{|t}}\text{Z}^{r+t}+\&\text{c.}\quad(15)$$

We have here another binomial expression similar to that in C; and we can write symbolically

$$\text{G.}\quad {}_np_{\overset{r}{\overline{wxyz\ldots(m)}}}=\text{Z}^r(1+\text{Z})^{-r}=\frac{\text{Z}^r}{(1+\text{Z})^r}\quad .\ .\ .\ .\ .\ .\ .\quad(16)$$

31. As an example of Formula F, let it be required to find the values of ${}_np_{\overset{1}{\overline{wxyz}}}$, ${}_np_{\overset{2}{\overline{wxyz}}}$, ${}_np_{\overset{3}{\overline{wxyz}}}$, and ${}_np_{\overset{4}{\overline{wxyz}}}$.

We have ${}_np_{\overset{1}{\overline{wxyz}}}=\text{Z}-\text{Z}^2+\text{Z}^3-\text{Z}^4$, and, assigning the proper values to Z and its powers,

$$\begin{aligned}{}_np_{\overset{1}{\overline{wxyz}}}&={}_np_w+{}_np_x+{}_np_y+{}_np_z\\&\quad-({}_np_{wx}+{}_np_{wy}+{}_np_{wz}+{}_np_{xy}+{}_np_{xz}+{}_np_{yz})\\&\quad+({}_np_{wxy}+{}_np_{wxz}+{}_np_{wyz}+{}_np_{xyz})-{}_np_{wxyz}.\end{aligned}$$

Again, $${}_np_{\overset{2}{\overline{wxyz}}}=\text{Z}^2-2\text{Z}^3+3\text{Z}^4$$
$$=({}_np_{wx}+{}_np_{wy}+{}_np_{wz}+{}_np_{xy}+{}_np_{xz}+{}_np_{yz})$$
$$-2({}_np_{wxy}+{}_np_{wxz}+{}_np_{wyz}+{}_np_{xyz})+3{}_np_{wxyz}.$$

Again, $${}_np_{\overset{3}{\overline{wxyz}}}=\text{Z}^3-3\text{Z}^4$$
$$=({}_np_{wxy}+{}_np_{wxz}+{}_np_{wyz}+{}_np_{xyz})-3{}_np_{wxyz}.$$

Finally, ${}_np_{\overset{4}{\overline{wxyz}}}=\text{Z}^4={}_np_{wxyz}$.

This last result is evidently correct; because the probability that at least four lives out of four lives will survive a term, is the same thing as the probability that the joint lives will survive.

If we seek the probability that the survivor of two lives will live n years, we must in F take $m=2$, and $r=1$. We then have ${}_np_{\overline{xy}}=\text{Z}-\text{Z}^2={}_np_x+{}_np_y-{}_np_{xy}$, the same result as in equation 5.

32. It must be distinctly understood that the symbol Z, as here used, is not a symbol of quantity; and, in the ordinary sense of the words, it cannot be called a symbol of operation; because, for instance, the rth power of Z, Z^r, does not mean r repetitions of the operation represented by Z, in the same way as Δ^r means r repetitions of the operation of differencing. Yet it is very remarkable that, with certain limitations, we can treat the symbol Z exactly as if it represented a quantity. We can add, subtract, multiply, divide, raise to powers, and extract roots, just as with any algebraical quantity; and if, at the end of the operations, we interpret Z and its powers in accordance with the definitions given of them at the outset, the final result will be correct.

For example, we can find ${}_np_{\overline{wxyz\ldots(m)}}^{\,[r]}$ from ${}_np_{\overline{wxyz\ldots(m)}}^{\,r}$ as follows:

$$\begin{aligned}{}_np_{\overline{wxyz\ldots(m)}}^{\,[r]} &= {}_np_{\overline{wxyz\ldots(m)}}^{\,r} - {}_np_{\overline{wxyz\ldots(m)}}^{\,r+1}\\ &= \frac{Z^r}{(1+Z)^r} - \frac{Z^{r+1}}{(1+Z)^{r+1}}\\ &= \frac{Z^r}{(1+Z)^r}\left(1-\frac{Z}{1+Z}\right)\\ &= \frac{Z^r}{(1+Z)^{r+1}}.\end{aligned}$$

Here we have treated Z and its powers exactly as if they had been symbols of quantity, and we have produced a result which we have proved in D to be correct. We have only to expand the expression $\frac{Z^r}{(1+Z)^{r+1}}$ by the binomial theorem, and take in the terms which do not involve a higher power of Z than Z^m; and, if in the resulting series we interpret the powers of Z according to definition, we arrive at a correct formula for the probability sought.

33. As a special case of formula 15, we can determine the value of the probability when all the lives are of equal age, and when $r=1$; that is, the value of the probability that the last survivor of m lives, all aged x, will live n years. When all the lives are of equal age, then ${}_np_{xx}={}_np_x{}^2$, ${}_np_{xxx}={}_np_x{}^3$, and so on: also $Z=m\,{}_np_x$, $Z^2=\frac{m(m-1)}{\underline{|2}}\,{}_np_x{}^2$, $Z^3=\frac{m(m-1)(m-2)}{\underline{|3}}\,{}_np_x{}^3$, and so on. Therefore formula 15 becomes

$$\begin{aligned}{}_np_{\overline{xxx\ldots(m)}} &= m\,{}_np_x - \frac{m(m-1)}{\underline{|2}}\,{}_np_x{}^2 + \frac{m(m-1)(m-2)}{\underline{|3}}\,{}_np_x{}^3 - \&c.\\ &= 1-(1-{}_np_x)^m \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)\end{aligned}$$

Similarly, for formula 13 we shall find that

$$\begin{aligned}{}_np_{\overline{xxx\ldots(m)}}^{\,[1]} &= m\,{}_np_x - m(m-1)\,{}_np_x{}^2 + \frac{m(m-1)(m-2)}{\underline{|2}}\,{}_np_x{}^3 - \&c.\\ &= m\,{}_np_x(1-{}_np_x)^{m-1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)\end{aligned}$$

Expressions 17 and 18 are evidently correct. Since $(1-{}_np_x)$ is the probability that a life aged x will die within n years; therefore $(1-{}_np_x)^m$ is the probability that all the lives will so die; and $1-(1-{}_np_x)^m$ is the probability that all will not die, that is, that at least one will survive. So also ${}_np_x(1-{}_np_x)^{m-1}$ is the probability that one particular life will survive, and all the others fail. But of the m lives there can be m such combinations; therefore $m\,{}_np_x(1-{}_np_x)^{m-1}$ is the probability that exactly one life will survive, and all the other $m-1$ lives fail.

34. Suppose we have m lives, all aged x. Required the most probable number of deaths among them within a year. If we select r lives from among the m lives, the probability that that particular set of r lives will all fail, while all the other $(m-r)$ lives survive, is $q_x{}^r p_x{}^{m-r}$. But the number of distinct sets of r lives that can be selected out of m lives, is (m, r), or $\dfrac{m(m-1)\ldots(m-r+1)}{\underline{|r}}$; and therefore the probability that some one set of r lives will all fail, while the remaining $(m-r)$ lives all survive, that is, that exactly r lives will fail out of the m lives, is $\dfrac{m(m-1)\ldots(m-r+1)}{\underline{|r}}q_x{}^r p_x{}^{m-r}$. If, therefore, in this expression we make r successively equal to 0, 1, 2, &c., we shall have the following series, the successive terms of which represent the probability that there will be no deaths in the community of m persons, that there will be exactly one death, exactly two deaths, &c. :—

$$p_x{}^m + mp_x{}^{m-1}q_x + \frac{m(m-1)}{\underline{|2}}p_x{}^{m-2}q_x{}^2 + \&c.$$

$$+\frac{m(m-1)\ldots(m-r+1)}{\underline{|r}}p_x{}^{m-r}q_x{}^r + \&c.$$

This series is the expansion of $(p_x+q_x)^m$ by the Binomial Theorem; and the sum of all the terms is unity, since $(p_x+q_x)=1$.

In order to ascertain the most probable number of deaths, we must find the term in the above expansion which has the largest numerical value. To obtain the $(r+1)$th, we multiply the rth term by $\dfrac{m-r+1}{r}\cdot\dfrac{q_x}{p_x}$, that is by $\left(\dfrac{m+1}{r}-1\right)\div\dfrac{p_x}{q_x}$. As r increases, this multiplier diminishes; and it is greater than unity only so long as $\left(\dfrac{m+1}{r}-1\right)$ is greater than $\dfrac{p_x}{q_x}$; that is only so long as $\dfrac{m+1}{r}$ is greater than $1+\dfrac{p_x}{q_x}$, which $= \dfrac{1}{q_x}$. Therefore the multiplier is greater than unity, only so long as $(m+1)q_x$ is greater than r; that is, until r is the greatest integer in $(m+1)q_x$. Now the terms of the expansion continue to increase in magnitude as long as the multiplier is greater than unity; and they begin to diminish when the multiplier becomes less than unity. Therefore the $(r+1)$th term is the greatest, when r is the greatest integer in $(m+1)q_x$: that is the *most probable number of deaths* is r, when r is the greatest integer in $(m+1)q_x$.

35. From the foregoing demonstration we see that, if an assurance office have on its books at the beginning of a year m policies on the lives of as many persons all aged x, the most probable number of claims that will arise among them in the year is $(m+1)q_x$. It is usual, however, in practice to look, not at the most probable number of deaths, but at the *expected deaths*, which we shall find to be mq_x. The value of a benefit is (*see* Chap. vii, Arts. 1 to 3) the amount of the benefit, multiplied by the probability of its being received; and, similarly, the value of the possible claim on the office is, the amount of claim, multiplied by the probability of its being incurred. If, therefore, we take all possible amounts of claim, and multiply each by its respective probability, and then take the sum, we shall have the measure of the company's liability to loss, or the expected deaths. The possible claims are 0, 1, 2, &c., up to m, and the series in Art. 34 gives the respective probabilities. Multiplying each loss by its probability, and taking the sum, we have

$$0 \times p_x{}^m + 1 \times mp_x{}^{m-1}q_x + 2 \times \frac{m(m-1)}{\underline{|2}}\,p_x{}^{m-2}q_x{}^2 + \&\text{c.}$$

$$= mq_x\left\{p_x{}^{m-1} + (m-1)p_x{}^{m-2}q_x + \frac{(m-1)(m-2)}{\underline{|2}}\,p_x{}^{m-3}q_x{}^2 + \&\text{c.}\right\}$$

$$= mq_x(p_x + q_x)^{m-1}$$

$$= mq_x, \text{ since } p_x + q_x = 1.$$

[36] The function which has been called the *Instantaneous Rate of Mortality*, but which is now always known as the *Force of Mortality*, is one which is constantly met with in the higher branches of the science of life contingencies. It will be convenient here to explain fully the real nature of that function, and to investigate formulas for calculating approximately its value.*

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* The explanation is taken almost verbatim from a paper on the Law of Mortality, by W. M. Makeham, *J.I.A.* xiii, 325. The formula of approximation is given in a paper on the Adjustment of Mortality Tables, by W. S. B. Woolhouse, *J.I.A.*

[37] In the subjoined table, l_x denotes the number of persons living at age x in a mortality table (see Table I); and Δl_x, the difference corresponding to an increment of Δx in the age—in this case 10 years. The other characters will be explained immediately.

x (1)	l_x (2)	$l_{30}-l_{20}$ $-\Delta l_x$ (3)	$-\frac{1}{l_x}\frac{\Delta l_x}{\Delta x}$ (4)	$\frac{l_x - l_{x+1}}{l_x}$ (5)	$-\frac{1}{l_x}\frac{dl_x}{dx}$ (6)
20	96,061	6,376	·00664	·00572	·00550
30	89,685	7,408	·00826	·00771	·00768
40	82,277	9,482	·01152	·01001	·00990
50	72,795	13,953	·01917	·01572	·01542
60	58,842	20,865	·03546	·02983	·02920
70	37,977	23,990	·06317	·06410	·06353
80	13,987	12,714	·09090	·14426	·14909
90	1,273	1,269	·09969	·31579	·36230
100	4	4	·10000	·75000	·89366

The second column then, headed l_x, contains the numbers living at decennial intervals of age; the third, $(-\Delta l_x)$, shows the decennial decrements; the fourth, $\left(-\frac{1}{l_x}\frac{\Delta l_x}{\Delta x}\right)$, gives the ratio which the average annual decrement of each decennial period bears to the number living at the commencement of the period; while the fifth column, $\left(\frac{l_x - l_{x+1}}{l_x}\right)$, exhibits the series which we have called the rate of mortality, q_x; that is, the ratio of the number actually dying in one year to the number living at the commencement of that year.

Thus, out of 96,061 living at age 20, 6,376 die before reaching age 30—which is at the average rate of 637·6 per annum. Dividing this last by 96,061, we have ·00664 for the ratio which the average number of annual deaths bears to the number living at the commencement of the decennial interval.

Comparing this with the corresponding number in column 5, we see that it is only very little in excess of the rate of mortality for the *least* of the 10 ages included in the given interval—the rate of mortality at age 20 being ·00572. The same comparative deficiency in the average annual proportion of deaths of the decennial interval will be observed upon comparing the two columns for the respective ages 30, 40, 50, and 60. At the age of 70 and upwards, however, the deficiency in question becomes much more apparent;—the average annual rate of decrease during the 10 years being then actually *less* than the rate of mortality for the youngest age of the decennial period; and so rapidly does this go on, that at the age of 100 the former is little more than one-eighth of the latter.

This result is owing to the fact that, as the numbers living are gradually diminishing by the effect of mortality, the number of deaths must also experience a corresponding diminution; and the longer the interval between the ages observed, the greater must be the disturbance caused by this incessant reduction in the number exposed to risk of death. The average number of deaths taken for several years together, is therefore seen to be an extremely inadequate measure of the actual *intensity*, or *force*, of the operating causes by which the decrements of life are produced.

Now it will be observed that the function $\frac{l_x - l_{x+1}}{l_x}$ (representing the rate of mortality), of which the values are shown in col. 5, is of precisely the same form as the function $-\frac{1}{l_x}\frac{\Delta l_x}{\Delta x}$, from which the values in col. 4 are computed; for the latter is equivalent to $\frac{l_x - l_{x+10}}{10 l_x}$, which becomes identical with the former by substituting 1 for 10 as the increment of x. But it is evident that the same causes which render the series in col. 4 so imperfect a measure of the force of mortality, must also tend (in a less degree) to make the series in col. 5 unfit for the same purpose. If therefore, instead of making $\Delta x = 1$ in the formula $-\frac{1}{l_x}\frac{\Delta l_x}{\Delta x}$, we make it a fraction (say ·5, for instance), we shall obtain a still better expression for the measure in question. And if we go still further, and diminish Δx without limit, we shall evidently get rid of the disturbing element altogether, and thus obtain a perfect measure of the actual intensity, or force, of mortality at each age.

This is, in effect, the process adopted in the construction of col. 6; $\frac{dl_x}{dx}$ (which, in the language of the differential calculus, is called the differential coefficient of l_x with respect to x), being the limit of the ratio of the infinitely small decrement of l_x to the infinitely small increment of x by which it is produced. The limit in question, when taken positively and divided by l_x, is called the *force of mortality* at age x, and has assigned to it the symbol μ_x.

[38] The foregoing reasoning shows, that the force of mortality at age x, is the proportion of persons of that age who would die in a year, if the intensity of mortality remained constant for a year, and if the number of persons under observation also remained constant, the places of those who die being constantly occupied by fresh lives.

[39] We have seen that the force of mortality (taken negatively) at age x is the differential coefficient of l_x with respect to x, divided by l_x; that is

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

If the values in column l_x always progressed by some known mathematical law; that is, if we could always express the numbers living at age x by a formula in terms of x; then it would be easy in every case to calculate exactly the force of mortality; because if we know the form of a function, we can always find its differential coefficient. Some very plausible theories have been suggested for the law of mortality, and very useful formulas deduced therefrom; but although in some cases (notably in the case of the Institute of Actuaries' Mortality Tables), these formulas have been found to represent the table, throughout its greater portion, with abundantly sufficient accuracy for practical purposes, yet, so far, it has not been possible to represent, with absolute exactitude, by a formula, any table derived from actual observations. In Table I, at the end of this volume, the column μ_x gives accurately from age 10 upwards, as explained in Chap. vi, the force of mortality corresponding to the parallel column l_x; yet with the majority of mortality tables we are obliged to content ourselves with approximations only to the values of μ_x. The following method of approximation may be relied on to give results very nearly true in the fifth place of decimals.

[40] By Chap. xxiii, formula 8,

$$l_t = l_0 + t\left(a_0 + \frac{t}{2}b_0\right) + \frac{t(t^2-1)}{6}\left(c_0 + \frac{t}{4}d_0\right),$$

whence,

$$-\frac{l_0 - l_t}{t} = a_0 + \frac{t}{2}b_0 + \frac{t^2-1}{6}\left(c_0 + \frac{t}{4}d_0\right).$$

If now we diminish t without limit, we have the limit of the ratio of the infinitely small decrement of l_x, to the infinitely small increment of x by which it is produced; and this we have seen to be $l_x\mu_x$.

When we diminish t without limit, those terms vanish in which t or any of its powers is a coefficient; and we have

$$-l_0\mu=a_0-\frac{c_0}{6}.$$

But by Chap. xxiii, Art. 28,

$$\begin{aligned}a_0&=\tfrac{1}{2}(a_{-1}+a_{+1})\\ c_0&=\tfrac{1}{2}(c_{-1}+c_{+1})=\tfrac{1}{2}(b_{+1}-b_{-1})\\ &=\tfrac{1}{2}\{(a_{+2}-a_{+1})-(a_{-1}-a_{-2})\}\\ &=\tfrac{1}{2}\{(a_{-2}+a_{+2})-(a_{-1}+a_{+1})\}.\end{aligned}$$

Therefore, by substitution,

$$-l_0\mu=\tfrac{1}{2}(a_{-1}+a_{+1})+\tfrac{1}{12}\{(a_{-1}+a_{+1})-(a_{-2}+a_{+2})\}.$$

To abandon the notation of central differences, and to return to the notation of the mortality table,

$$l_0\mu=\tfrac{1}{2}(d_{-1}+d_0)+\tfrac{1}{12}\{(d_{-1}+d_0)-(d_{-2}+d_{+1})\},$$

or,

$$\mu=\frac{7(d_{-1}+d_0)-(d_{-2}+d_{+1})}{12l_0}\quad\ldots\ldots\quad(20)$$

If required to express this in terms of the numbers living, we have

$$\begin{aligned}d_{-1}+d_0&=l_{-1}-l_{+1}\\ d_{-2}+d_{+1}&=(l_{-2}-l_{+2})-(l_{-1}-l_{+1})\ ;\end{aligned}$$

hence,

$$l_0\mu=\tfrac{1}{2}(l_{-1}-l_{+1})+\tfrac{1}{6}\{(l_{-1}-l_{+1})-\tfrac{1}{2}(l_{-2}-l_{+2})\}$$

or,

$$\mu=\frac{8(l_{-1}-l_{+1})-(l_{-2}-l_{+2})}{12l_0}\quad\ldots\ldots\ldots\quad(21)$$

For joint lives formula 21 is better than formula 20.

[41] If in the first expression for the value of $-l_0\mu$, we stop at first differences, and omit the term $\frac{c_0}{6}$, we have $-l_0\mu=a_0=\frac{1}{2}(a_{-1}+a_{+1})$; and hence, in the notation of the mortality table,

$$\left.\begin{aligned}\mu&=\frac{d_{-1}+d_0}{2l_0}\\ &=\frac{l_{-1}-l_{+1}}{2l_0}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(22)$$

This last is a rough approximation to the value of μ; but for practical purposes it is usually sufficiently accurate. From it we see that the central death rate at age x, the m_x of Chap. i, Art. 12, is an approximate value of the force of mortality at an age half a year older. That is, approximately,

$$m_x = \mu_{x+\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

[42] The following numerical examples will show how far formulas 20 and 22 are to be depended upon:—

Age.	Force of Mortality		
	By Formula 20.	By Formula 22.	True Value.
20	·00550	·00550	·00550
30	·00768	·00768	·00768
40	·00990	·00989	·00990
50	·01541	·01542	·01542
60	·02921	·02921	·02920
70	·06353	·06349	·06353
80	·14907	·14885	·14909
90	·36240	·36489	·36230

It is seen that up to age 60 both formulas give results that do not appreciably deviate from the truth, and that even at age 90 the first scarcely loses in accuracy. With a table less smoothly graduated, the superior advantages of the first over the second would be still more apparent.

[43] By formula 19, $\mu = -\frac{1}{l}\cdot\frac{dl}{dt}$, which, by an elementary principle of the differential calculus, may be written,

$$\mu = -\frac{d\log l}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

[44] The reasoning by which formula 24 is obtained is perfectly general, and is not affected if two or more joint lives be substituted for a single life. We may therefore write,

$$\mu_{xyz} = -\frac{d\log l_{xyz}}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

But

$$\frac{d\log l_{xyz}}{dt} = \frac{d(\log l_x + \log l_y + \log l_z)}{dt} = \frac{1}{l_x}\cdot\frac{dl_x}{dt} + \frac{1}{l_y}\cdot\frac{dl_y}{dt} + \frac{1}{l_z}\cdot\frac{dl_z}{dt}$$

Therefore,

$$\mu_{xyz} = \mu_x + \mu_y + \mu_z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and an expression similar in form holds for any number of joint lives.

CHAPTER III.

EXPECTATIONS OF LIFE.

1. ASSUMING that a given Table exactly represents the rate of mortality which will for the future prevail in a community; what is the total future lifetime that will be enjoyed by l_x persons in that community, who are now of the precise age x?

Of the l_x persons, d_x will die in the first year; and, on the supposition of a uniform distribution of deaths throughout each year of age, the d_x persons will live between them $\frac{1}{2}d_x$ years. There will survive the first year, and die in the second year, d_{x+1} persons. As each of these persons will live one year in the first year, the number of years lived by them in the aggregate in the first year, will be d_{x+1}; and, as before, in the year in which they die they will live between them $\frac{1}{2}d_{x+1}$ years. The total future lifetime of these d_{x+1} persons will therefore be $\frac{3}{2}d_{x+1}$ years. By similar reasoning we shall find that the total future lifetime of the d_{x+2} persons who will die in the third year, will be $\frac{5}{2}d_{x+2}$; of the d_{x+3} persons who will die in the fourth year, $\frac{7}{2}d_{x+3}$; and so on; until all the lives become extinct. The total future lifetime of all the l_x persons will therefore be

$$\begin{aligned}\tfrac{1}{2}d_x+\tfrac{3}{2}d_{x+1}+\tfrac{5}{2}d_{x+2}+\tfrac{7}{2}d_{x+3}+\&c.\\ &=\tfrac{1}{2}(d_x+d_{x+1}+d_{x+2}+\&c.)\\ &\quad+(d_{x+1}+2d_{x+2}+3d_{x+3}+\&c.)\end{aligned}$$

Now, remembering (Chap. i, Eq. 2) that $\Sigma d_x=l_x$; and that

$$\begin{aligned}(d_{x+1}+2d_{x+2}+3d_{x+3}+\&c.)\\ &=(d_{x+1}+d_{x+2}+d_{x+3}+\&c.)\\ &\quad+\qquad(d_{x+2}+d_{x+3}+\&c.)\\ &\quad+\qquad\qquad(d_{x+3}+\&c.)\\ &\quad+\&c.\\ &=l_{x+1}+l_{x+2}+l_{x+3}+\&c.,\end{aligned}$$

the above expression reduces to

$$\tfrac{1}{2}l_x+l_{x+1}+l_{x+2}+l_{x+3}+\&c.,$$

which is the total future lifetime sought.

2. If we divide the total future lifetime of all the l_x persons, equally among those persons, the share of each will be

$$\frac{1}{2}+\frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_x}$$

which share is called the *Complete Expectation of Life* at age x, and denoted by the symbol $\mathring{e}_x$. Therefore

$$\mathring{e}_x=\frac{1}{2}+\frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_x} \quad . \ . \ . \ . \ . \quad (1)$$

3. The same result may be arrived at more briefly, in another way. Of the l_x persons under observation, l_{x+1} will each live a complete year in the first year; or, among them, l_{x+1} complete years; l_{x+2} will live another complete year each in the second year; or, among them, a further l_{x+2} complete years; and so on in future years. The total number of complete years which will be lived in the aggregate by the l_x persons, will therefore be $l_{x+1}+l_{x+2}+l_{x+3}+\&c.$; giving for each person an average number of complete years of $\frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_x}$. This average number of complete years is called the *Curtate Expectation of Life* at age x, and denoted by the symbol e_x; so that

$$e_x=\frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_x} \quad . \ . \ . \ . \ . \quad (2)$$

4. Here we have not taken any account of that portion of lifetime lived by each individual in the year of his death, but have confined our attention to completed years. It may be assumed, however, that, as some of these persons will live only a few days in the year of their death, while others will live nearly a whole year, taking one person with another, the l_x persons will live in the aggregate $\tfrac{1}{2}l_x$ years in the year of death, giving an average of half a year for each. Adding, therefore, this extra half-year to the curtate expectation of life found above, we have, as before, the complete expectation of life equal to

$$\frac{1}{2}+\frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_x}$$

From the foregoing demonstrations we derive the relation

$$\mathring{e}_x = \tfrac{1}{2} + e_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

5. The term "expectation of life" is apt to mislead unthinking persons, who not unfrequently take it for granted that a man's expectation of life represents the number of years he may "reasonably expect" still to live. But when we analyze the phrase "reasonably expect", we see that it is wanting in precision. If it has any meaning, it implies that the probability is far greater of a man's living the term of the expectation, than of his living any other term. This, however, is not so. Very few of the lives, if any, will fail when they have exactly completed the expectation of life. Many will fall far short of it; and many will long survive it. In fact, to form the expectation of life, we take the excess from those who live long, and distribute it among those who die early, so as to place all on an equality; and the expectation of life has no relation whatever to the most probable lifetime of any given individual.

6. Several other names have been proposed for the expectation of life. It has been called the "mean after-lifetime"; the "average after-lifetime"; the "mean duration of life"; and the "average duration of life". Perhaps to adopt the first or second of these terms (preferably the second: *J.I.A.*, xv, 194), would be an improvement; but as the name "expectation of life" is that most commonly used, we shall retain it. If we place clearly before our minds the real nature of the functions with which we are dealing, we shall not be misled by ambiguous phrases.

7. The expectation of life, then, is not the most probable after-lifetime. The *most probable after-lifetime* is, the difference between the present age and that year of age in which the life is most likely to fail. Now, the probability that a life aged x will fail in the $(n+1)$th year is $\frac{d_{x+n}}{l_x}$; and this probability is largest when d_{x+n} has its largest value. By the mortality table at the end of this volume, d_x, after age 1, is largest at age 73; and therefore, the most probable year of death for all persons aged between 2 and 73, is the year of age 73 to 74. This gives a duration very different from the expectation of life. Thus at age 10, the most probable lifetime is between 63 and 64 years; while $\mathring{e}_{10}$ is only 50·257 years; and at birth the most probable lifetime is less than a year; while $\mathring{e}_0$ is 47·784 years.

omit

8. Another function, which goes by the French name *Vie Probable*,—in English, *Probable Lifetime*,—has sometimes been confused with the Expectation of Life. The "vie probable", or "probable lifetime"—the name is an unfortunate one, and has no meaning in itself, since every possible duration of life has a determinate probability, and is therefore a "probable lifetime"—is the time that will elapse until the number living is reduced to exactly one half the original number. It is therefore an even chance whether a given life will survive or fall short of the "vie probable". For example, by Table I, the number living at age 10 is 100,000; and this is reduced to 50,000 somewhere between ages 64 and 65, say at age 64$\frac{2}{3}$. The "vie probable" at age 10 is therefore 54$\frac{2}{3}$ years, which is more than four years in excess of the expectation of life at that age. It will be found that up to age 56 the "vie probable" is greater than the complete expectation; and above that age it is less.

9. We have supposed that the deaths are equally distributed throughout each year of age. If we extend the hypothesis, and suppose that the deaths are equally distributed throughout the whole of life; or, in other words, that the column l_x consists of a decreasing arithmetical progression, the values in the column d_x being therefore constant; we shall find that the complete expectation of life, and the "vie probable", are equal. Let us suppose that, by the mortality table, there are n persons aged x; and that one of them will die each year, until by the end of n years all become extinct; then the "vie probable" is, by definition, $\frac{1}{2}n$. We have also $l_x=n$, $l_{x+1}=(n-1)$, &c., and

$$\mathring{e}_x=\frac{1}{2}+\frac{(n-1)+(n-2)+\ .\ .\ .\ +2+1}{n}$$

$$=\frac{1}{2}+\frac{1}{n}\times\frac{n(n-1)}{2}$$

$$=\frac{1}{2}+\frac{n-1}{2}=\frac{n}{2}$$

which, as we have already seen, is the value of the "vie probable".

omit

10. On the hypothesis of equal decrements throughout life, there is no such function as the most probable lifetime; because, d_x being constant, the probability of death is as great in any one year of age as in any other.

11. The hypothesis of equal decrements was first introduced by De Moivre, and is called *De Moivre's hypothesis*. As it affords great

facilities for the calculation of annuity-values, it was very useful in the early days of the science of life contingencies, when there were few or no tables of annuity-values published; but the results given by it are not sufficiently correct to be of use now. We shall return to this subject in Chap. vi, Arts. 1 to 4, and Chap. vii, Arts. 104 to 109.

12. The expectation of life at age x being $\mathring{e}_x$, the average age at death of the l_x persons will be $x+\mathring{e}_x$. Now it will be found on reference to the table that the higher the age, x, from which we count, the greater will be the average age at death. Thus, at age 10, the average age at death is 60·257 years; at age 20, it is 62·101; at age 30, it is 64·726; and at age 60, it is 73·808. This fact exposes a fallacy which is very often current. It is frequently stated by shallow reasoners that some profession, such as that of the lawyer, must be conducive to longevity; or that some class in society, such as the peerage, must be peculiarly long-lived; because the average age at death of the members of the profession or class, is much higher than that of the general population. But the general population starts from age 0; and, starting from age 0, the average age at death, if the mortality were to follow the table, would be only 47·784 years; whereas a peer does not become a peer until the death of his predecessor, when he himself is perhaps 30, 40, or even 50 or 60 years of age; or until he is created a peer, when also he is usually advanced in life. The peer then does not become a peer until he is, on the average, say, 40 years of age; so that, measured by the same table as the general population, the average age of peers at death should be about 67 or 68 years.

In the same way, a lawyer does not enter the profession until he reaches manhood; and usually, it is not until many years later that he attains sufficient eminence for his death to be commented upon. Therefore, even if the rate of mortality among lawyers be not more favourable than among the general population, the average age at death of those whose deaths attract notice, must be greater.

To compare the average ages at death of different classes or groups of persons does not prove anything, unless we know the average ages at which the persons come under observation, so as to be able to make allowances for differences in this respect. In order to form a correct idea of the relative mortality of different classes of persons, we must ascertain and compare the rate of mortality at each age in each class.

13. Here we may take the opportunity of exposing another fallacy which is sometimes current; although perhaps it does not bear directly on the subject-matter of the present chapter. A favourite mode of displaying vital statistics is, to tabulate the number of deaths per annum per thousand of the total population. Now, if two communities are similarly constituted as regards the proportionate number living in each year of age, it may give a sufficiently accurate idea of the relative salubrity of the districts in which they are placed, if the rates of mortality per thousand of population be compared. Thus, if we were to take the rate of mortality per thousand in London, and compare it with the rate of mortality per thousand in Edinburgh, we should probably not fall into a great mistake. But if the distribution of the population as regards age were very different,—if, for instance, a great many persons in the prime of life were to migrate from one community to the other, leaving behind them a large proportion of elderly persons, among whom the rate of mortality is normally heavy,—it would be inadmissible any longer to compare the rates of mortality per thousand. The community which had lost members by emigration would show a higher death rate, on account of the greater average age of the remaining population, than the community which had received the migrants. Such are the relative positions of Britain and some of her Colonies. The average number of annual deaths per thousand of population in Victoria, Australia, was 16·6 during the years 1853 to 1878, and 22·1 in England and Wales during the years 1850 to 1874; but, for the reasons explained above, these figures do not tell us whether Victoria or England is the more healthy. To form a just comparison we must, as we have said before, ascertain the rate of mortality at each age. (See also Chap. v, Arts. 14 to 16.)

14. We have seen that, to obtain the expectation of life at age x, we find the total number of years that will be lived by the l_x persons, and divide them equally among those persons. In the same way we can find the temporary or the deferred expectation of life; that is, the expectation of life during the next n years, or after the next n years. In accordance with the system of notation explained in the note to Art. 10 of Chap. ii, we shall write ${}_{|n}\mathring{e}_x$ and ${}_{|n}e_x$ for the complete and the curtate temporary expectations of life; and ${}_{n|}\mathring{e}_x$ and ${}_{n|}e_x$ for the corresponding deferred expectations.

Taking first the curtate expectation, and looking at equation 2 of this chapter, it is evident that

$$ {}_{|n}e_x = \frac{l_{x+1} + l_{x+2} + \ldots + l_{x+n}}{l_x} = l_{x\overline{n}} \quad \ldots \quad (4)$$

$$ {}_{n|}e_x = \frac{l_{x+n+1} + l_{x+n+2} + \&c.}{l_x} \quad \ldots\ldots \quad (5)$$

and therefore, $$e_x = {}_{|n}e_x + {}_{n|}e_x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

As regards the complete expectation, to find ${}_{|n}\mathring{e}_x$ and ${}_{n|}\mathring{e}_x$ we cannot in the same way sum a portion of the terms of the series in equation 1; but we must proceed differently. We can obtain ${}_{|n}\mathring{e}_x$ from ${}_{|n}e_x$ by taking account of the portion of time that will be lived in the year of death, by the lives that will fail during the n years. Now $l_x - l_{x+n}$ persons will die within the term; and they will live half a year each, on the average, in the year of death; or $\frac{1}{2}(l_x - l_{x+n})$ years in all; giving $\dfrac{\frac{1}{2}(l_x - l_{x+n})}{l_x}$ as the portion for each of the l_x persons. Adding this portion to the temporary curtate expectation, we have

$${}_{|n}\mathring{e}_x = {}_{|n}e_x + \frac{\frac{1}{2}(l_x - l_{x+n})}{l_x}$$

$$= \tfrac{1}{2} + {}_{|n}e_x - \tfrac{1}{2}\frac{l_{x+n}}{l_x} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

15. By simple algebraical transformation, with the help of equations 6 and 3, this last result may be written,

$${}_{|n}\mathring{e}_x = \mathring{e}_x - {}_np_x \times \mathring{e}_{x+n} \quad . \; . \; . \; . \; . \; . \quad (8)$$

whence $${}_{n|}\mathring{e}_x = {}_np_x \times \mathring{e}_{x+n} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

We may write, $$e_x = \frac{\Sigma l_{x+1}}{l_x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

$${}_{n|}e_x = \frac{\Sigma l_{x+n+1}}{l_x} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

and $${}_{|n}e_x = \frac{\Sigma l_{x+1} - \Sigma l_{x+n+1}}{l_x} \quad . \; . \; . \; . \; . \quad (12)$$

where the symbol Σ means the summation of the function l, from the age indicated by the suffix, to the oldest age in the table, both inclusive.

From these last three equations it will be seen that, if we have a column giving opposite each age the sum of the numbers living at all higher ages, we can, by simple division, obtain e_x or ${}_{n|}e_x$; and by a subtraction and a division, obtain ${}_{|n}e_x$. Such is the column headed N'_x in Table No. I.*

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* We use the symbol N′ to distinguish this column from the column N explained in Chapter vii. The columns are identical in principle; the only difference being that N involves the rate of interest, while N′ does not.

16. In Chap. i, Art. 9, we defined L_x as $\frac{1}{2}(l_x+l_{x+1})$. Now

$$\begin{aligned}\mathring{e}_x&=\frac{\frac{1}{2}l_x+l_{x+1}+l_{x+2}+\&c.}{l_x}\\&=\frac{\frac{1}{2}(l_x+l_{x+1})+\frac{1}{2}(l_{x+1}+l_{x+2})+\&c.}{l_x}\\&=\frac{\Sigma L_x}{l_x}\quad\ldots\ldots\ldots\quad(13)\end{aligned}$$

We can similarly show that

$$_{n|}\mathring{e}_x=\frac{\Sigma L_{x+n}}{l_x}\quad\ldots\ldots\quad(14)$$

and

$$_{|n}\mathring{e}_x=\frac{\Sigma L_x-\Sigma L_{x+n}}{l_x}\quad\ldots\ldots\quad(15)$$

Therefore, if we have a column giving opposite each age the sum of the population at that age and all higher ages, we can, by a simple division, obtain $\mathring{e}_x$ or $_{n|}\mathring{e}_x$; and, by a subtraction and a division, obtain $_{|n}\mathring{e}_x$. Such is the column headed T_x in Table No. I. The columns N'_x and T_x we shall illustrate further in Chap. v.

17. Since, by the last article, $\mathring{e}_x=\frac{\Sigma L_x}{l_x}$; therefore

$$\begin{aligned}\mathring{e}_x&=\frac{L_x+L_{x+1}+L_{x+2}+\&c.}{l_x}\\&=\frac{L_x+L_{x+1}+L_{x+2}+\&c.}{(l_x+l_{x+1})-(l_{x+1}+l_{x+2})+(l_{x+2}+l_{x+3})-\&c.}\\&=\frac{1}{2}\cdot\frac{L_x+L_{x+1}+L_{x+2}+\&c.}{L_x-L_{x+1}+L_{x+2}-\&c.}\quad\ldots\ldots\quad(16)\end{aligned}$$

18. Thus far, we have investigated the methods of finding the expectation of life when the mortality table is given. The inverse problem is also of some interest; namely, to deduce the mortality table, having given a complete table of the expectation of life.

Looking first at the curtate expectation:—Having given a complete table of e_x, to find the mortality table on which it is based.

We have

$$\begin{aligned}e_x&=\frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_x}\\&=\frac{l_{x+1}}{l_x}\times\frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_{x+1}}\\&=p_x(1+e_{x+1})\quad\ldots\ldots\ldots\quad(17)\end{aligned}$$

Therefore $$p_x = \frac{e_x}{1+e_{x+1}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

By means of equation 18 we can, from the table of e_x, construct the mortality table; because we can form a complete table of p_x; and then, starting with any convenient radix, we can pass, as explained in Chap. i, Art. 11, to the column l_x. The values in the column of l_x so formed will not be the same as in the corresponding column from which the given table of e_x was originally calculated, unless the radix we choose is the same as in that table: but the *ratios* between the values will be the same; and that is all that is required. As before remarked, the column l_x does not give absolute numbers living, but only relative numbers.

Taking now the complete expectation; when we have a complete table of $\mathring{e}_x$, we may deduct ·5 from each of the values, and so form the column e_x; and then proceed as above.

19. The expectation of life may be looked at from quite a different point of view from that hitherto adopted; and one which explains the origin of the name. If there be a lottery with $(c+k)$ tickets, and c prizes of 1 each; it is evident that the value to each of the $(c+k)$ ticket-holders of his expectation of a prize is $c \div (c+k)$; because a person buying from them all the tickets, could afford to give $c \div (c+k)$ to each, and in the result he would neither gain nor lose. We may therefore say that each ticket-holder has an *expectation*, the value of which is $c \div (c+k)$, of winning a prize. Now, life may be looked upon as a lottery. The l_x persons alive at age x correspond to the $(c+k)$ ticket-holders. Of these persons, l_{x+n} will complete a year of life in the nth year; the l_{x+n} years of life thus corresponding to the c lottery prizes. Therefore the *expectation* that each individual has of completing a year of life in the nth year is $\frac{l_{x+n}}{l_x} = {}_np_x$. Now, giving to n every integral value from 1 upwards, we have as the total expectation

$$e_x = \frac{l_{x+1}+l_{x+2}+l_{x+3}+\&c.}{l_x}$$
$$= p_x + {}_2p_x + {}_3p_x + \&c.$$

We therefore may write

$$e_x = \underset{n=1}{\Sigma}\, {}_np_x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

where the function ${}_np_x$ is to be summed for the successive integral values of n, commencing at unity.

20. This last equation is, algebraically, of great importance; and we shall make frequent use of it in the following pages. It shows that, where we have the sum of a series consisting of the probability of living 1, 2, 3, &c., years, we may substitute for it a curtate expectation of life. In fact, the curtate expectation of life is the sum of a series of probabilities.

21. To find the joint expectation of life of two or more persons. Let there be two persons, aged x and y respectively. Out of l_x+l_y persons, of whom l_x are aged x, and l_y aged y, the total number of pairs of persons, one of whom is aged x, and the other y, is $l_x \times l_y$, or l_{xy}. Of these, the number of pairs unbroken at the end of the first year will be $l_{x+1:y+1}$, or ${}^1l_{xy}$; and these pairs will each have completed a year of joint life in the first year, or ${}^1l_{xy}$ years in all. In the same way, the number of completed years of joint life which will be lived in the second year, is ${}^2l_{xy}$; and in the third year, ${}^3l_{xy}$; and so on. The total number of completed years of joint life which will be lived by all the l_{xy} pairs of persons, is the sum of the series ${}^1l_{xy}+{}^2l_{xy}+{}^3l_{xy}+$ &c. to the end of the table. Dividing this number equally among the l_{xy} pairs, we have

$$e_{xy}=\frac{{}^1l_{xy}+{}^2l_{xy}+{}^3l_{xy}+\&c.}{l_{xy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Just as we found $e_x=\Sigma_n p_x$, so now we see that, since $\frac{{}^nl_{xy}}{l_{xy}}={}_np_{xy}$, equation 20 may be written

$$e_{xy}=\Sigma_n p_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

22. In the last article we have spoken of only two persons; but the same principles evidently apply, whatever may be the number. We may therefore write

$$e_{xyz\ldots(r)}=\Sigma_n p_{xyz\ldots(r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the letter (r), in brackets, refers to the number of lives in question, and not to the age of a life.

23. In discussing the expectation of life for single lives, we found that, on the supposition of a uniform distribution of deaths throughout each year of life, $\mathring{e}_x=\frac{1}{2}+e_x$. A little consideration will now show us, that the relation $\mathring{e}=\frac{1}{2}+e$ does *not* hold for joint lives; that, in fact, if the distribution of deaths be uniform, $\mathring{e}_{xy}$ cannot be equal to $\frac{1}{2}+e_{xy}$. There can be $l_x \times l_y$ pairs of lives, one aged x and the other y, formed from l_x+l_y persons, of whom l_x are aged x, and l_y are aged y; and of these, at the end of a year, there will remain unbroken $l_{x+1} \times l_{y+1}$. Let us suppose that the first death in the year will be that of one of the persons aged x. Each of the persons aged x having been combined to form a pair with each of the persons aged y, this first death will cause l_y pairs to be broken. Among the l_y persons, d_y will die during the year, the deaths taking place at equal intervals. Therefore, when the last of the deaths of the year among the l_x group takes place, there will be fewer than l_y persons of the other group remaining; say l_y-r, where $r <$ or $= d_y$; and the number of pairs broken by this last death in the l_x group will be only (l_y-r), which is less than l_y. Therefore the first death in the l_x group will break more pairs than the last; and in the same way the first death in the l_y group will break more pairs than the last; or, in more general language, the earlier deaths in the year will break more pairs than the later; and the average duration of the pairs that fail in the year will be less than $\frac{1}{2}$. Therefore $\mathring{e}_{xy} < \frac{1}{2} + e_{xy}$.

Precisely similar reasoning applies where more than two lives are involved; and therefore formula 3 does not hold for joint lives.

[24] With the help of Lubbock's formula of summation, which is based on the Calculus of Finite Differences, Chap. xxiv, formula 15, we can, as follows, deduce an expression for the value of $\mathring{e}$, which is independent of any supposition as to the distribution of deaths; and which is therefore equally applicable to single lives and to joint lives.

In formula 2 we have $e_x = \frac{1}{l_x}(l_{x+1}+l_{x+2}+\&c.)$, which is the average number of complete *years* of life for each of l_x persons aged x. Similarly, $\frac{1}{l_x}(l_{x+\frac{1}{t}}+l_{x+\frac{2}{t}}+\&c.+l_{x+1}+l_{x+1+\frac{1}{t}}+\&c.)$ is the average number of complete tth parts of years for each of l_x persons aged x. Dividing this last expression by t, we have

$$(A) \quad . \quad . \quad . \quad . \quad \frac{1}{l_x}\cdot\frac{1}{t}(l_{x+\frac{1}{t}}+l_{x+\frac{2}{t}}+\&c.+l_{x+1}+l_{x+1+\frac{1}{t}}+\&c.),$$

which is the average number of *years* of lifetime for each of the l_x persons, due to completed tth parts of years. Applying now Lubbock's formula of summation (No. 15 of Chap. xxiv), and writing in it $u_x = l_x$, and $t = \frac{1}{m}$, and $n = m$, and also deducting u_0 from each side of the equation, and, for compactness, writing Δ_x and Δ_x^2 for Δl_x and $\Delta^2 l_x$ respectively, and, after reduction, replacing m by t, expression A becomes

$$(B) \quad . \quad . \quad \frac{1}{l_x}\left\{(l_{x+1}+l_{x+2}+\&c.)+\frac{t-1}{2t}l_x+\frac{t^2-1}{12t^2}\Delta_x-\frac{t^2-1}{24t^2}\Delta_x^2\right\}$$

If, now, t become indefinitely great, then $\frac{t-1}{2t}=\frac{1}{2}$, $\frac{t^2-1}{12t^2}=\frac{1}{12}$, $\frac{t^2-1}{24t^2}=\frac{1}{24}$. Also $\frac{1}{l_x}(l_{x+1}+l_{x+2}+\&c.)=e_x$. Therefore, expression B (which now represents the average lifetime for each of the l_x persons, due to completed moments, in other words the complete expectation of life) becomes

$$\mathring{e}_x=e_x+\tfrac{1}{2}+\frac{\Delta_x-\frac{1}{2}\Delta_x^2}{12l_x} \quad \ldots\ldots\ldots \quad (23)$$

$$=e_x+\tfrac{1}{2}-\frac{d_x+\frac{1}{2}(d_x-d_{x+1})}{12l_x} \quad \ldots\ldots \quad (24)$$

$$=e_x+\tfrac{1}{2}-\frac{3l_x-4l_{x+1}+l_{x+2}}{24l_x} \quad \ldots\ldots \quad (25)$$

[25] Although in the last preceding article we have used the symbols l_x, d_x, &c., yet our results are perfectly general. Instead of writing l_x, and speaking of persons, we might have written l_{xy}, and spoken of pairs; d_x being in this case replaced by $l_{xy}-{}^1l_{xy}$, and d_{x+1} by ${}^1l_{xy}-{}^2l_{xy}$; or we might have written l_{xyz}, and spoken of triplets, &c. For the case of r joint lives formula 25 becomes

$$\mathring{e}_{xyz\ldots(r)}=\tfrac{1}{2}+e_{xyz\ldots(r)}-\frac{3l_{xyz\ldots(r)}-4\,{}^1l_{xyz\ldots(r)}+{}^2l_{xyz\ldots(r)}}{24l_{xyz\ldots(r)}} \quad \ldots \quad (26)$$

[26] In obtaining formula 23 we have neglected the differences of l_x of higher orders than the second; or, what is the same thing, the differences of d_x of higher orders than the first. It therefore appears that we have assumed that the first differences of d_x are constant. But it is only at the initial age, x, that the differences remain in the final formula, the differences at the higher ages vanishing in the process of summation. Now, in all tables based on actual observations, the first differences of d_x are small, and very nearly constant for each small section of the table; and the third and higher differences almost vanish. Also, if we were to continue the series further, we should find that the denominators would rapidly become very large. Therefore the formula is a very close approximation to the truth.

[27] We can write the last term of the expression in equation 23,

$$\frac{1}{12}\,\frac{2d_x-(d_{x+1}-d_x)}{2l_x},\ \text{or}\ \frac{1}{12}\,\frac{2d_x-\Delta d_x}{2l_x},\ \text{or}\ \frac{1}{12}\,\frac{d_x+(d_x-\Delta d_x)}{2l_x}.$$

But, on the supposition that the differences of d_x are constant, $d_x - \Delta d_x = d_{x-1}$; therefore for the expression in question we may write $\frac{1}{12}\frac{d_{x-1}+d_x}{2l_x}$. By Chap. ii, formula 22, we have approximately $\frac{d_{x-1}+d_x}{2l_x} = \mu_x$; and we can therefore write

$$\mathring{e}_x = \tfrac{1}{2} + e_x - \tfrac{1}{12}\mu_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

[28] Equation 27 may be obtained more directly by using Woolhouse's formula of summation, No. 25 of Chap. xxiv, instead of Lubbock's. By Woolhouse's formula, expression B in Art. 24 becomes

$$\text{(C)} \quad . \quad . \quad . \quad \frac{1}{l_x}\left\{(l_{x+1}+l_{x+2}+\&\text{c.}) + \frac{t-1}{2t}\,l_x + \frac{t^2-1}{12t^2}\cdot\frac{dl_x}{dx}\right\};$$

or, when t becomes indefinitely great,

$$\text{(D)} \quad . \quad . \quad . \quad . \quad \frac{1}{l_x}\left\{(l_{x+1}+l_{x+2}+\&\text{c.}) + \frac{l_x}{2} + \frac{1}{12}\cdot\frac{dl_x}{dx}\right\}.$$

But, by Chap. ii, formula 19, $\frac{1}{l_x}\cdot\frac{dl_x}{dx} = -\mu_x$. Therefore $\mathring{e}_x = e_x + \tfrac{1}{2} - \tfrac{1}{12}\mu_x$ as before.

Also, for r joint lives, since, by Chap. ii, formula 26,

$$\mu_{xyz\ldots(r)} = \mu_x + \mu_y + \mu_z + \&\text{c.};$$

therefore, substituting this value for μ_x in the last expression,

$$\mathring{e}_{xyz\ldots(r)} = e_{xyz\ldots(r)} + \tfrac{1}{2} - \tfrac{1}{12}(\mu_x + \mu_y + \mu_z + \&\text{c.}) \quad . \quad . \qquad (28)$$

29. To find $e_{\overset{r}{\overline{wxyz\ldots(m)}}}$, the curtate expectation of life of the last r survivors of m lives.

We have seen, equation 19, that $e_x = \Sigma_n p_x$; where $\Sigma_n p_x$ represents $p_x + {}_2p_x + {}_3p_x + \&\text{c.}$ The reasoning by which we arrived at this formula may be extended; and for the life aged x we may substitute any combination of lives we please. The curtate expectation of life of the combination will be the sum of the probabilities that the combination will survive 1, 2, 3, &c., years. Let the combination in question be the last r survivors of m lives. Then

$$e_{\overset{r}{\overline{wxyz\ldots(m)}}} = \Sigma_n p_{\overset{r}{\overline{wxyz\ldots(m)}}}$$

But (Chap. ii, Formula 15)

$$_np_{\overline{wxyz\ldots(m)}}^{\,r}=\mathrm{Z}^r-r\mathrm{Z}^{r+1}+\frac{r(r+1)}{2}\mathrm{Z}^{r+2}-\&\mathrm{c}.$$

where each power of Z represents the sum of certain probabilities of the form $_np_{wxyz\ldots(t)}$. It therefore appears that to obtain $\Sigma_np_{\overline{wxyz\ldots(m)}}^{\,r}$ we must take the sum for all values of n of each of the probabilities included in each power of Z. But by formula 22, $\Sigma_np_{wxyz\ldots(t)}=e_{wxyz\ldots(t)}$; and therefore when we take the sum for n of the probabilities included in each power of Z, we have simply to replace the probabilities by the corresponding curtate expectations of joint lives. Extending in this sense the meaning of Z; so that, when dealing with expectations of life, Z^r represents the sum of the curtate expectations of r joint lives, for all the combinations of r lives that can be made out of m lives, we have

$$e_{\overline{wxyz\ldots(m)}}^{\,r}=\mathrm{Z}^r-r\mathrm{Z}^{r+1}+\frac{r(r+1)}{\underline{|2}}\mathrm{Z}^{r+2}-\frac{r(r+1)(r+2)}{\underline{|3}}\mathrm{Z}^{r+3}+\&\mathrm{c}. \quad . \quad (29)$$

and, as is done in Chap. ii, Art. 30, this formula may be concisely written

$$e_{\overline{wxyz\ldots(m)}}^{\,r}=\frac{\mathrm{Z}^r}{(1+\mathrm{Z})^r}\cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

30. As an example, let it be required to determine, when $m=4$, the values of $e_{\overline{wxyz}}^{\,1}$, $e_{\overline{wxyz}}^{\,2}$, $e_{\overline{wxyz}}^{\,3}$, and $e_{\overline{wxyz}}^{\,4}$.

$$\begin{aligned}e_{\overline{wxyz}}^{\,1}&=\mathrm{Z}-\mathrm{Z}^2+\mathrm{Z}^3-\mathrm{Z}^4\\&=e_w+e_x+e_y+e_z\\&\quad-(e_{wx}+e_{wy}+e_{wz}+e_{xy}+e_{xz}+e_{yz})\\&\quad+(e_{wxy}+e_{wxz}+e_{wyz}+e_{xyz})-e_{wxyz}.\end{aligned}$$

Again, $$\begin{aligned}e_{\overline{wxyz}}^{\,2}&=\mathrm{Z}^2-2\mathrm{Z}^3+3\mathrm{Z}^4\\&=(e_{wx}+e_{wy}+e_{wz}+e_{xy}+e_{xz}+e_{yz})\\&\quad-2(e_{wxy}+e_{wxz}+e_{wyz}+e_{xyz})+3e_{wxyz}.\end{aligned}$$

Again, $$\begin{aligned}e_{\overline{wxyz}}^{\,3}&=\mathrm{Z}^3-3\mathrm{Z}^4\\&=(e_{wxy}+e_{wxz}+e_{wyz}+e_{xyz})-3e_{wxyz}.\end{aligned}$$

Finally, $$e_{\overline{wxyz}}^{\,4}=\mathrm{Z}^4=e_{wxyz}.$$

31. If the problem be to find the curtate expectation of life, not of the *last* r survivors of m lives, but of *exactly* r survivors of m lives, we must use formulas 13 and 14 of Chap. ii, instead of formula 15. The expectation may be symbolized by $e_{\overline{wxyz\ldots(m)}}^{[r]}$, and its value we see to be

$$e_{\overline{wxyz\ldots(m)}}^{[r]}=\mathrm{Z}^{r}-(r+1)\mathrm{Z}^{r+1}+\frac{(r+1)(r+2)}{\underline{|2}}\mathrm{Z}^{r+2}-\&c.$$

$$=\frac{\mathrm{Z}^{r}}{(1+\mathrm{Z})^{r+1}}\quad.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad(31)$$

32. In the expectation in question, we do not begin to take into account the existence of the lives until $(m-r)$ of the lives have failed; and we cease to take account of the existence, immediately on the failure of the $(m-r+1)$th life. It is evident, therefore, that we may write

$$e_{\overline{wxyz\ldots(m)}}^{[r]}=e_{\overline{wxyz\ldots(m)}}^{r}-e_{\overline{wxyz\ldots(m)}}^{r+1}\quad.\ .\ .\ .\ .\quad(32)$$

33. We have examined with some detail, problems connected with expectations of life, although the subject in itself is not of much practical use. Our purpose has been to afford a very valuable exercise to the student. He will find that a thorough familiarity with the contents of this chapter will greatly assist him when he comes to read Chap. vii on life annuities and assurances; for, in fact, the curtate expectation of life is the value of an annuity on the supposition that capital yields no interest.

CHAPTER IV.

PROBABILITIES OF SURVIVORSHIP.

1. To find the probability that (x) will die in the nth year from the present time, (y) being alive at the moment of the death of (x).

This probability, which we shall symbolize by ${}_{n-1|}q^{1}_{xy}$, is composed of two partial probabilities; namely, 1st, that (x) will die in the nth year, and (y) survive to the end of the year; the value of which is $({}_{n-1}p_x - {}_np_x){}_np_y$; and, 2nd, that both lives will fail in the nth year, (x) dying first. The value of this second portion is arrived at as follows. The probability that both lives will fail in the nth year is $({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y - {}_np_y)$. If both the lives are to fail in the same year, each of them, on the supposition of a uniform distribution of deaths, has the same probability of failing in any one portion of the year as in any other equal portion; for the measure of each such probability is a fraction, of which both numerator and denominator are constant. Both lives, then, being by the supposition certain to fail in the year, and each being, by the mode of distribution of deaths, as likely to fail in any one portion of it as in any other equal portion; it follows that it is an even chance which of them shall fail first; because for every chance that (x) will die first, there is a corresponding and equal chance that he will die second. The chance, then, that the first life to fail will be (x), is $\frac{1}{2}$; which, multiplied by the probability of both lives failing in the year, gives $\frac{1}{2}({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y - {}_np_y)$.

Therefore the entire probability that (x) will die in the nth year, and that (y) will be alive at the moment of his death, is

$$\begin{aligned} {}_{n-1|}q^{1}_{xy} &= ({}_{n-1}p_x - {}_np_x){}_np_y + \tfrac{1}{2}({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y - {}_np_y) \\ &= \tfrac{1}{2}({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y + {}_np_y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

Written in terms of l_x and d_x, the equation becomes

$$ {}_{n-1|}q^1_{xy} = \frac{(l_{x+n-1} - l_{x+n})(l_{y+n-1} + l_{y+n})}{2l_{xy}} $$

$$ = \frac{d_{x+n-1} \times l_{y+n-\frac{1}{2}}}{l_{xy}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2) $$

This shows that the probability that (x) will die before (y) in the nth year, is equal to the probability that (x) will die in that year, and that (y) will survive to the middle of the year.

2. If the question be, to find the probability that (x) will die before (y) in the first year; n becomes unity in equation 1, and we have, because ${}_0p_x = {}_0p_y = 1$,

$$ q^1_{xy} = \tfrac{1}{2}(1 - p_x)(1 + p_y) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3) $$

$$ = p_{xy} \times \tfrac{1}{2}(p_x^{-1} - 1)(p_y^{-1} + 1) \; . \quad . \; . \; . \; . \; . \quad (4) $$

The probability is put in the form of equation 4, to facilitate certain calculations; as will appear in the chapter on the construction of tables.

3. To find the total probability that (x) will die before (y), we must, in formula 1, give to n successively every integral value from unity upwards, and take the sum. This total probability we shall write Q^1_{xy}. Formula 1 may be written

$$ \tfrac{1}{2}({}_{n-1}p_{xy} - {}_np_{xy} - {}_np_x \times {}_{n-1}p_y + {}_{n-1}p_x \times {}_np_y), $$

or (in accordance with Chap. ii, Art. 24)

$$ \frac{1}{2}\left({}_{n-1}p_{xy} - {}_np_{xy} - \frac{{}_np_{x:y-1}}{p_{y-1}} + \frac{{}_np_{x-1:y}}{p_{x-1}}\right). $$

Summing this last expression for every integral value of n from unity upwards, we have, for the first two terms,

$$ \Sigma({}_{n-1}p_{xy} - {}_np_{xy}) = {}_0p_{xy} + \Sigma({}_np_{xy} - {}_np_{xy}) = 1, $$

because ${}_0p_{xy} = 1$, and the summations of the two terms ${}_np_{xy}$ cancel each other; and for the remaining two terms,

$$ \Sigma\frac{{}_np_{x:y-1}}{p_{y-1}} = \frac{e_{x:y-1}}{p_{y-1}}, \text{ and } \Sigma\frac{{}_np_{x-1:y}}{p_{x-1}} = \frac{e_{x-1:y}}{p_{x-1}} $$

(by Chap. iii, formula 22). Therefore

$$ Q^1_{xy} = \frac{1}{2}\left(1 - \frac{e_{x:y-1}}{p_{y-1}} + \frac{e_{x-1:y}}{p_{x-1}}\right) \quad . \; . \; . \; . \; . \quad (5) $$

In the same way

$$Q_{x\overset{1}{y}}=\frac{1}{2}\left(1-\frac{e_{x-1:y}}{p_{x-1}}+\frac{e_{x:y-1}}{p_{y-1}}\right)$$

In equation 35 of Chap. xiii, another expression will be given for $Q_{\overset{1}{x}y}$, derived by means of the Differential Calculus.

4. It should be noticed that $Q_{\overset{1}{x}y}+Q_{x\overset{1}{y}}=1$; so that $Q_{x\overset{1}{y}}=1-Q_{\overset{1}{x}y}$. Therefore, if a table of the function be required, it need be constructed only for $x>y$, or for $x<y$. To tabulate both sets of values would be an unnecessary labour.

If $x=y$, then $Q_{\overset{1}{x}y}=\frac{1}{2}$.

omit

5. The probability ${}_{n-1|}q_{\overset{1}{x}y}$ may be obtained in another way, which to some minds may perhaps be more conclusive. Let the nth year be divided into t equal parts. Then, on the supposition of a uniform distribution of deaths, and assuming for convenience that there are l_x persons aged x, and l_y persons aged y, the number of lives of the (x) group who will die in each of such parts, is $\frac{1}{t}d_{x+n-1}$; and of the (y) group, $\frac{1}{t}d_{y+n-1}$; and the probability that (x) will die in any particular one of these parts, say the sth, and that (y) will survive to the end of that part, is $\frac{d_{x+n-1}}{tl_x}\times\frac{l_{y+n-1}-\frac{s}{t}d_{y+n-1}}{l_y}$, which is equal to $\frac{d_{x+n-1}}{t^2l_xl_y}(tl_{y+n-1}-sd_{y+n-1})$. If in the last expression we make s equal to 1, 2, 3, ... t, successively, and add the results; the sum will be the probability that (x) will die in some one or other of the parts into which the year is supposed to be divided, and that (y) will be alive at the end of the same part. Performing the operation, we have for the probability in question, $\frac{d_{x+n-1}}{t^2l_xl_y}\left(t^2l_{y+n-1}-\frac{t(t+1)}{2}d_{y+n-1}\right)$, or $\frac{d_{x+n-1}}{l_xl_y}\left\{l_{y+n-1}-\frac{1}{2}\left(1+\frac{1}{t}\right)d_{y+n-1}\right\}$. This, as just stated, is the probability that (x) will die in the nth year, and that (y) will survive him by at least a portion of time which does not exceed $\frac{1}{t}$ of a year. If now we make t infinitely great, we arrive at the probability that (x) will die in the nth year, and that (y) will be alive at the moment of the death of (x). By making t infinitely great, the term involving t in the denominator is caused to vanish; with the result that

$${}_{n-1|}q_{\overset{1}{x}y}=\frac{d_{x+n-1}}{l_xl_y}\left(l_{y+n-1}-\tfrac{1}{2}d_{y+n-1}\right).$$

This is identical with equation 2, because $l_{y+n-1}-\frac{1}{2}d_{y+n-1}=l_{y+n-\frac{1}{2}}$.

[6] In the foregoing investigation, we have found ${}_{n-1|}q^1_{xy}$ and Q^1_{xy} on the hypothesis that the deaths are equally distributed throughout each year of age: but it will be well now to extend the enquiry, and find an expression for the error which that hypothesis involves. At present we shall merely amplify the last article, making use only of the Calculus of Finite Differences. We shall include only the second differences of the l column, and discard the higher orders.

Fixing our attention to begin with on the first year, and dividing that year as in Art. 5 into t parts; the probability that (x) will survive $(s-1)$ of these parts, but die in the next part, and that (y) will survive the s parts is

$$\frac{1}{l_xl_y}\left(l_{x+\frac{s-1}{t}}-l_{x+\frac{s}{t}}\right)\times l_{y+\frac{s}{t}}.$$

Retaining the notation of Chap. iii, this may be written,

$$\frac{1}{l_xl_y}\left\{l_x+\frac{s-1}{t}\Delta_x-\frac{(s-1)(t-s+1)}{2t^2}\Delta^2{}_x\right.$$

$$\left.-l_x-\frac{s}{t}\Delta_x+\frac{s(t-s)}{2t^2}\Delta^2{}_x\right\}\left\{l_y+\frac{s}{t}\Delta_y-\frac{s(t-s)}{2t^2}\Delta^2{}_y\right\}$$

$$=\frac{1}{l_xl_y}\left\{-\frac{1}{t}\Delta_x+\frac{t-2s+1}{2t^2}\Delta^2{}_x\right\}\left\{l_y+\frac{s}{t}\Delta_y-\frac{s(t-s)}{2t^2}\Delta^2{}_y\right\}$$

$$=-\frac{1}{t}\frac{1}{l_xl_y}\left\{l_y\Delta_x+\frac{s}{t}\Delta_x\Delta_y-\frac{s(t-s)}{2t^2}\Delta_x\Delta^2{}_y-\frac{t-2s+1}{2t}l_y\Delta^2{}_x\right.$$

$$\left.-\frac{s(t-2s+1)}{2t^2}\Delta^2{}_x\Delta_y+\frac{s(t-s)(t-2s+1)}{4t^3}\Delta^2{}_x\Delta^2{}_y\right\}.$$

If now in the last expression we give to s successively the values 1, 2, 3, &c., up to t, and take the sum by means of the formula for summation given in Chap. xxiv, Art. 17, we shall have

$$-\frac{1}{l_xl_y}\left\{l_y\Delta_x+\frac{t+1}{2t}\Delta_x\Delta_y-\frac{t^2-1}{12t^2}\left(\Delta^2{}_y\Delta_x-\Delta^2{}_x\Delta_y-\frac{1}{2t}\Delta^2{}_x\Delta^2{}_y\right)\right\}$$

and this is the probability that (x) will die in the year, and that (y) will survive to the end of that tth part of the year in which (x) dies. By making t indefinitely great, the probability becomes that which we require; namely, the probability of (x) dying in the year, and (y) being alive at the moment of his death. Increasing, then, t without limit, we have

$$-\frac{1}{l_xl_y}\left\{l_y\Delta_x+\tfrac{1}{2}\Delta_x\Delta_y-\tfrac{1}{12}(\Delta^2{}_y\Delta_x-\Delta^2{}_x\Delta_y)\right\}$$

Returning now to the notation of the mortality table, where

$$\Delta_x = l_{x+1} - l_x,$$
$$\Delta^2{}_x = (l_{x+2} - l_{x+1}) - (l_{x+1} - l_x),$$

and similarly for Δ_y and $\Delta^2{}_y$, we have

$$\begin{aligned} &\frac{1}{l_x l_y}\{l_y(l_x - l_{x+1}) - \tfrac{1}{2}(l_x - l_{x+1})(l_y - l_{y+1}) \\ &\qquad + \tfrac{1}{12}([l_{y+1} - l_{y+2}] - [l_y - l_{y+1}])(l_x - l_{x+1}) \\ &\qquad - \tfrac{1}{12}([l_{x+1} - l_{x+2}] - [l_x - l_{x+1}])(l_y - l_{y+1})\} \\ &= \frac{1}{l_x l_y}\{\tfrac{1}{2}(l_x - l_{x+1})(l_y + l_{y+1}) \\ &\qquad + \tfrac{1}{12}(l_x - l_{x+1})(l_{y+1} - l_{y+2}) - \tfrac{1}{12}(l_y - l_{y+1})(l_{x+1} - l_{x+2})\} \quad . \quad . \quad (6) \end{aligned}$$

This is identical with expression 2, if $\dfrac{l_x - l_{x+1}}{l_{x+1} - l_{x+2}} = \dfrac{l_y - l_{y+1}}{l_{y+1} - l_{y+2}}$.

If we wish to find the probability that (x) will die before (y) in the nth year instead of in the first year; we must, as in Art. 5, in the numerator of expression 6, change x and y into $x+n-1$ and $y+n-1$, respectively. The expression then becomes

$$\begin{aligned} {}_{n-1|}q^1_{xy} &= \frac{(l_{x+n-1} - l_{x+n})(l_{y+n-1} + l_{y+n})}{2l_{xy}} \\ &+ \frac{(l_{x+n-1} - l_{x+n})(l_{y+n} - l_{y+n+1}) - (l_{y+n-1} - l_{y+n})(l_{x+n} - l_{x+n+1})}{12l_{xy}} \quad . \quad (7) \end{aligned}$$

[7] It will be noticed that if we omit the second term of the right-hand member of the last equation, we have the expression of Art. 1. The second term, therefore, gives approximately the amount of the error due to the hypothesis of a uniform distribution of deaths throughout the year of life; and it is easy to see that the error is very small. The numerator consists of the difference of two products, both of which are small, and which are very nearly equal, each factor in the one product having a corresponding, and nearly equal, factor in the other. Also, the difference of the products is divided by $12l_x l_y$, a comparatively very large quantity.

[8] To find the total probability that (x) will die before (y), we must, in expression 7, give to n every integral value from unity upwards. We thus have, after reduction,

$$Q^{1}_{xy}=\frac{1}{2}\left(1-\frac{e_{x:y-1}}{p_{y-1}}+\frac{e_{x-1:y}}{p_{x-1}}\right)$$
$$+\frac{1}{12}\left\{\frac{e_{x-1:y}-p_y e_{x-1:y+1}}{p_{x-1}}-\frac{e_{x:y-1}-p_x e_{x+1:y-1}}{p_{y-1}}\right.$$
$$\left.+p_y e_{x:y+1}-p_x e_{x+1:y}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

9. The total probability that (x) will die after (y), Q^{2}_{xy}, is evidently the probability that (x) will die, less the probability that he will die before (y). But in dealing with probabilities for the whole of life the probability that (x) will die is certainty, denoted by unity; therefore

$$Q^{2}_{xy}=1-Q^{1}_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We have seen also that $Q_{x\overset{1}{y}}=1-Q^{1}_{xy}$, so that $Q^{2}_{xy}=Q_{x\overset{1}{y}}$.

10. For a period limited to n years the formulas are different. They are as follows. The probability that (x) will die after (y) within n years, is the probability that x will die within the term, less the probability that he will die first in the term; that is

$${}_{|n}q^{2}_{xy}={}_{|n}q_x-{}_{|n}q^{1}_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The probability that (y) will die before (x) within n years, is the probability that the joint lives will fail within the term, less the probability that the failure will be caused by the death of (x); that is,

$${}_{|n}q_{x\overset{1}{y}}={}_{|n}q_{xy}-{}_{|n}q^{1}_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Also

$${}_{|n}q_{x\overset{2}{y}}={}_{|n}q_y-{}_{|n}q_{x\overset{1}{y}}$$
$$={}_{|n}q_y-{}_{|n}q_{xy}+{}_{|n}q^{1}_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

11. In formulas 10, 11, and 12, the values of ${}_{|n}q_x$, ${}_{|n}q_y$, and ${}_{|n}q_{xy}$ are easily deduced from the mortality table, because ${}_{|n}q_x=1-\frac{l_{x+n}}{l_x}$, ${}_{|n}q_y=1-\frac{l_{y+n}}{l_y}$, and ${}_{|n}q_{xy}=1-\frac{l_{x+n}l_{y+n}}{l_x l_y}$. It therefore only remains to assign a value to ${}_{|n}q^{1}_{xy}$. We have, Art. 3, the probability that (x) will die before (y) in the nth year,

$$\frac{1}{2}\left({}_{n-1}p_{xy}-{}_{n}p_{xy}-\frac{{}_{n}p_{x:y-1}}{p_{y-1}}+\frac{{}_{n}p_{x-1:y}}{p_{x-1}}\right);$$

and to obtain ${}_{|n}q^{1}_{xy}$ we must sum the expression for all integral values of n, from 1 to n. The result is

$${}_{|n}q^{1}_{xy}=\frac{1}{2}\left\{1-{}_{n}p_{xy}-\frac{{}_{|n}e_{x:y-1}}{p_{y-1}}+\frac{{}_{|n}e_{x-1:y}}{p_{x-1}}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

Remembering that ${}_{|n}e_{xy} = e_{xy} - {}_np_{xy} \times e_{x+n:y+n}$; the value of the expression in equation 13 can be calculated, having given a table of e_{xy} for all values of x and y.

12. We have seen, formula 2, that

$$ {}_{n-1|}q^1_{xy} = \frac{d_{x+n-1} \times l_{y+n-\frac{1}{2}}}{l_x l_y}, $$

whence, giving to n every integral value from unity upwards,

$$ Q^1_{xy} = \frac{d_x l_{y+\frac{1}{2}} + d_{x+1} l_{y+\frac{3}{2}} + \&c.}{l_x l_y}, $$

and also

$$ {}_{|n}q^1_{xy} = \frac{d_x l_{y+\frac{1}{2}} + d_{x+1} l_{y+\frac{3}{2}} + \&c. + d_{x+n-1} l_{y+n-\frac{1}{2}}}{l_x l_y}. $$

These two expressions show us how we may construct a table by means of which to calculate, with great facility, the survivorship probabilities, whether for the whole of life, or temporary or deferred. We must form the product $d_x \times l_{y+\frac{1}{2}}$ for each year of life, the difference $x-y$ remaining constant; and having placed the values in a column, we must continuously sum them from the bottom upwards, entering the results in a second column. For each difference $x-y$ a distinct set of columns must be prepared. The following is the concluding portion of such a table for the difference, $x-y=5$.

(1) x	(2) y	(3) $d_x \times l_{y+\frac{1}{2}}$	(4) $\Sigma(d_x \times l_{y+\frac{1}{2}})$
90	85	1,966,986	4,245,325
91	86	1,125,392	2,278,339
92	87	603,174	1,152,947
93	88	307,440	549,773
94	89	142,941	242,333
95	90	62,176	99,392
96	91	24,582	37,216
97	92	8,460	12,634
98	93	2,940	4,174
99	94	880	1,234
100	95	300	354
101	96	54	54

To find from the table the probability that a life aged 90 will die before a life aged 85, we must divide the number in col. 4, opposite age 90 in col. 1, by $l_{90} \times l_{85}$. The result is ·60822. If we wish to find the probability that a life aged 90 will die before a life aged 85 within the next 5 years, we must, from the number in col. 4 opposite age 90 in col. 1, deduct the number opposite age 95, and divide the difference by $l_{90} \times l_{85}$. The result is ·59398. If we wish to find the probability that a life aged 90 will die before a life aged 85 after 5 years, we divide the number in col. 4 opposite age 95 in col. 1, by $l_{90} \times l_{85}$. The result is ·01424.

The labour of constructing a complete table of the kind described would be very great; and as its uses would be limited, no one has ever undertaken the task. But when the rate of interest is also involved, such tables are most valuable for the purpose of calculating contingent assurances; and David Chisholm has given a complete set for the Carlisle Mortality Table, at several rates of interest, in his great work on Life Contingencies, published in 1858.

13. To find the probability that (x) will die, either before (y), or within t years after the death of (y). This probability we shall denote by the symbol $Q^1_{x:\overline{y(t\rceil)}}$. The mark $\rceil$ round the t signifies, that t represents a term certain. The brackets ( ) round t tell us, that that term certain is to commence only on the death of (y); and the bar —— over $y(\overline{t}\rceil)$, like the bar in $e_{\overline{xyz}}$, means that the last survivor is in question; that is, that in order that the conditions of the probability may be satisfied, (x) must die before (y), or before the end of the t years that follow the death of (y).

The probability sought consists of two portions; namely, 1st, that (x) will die within the first t years; and 2nd, that he will survive the t years, and die subsequently, the other conditions of the case being also fulfilled. If (x) die at any time within the first t years, he must of necessity die either before (y), or within t years after (y); and the first part of the required probability therefore is simply $(1-{}_tp_x)$. If (x) die in the nth year after t years, it is necessary, in order that the other conditions of the case may be fulfilled, that (y) shall live at least so long that his death shall not take place more than t years before that of (x). He must therefore either survive the nth year from the present time; or die in that year, but at such a time of the year that the interval between the two deaths shall not exceed t years. The probability that (x) will die in the $(n+t)$th year, and that (y) will survive the nth year is $({}_{n+t-1}p_x - {}_{n+t}p_x)\,{}_np_y$, or ${}_tp_x \times {}_np_y({}_{n-1}p_{x+t} - {}_np_{x+t})$. The probability that (x) will die in the $(n+t)$th year, and (y) in the nth year, the deaths so falling that there shall not be more than

t years between them is, on the usual supposition of a uniform distribution of deaths in each year of age, $\frac{1}{2}({}_{n+t-1}p_x - {}_{n+t}p_x)({}_{n-1}p_y - {}_{n}p_y)$, or $\frac{1}{2}{}_{t}p_x({}_{n-1}p_{x+t} - {}_{n}p_{x+t})({}_{n-1}p_y - {}_{n}p_y)$. The total probability for the nth year after t years, is the sum of the two partial probabilities, namely,

$$\begin{aligned}&{}_{t}p_x \times {}_{n}p_y({}_{n-1}p_{x+t} - {}_{n}p_{x+t}) + \tfrac{1}{2}{}_{t}p_x({}_{n-1}p_{x+t} - {}_{n}p_{x+t})({}_{n-1}p_y - {}_{n}p_y)\\ &= \tfrac{1}{2}{}_{t}p_x({}_{n-1}p_{x+t} - {}_{n}p_{x+t})({}_{n-1}p_y + {}_{n}p_y).\end{aligned}$$

If now we take the sum of the last expression for every integral value of n, we shall have the second portion of the required probability. This sum evidently is ${}_{t}p_x \times \mathrm{Q}_{\overset{1}{x+t}:y}$. Thus the total probability which we seek, is

$$\begin{aligned}\mathrm{Q}^1_{x:\overline{y(t)}} &= 1 - {}_{t}p_x + {}_{t}p_x \times \mathrm{Q}_{\overset{1}{x+t}:y}\\ &= 1 - {}_{t}p_x(1 - \mathrm{Q}_{\overset{1}{x+t}:y}) \quad . \quad . \quad . \quad . \quad . \quad (14)\end{aligned}$$

14. There are two similar but distinct probabilities which must not be confused; namely, α, that (y) will be alive at the death of (x); and, β, that (y) will be alive at the end of the year in which (x) shall die: or, to take the more general case; α, that (y) will be alive t years after the death of (x); and, β, that (y) will be alive at the end of the tth year succeeding the year in which (x) shall die. For α, (y) will be alive t years after the death of (x), if (x) die in the nth year, and (y) survive $(n+t)$ years, or also if (x) die in the nth year and (y) in the $(n+t)$th year, but at such times that the interval between the deaths is not less than t years. This probability is, for the nth year, $\frac{1}{2}{}_{t}p_y({}_{n-1}p_x - {}_{n}p_x)({}_{n-1}p_{y+t} + {}_{n}p_{y+t})$ and the total for all values of n is ${}_{t}p_y \times \mathrm{Q}^1_{x:y+t}$, as in Art. 13. For β; (y) will be alive at the end of the tth year succeeding the year of the death of (x), if (x) die in the nth year, and (y) survive $(n+t)$ years. For the nth year, therefore, the probability is ${}_{n+t}p_y({}_{n-1}p_x - {}_{n}p_x)$; and the total for all values of n, is ${}_{t}p_y\left(\frac{e_{x-1:y+t}}{p_{x-1}} - e_{x:y+t}\right)$.

15. We have investigated in Chap. iii, Art. 15, the expectation of life of (x), deferred for a term of years certain. We may now extend the inquiry, and ask what is the expectation of life of a person aged x, after the death of a person aged y. The meaning of the question is, having given a large number of pairs of persons, one aged x, and the other aged y, if we take all the years of life completed by the members of the (x) group, after the death of their partners of the (y) group, and divide the total equally among all the members of the (x) group who come under observation at the commencement, what will be the share of each? We may denote this reversionary expectation of life by the symbol $e_{y|x}$; meaning, the expectation of life of (x) after the death of (y). In order that any year may count, (x) must survive to the end of the year, and (y) must die before the end of the year; and the probability of this double event is ${}_np_x(1-{}_np_y)={}_np_x-{}_np_{xy}$. The reversionary expectation of life which we seek, is the sum of the values of this probability for every integral value of n. That is,

$$\begin{aligned} e_{y|x} &= \Sigma({}_np_x-{}_np_{xy}) \\ &= e_x-e_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15) \end{aligned}$$

16. In Art. 13 of this chapter we inquired what is the probability that (x) will die before (y), or within t years after the death of (y)? The cognate question may be asked, What is the expectation of life of (x) during the life of (y), and for t years after the death of (y)? This expectation we shall denote by $e_{x:\overline{y(\overline{t|})}}$. Following the reasoning of Art. 13, we may divide the required expectation into two parts; namely, the portion within the first t years, and the portion after t years. The portion within the first t years is ${}_{|t}e_x$. After t years, that any year, say the nth, may count, (x) must live to the end of the year, and (y) must have lived to within at least t years of the end of the year. The probability of this double event is ${}_{n+t}p_x \times {}_np_y$; and, taking the sum for all values of n, we have ${}_tp_x \times e_{x+t:y}$, as the portion of the required expectation for the period after t years. The total required expectation therefore is

$$\begin{aligned} e_{x:\overline{y(\overline{t|})}} &= {}_{|t}e_x + {}_tp_x \times e_{x+t:y} \\ &= e_x - {}_tp_x(e_{x+t}-e_{x+t:y}) \quad . \quad . \quad . \quad . \quad . \quad (16) \end{aligned}$$

17. We shall now briefly consider the probabilities of survivorship in which three lives are involved; but before doing so it will be convenient to introduce for the moment a notation, which will be a little less cumbrous, though less graphic, than the notation hitherto employed. We shall, in dealing with this branch of the subject, write

$$\begin{array}{lll} \alpha \text{ for } {}_{n-1}p_x & \beta \text{ for } {}_{n-1}p_y & \gamma \text{ for } {}_{n-1}p_z \\ {}'\alpha \quad ,, \quad {}_np_x & {}'\beta \quad ,, \quad {}_np_y & {}'\gamma \quad ,, \quad {}_np_z \end{array}$$

so that the probabilities that (x), (y) and (z), respectively, will die in the nth year are, $(\alpha-{}'\alpha)$, $(\beta-{}'\beta)$ and $(\gamma-{}'\gamma)$.

18. To determine the probability that (x) will die first of the three lives, (x), (y), and (z).

In respect of the nth year there are four ways in which the event may happen. First, (x) alone may die in the nth year, (y) and (z) both surviving it; the probability of which is $(\alpha-{}'\alpha)'\beta'\gamma$. Secondly, (x) and (y) may both die in the nth year, (x) dying first, and (z) surviving the year; the probability of which is $\frac{1}{2}(\alpha-{}'\alpha)(\beta-{}'\beta)'\gamma$. Thirdly, (x) and (z) may both die in the year, (x) dying first, and (y) surviving the year; the probability of which is $\frac{1}{2}(\alpha-{}'\alpha)'\beta(\gamma-{}'\gamma)$. Fourthly, all three lives may fail in the year, (x) dying first. Now, there are six different orders in which the deaths may occur in the year; and, on the supposition of a uniform distribution of deaths, all of these are equally likely. Therefore the probability of each one of them is one-sixth of the probability that all three lives will fail in the year. Out of these six possible arrangements, two are in favour of (x) dying first; and therefore the probability that (x) will be the first to die, is one-third of the probability that all three lives will die in the year; that is, $\frac{1}{3}(\alpha-{}'\alpha)(\beta-{}'\beta)(\gamma-{}'\gamma)$. Therefore ${}_{n-1|}q^{1}_{xyz}$, the probability that (x) will die in the nth year, while both (y) and (z) survive him, is

$$\begin{aligned}{}_{n-1|}q^{1}_{xyz}&=(\alpha-{}'\alpha)'\beta'\gamma+\tfrac{1}{2}\{(\alpha-{}'\alpha)(\beta-{}'\beta)'\gamma+(\alpha-{}'\alpha)'\beta(\gamma-{}'\gamma)\}\\&\qquad+\tfrac{1}{3}(\alpha-{}'\alpha)(\beta-{}'\beta)(\gamma-{}'\gamma)\\&=\tfrac{1}{3}(\alpha\beta\gamma-{}'\alpha'\beta'\gamma)-\tfrac{1}{3}({}'\alpha\beta\gamma-\alpha'\beta'\gamma)\\&\qquad+\tfrac{1}{6}(\alpha\beta'\gamma-{}'\alpha'\beta\gamma+\alpha'\beta\gamma-{}'\alpha\beta'\gamma)\,.\quad\ldots\ldots\quad(17)\end{aligned}$$

To find Q^{1}_{xyz}, the total probability that (x) will die first of the three lives, (x), (y), and (z), we must sum the expression in equation 17 for all the integral values of n. We then have, after making what now must be to the student self-evident modifications,

$$\begin{aligned}Q^{1}_{xyz}&=\frac{1}{3}\left\{1-\frac{e_{x:y-1:z-1}}{p_{y-1:z-1}}+\frac{e_{x-1:y:z}}{p_{x-1}}\right\}\\&\qquad+\frac{1}{6}\left\{\frac{e_{x-1:y-1:z}}{p_{x-1:y-1}}-\frac{e_{x:y:z-1}}{p_{z-1}}+\frac{e_{x-1:y:z-1}}{p_{x-1:z-1}}-\frac{e_{x:y-1:z}}{p_{y-1}}\right\}\;.\;.\quad(18)\end{aligned}$$

In equation 36 of Chap. xiii, a much shorter expression will be given for Q^{1}_{xyz}, deduced by means of the Differential Calculus.

19. When we were dealing in former chapters with probabilities and expectations of life, we found that, when we had so reduced the expressions as to involve only single and joint lives, we could not make further simplifications. Thus we found that $e_{\overline{xyz}}$, expressed in its simplest terms, is $e_x+e_y+e_z-e_{xy}-e_{xz}-e_{yz}+e_{xyz}$. Now, in a similar manner, we shall find that, in dealing with those functions which we may designate as simple survivorship probabilities,—to distinguish them from the compound survivorship probability to be briefly discussed in Arts. 26 and 27 of this chapter,—we can reduce the expressions to probabilities which determine on the first death; but that then we cannot go any further.

20. Thus, the probability that (x) will die second of the three lives (x), (y), and (z), (y) having died first, which we may write $Q^2_{x\underset{1}{y}z}$, is evidently the probability that (x) will die before (z), less the probability that he will die before both (y) and (z). That is,

$$Q^2_{x\underset{1}{y}z}=Q^1_{xz}-Q^1_{xyz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

21. Again, the probability that (x) will die second of the three lives, (x), (y), and (z), is the probability that he will die before (y) and after (z); together with the probability that he will die before (z) and after (y). That is,

$$\begin{aligned} Q^2_{xyz}&=Q^2_{x\underset{1}{y}z}+Q^2_{xy\underset{1}{z}} \\ &=Q^1_{xy}+Q^1_{xz}-2Q^1_{xyz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20) \end{aligned}$$

22. Also, the probability that (x) will die third of the three lives is the probability that he will die, less the sum of the probabilities that he will die first, and that he will die second. The probability that (x) will die is certainty, that is, unity. Therefore

$$\begin{aligned} Q^3_{xyz}&=1-Q^1_{xyz}-Q^2_{xyz} \\ &=1-Q^1_{xy}-Q^1_{xz}+Q^1_{xyz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21) \end{aligned}$$

23. The probability that (x) will die before the survivor of (y) and (z), is the probability that he will die either first or second of the three lives. That is,

$$\begin{aligned} Q^1_{x:\overline{yz}}&=Q^1_{xyz}+Q^2_{xyz} \\ &=Q^1_{xy}+Q^1_{xz}-Q^1_{xyz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22) \end{aligned}$$

24. The probability that the survivor of (x), and (y), will die before (z), is the sum of the probabilities that (x) will die second, (y) having died first, and that (y) will die second, (x) having died first. That is

$$\begin{aligned} Q_{\overset{1}{\overline{xy}}:z}&=Q^2_{x\underset{1}{y}z}+Q_{\underset{1}{x}\overset{2}{y}z} \\ &=Q^1_{xz}+Q^1_{yz}-Q^1_{xyz}-Q_{x\overset{1}{y}z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23) \end{aligned}$$

It can easily be proved that this is equal to $Q_{xy\overset{3}{z}}$, the probability that (z) will die last of the three lives.

25. The problems are almost endless that can be set from the various combinations of three lives; but the foregoing examples are sufficient for purposes of illustration. As an exercise, the student may ask himself many questions similar to those that have been answered above.

26. Hitherto in this chapter we have considered probabilities in which only one order of survivorship is in question; the only apparent exception being $Q_{\overset{2}{x}\underset{1}{y}z}$, the probability that (x) will die before (z), and after (y); and we have been able to reduce all the probabilities investigated to expressions including only terms of the forms $Q_{\overset{1}{x}y}$ and $Q_{\overset{1}{x}yz}$. There is one case involving three lives which is of a more complex nature; namely, the probability that (x) will survive to the end of the nth year, and that the joint lives (y) and (z) will fail within the n years, the failure being caused by the death of (y). In all the previous cases that have engaged our attention, the probability as regards the nth year depended on the sum, or the product, of certain simple probabilities; the values of which could be calculated without difficulty from the l_x and d_x columns of the mortality table; but the present case is of a more complicated character. The probability now sought, consists of the product of two probabilities; 1st, that (x) will survive n years, the value of which is ${}_np_x$; and 2nd, that (y) will die before (z) within n years, ${}_{|n}q_{\overset{1}{y}z}$, the value of which, equation 13, we found to be the sum, for all integral values of n from 1 to n, of the expression

$$\frac{1}{2}\left({}_{n-1}p_{yz}-{}_np_{yz}-\frac{{}_np_{y:z-1}}{p_{z-1}}+\frac{{}_np_{y-1:z}}{p_{y-1}}\right).$$

The probability which we require therefore is

$${}_np_x\times{}_{|n}q_{\overset{1}{y}z}={}_np_x\times\Sigma_1^n\frac{1}{2}\left({}_{n-1}p_{yz}-{}_np_{yz}-\frac{{}_np_{y:z-1}}{p_{z-1}}+\frac{{}_np_{y-1:z}}{p_{y-1}}\right)\quad.\quad.\quad(24)$$

We must multiply the simple probability, ${}_np_x$, by the result of a summation of a series of values of other simple probabilities.

27. If now we ask for the expectation of life of (x) after the failure of the joint lives (y) and (z), that failure being caused by the death of (y), we shall have to perform a double process of summation. If the order of survivorship between (y) and (z) were not part of the problem, we should merely have to determine $e_{yz|x}$, the expectation of life of (x)

after the failure of the joint lives (y) and (z) ; and this is easily done, because

$$\begin{aligned} e_{yz|x} &= \Sigma\,{}_np_x(1-{}_np_{yz}) \\ &= e_x - e_{xyz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25) \end{aligned}$$

But when we introduce the order of survivorship between (y) and (z), we have to determine $e^1_{yz|x}$ from the equation

$$e^1_{yz|x} = \Sigma\,{}_np_x \times {}_{|n}q^1_{yz},$$

and by equation 24 this becomes

$$e^1_{yz|x} = \Sigma\left\{{}_np_x \times \Sigma\frac{1}{2}\left({}_{n-1}p_{yz} - {}_np_{yz} - \frac{{}_np_{y:z-1}}{p_{z-1}} + \frac{{}_np_{y-1:z}}{p_{y-1}}\right)\right\} \quad . \quad . \quad (26)$$

The value of ${}_np_x$ for each value of n, must be multiplied by a distinct sum of a series of other probabilities; and the sum of the results taken; and to calculate the numerical value the labour is very great. No formula has yet been devised to render the process short and easy.

In Chap. xv, on Compound Survivorship Annuities and Assurances, we shall show that all the cases involving not more than three lives depend on the annuity which corresponds to the expectation $e^1_{yz|x}$; and we shall discuss the general question, where the number of lives is unlimited.

CHAPTER V.

STATISTICAL APPLICATIONS OF THE MORTALITY TABLE.

1. The object of the present chapter is to explain more fully and illustrate the nature and the statistical applications of the Mortality Table. A certain amount of repetition is unavoidable; but good purpose will be served, if the student thereby gain a thorough knowledge of the groundwork of the science of Life Contingencies. A perfect acquaintance with the properties of the Mortality Table is indispensable to a full understanding of the chapters that follow.

2. The Mortality Table represents a stationary population, kept up by births alone, and not disturbed by emigration or immigration. From problems relating to such an imaginary population we can pass, by making suitable modifications, to others relating to communities that actually exist.

3. Hitherto we have used the name *Mortality Table* to denote the two columns l_x and d_x. We are now, however, about to discuss a number of other columns, which, for statistical purposes, it is convenient to include in the table; and it is desirable to apply another name to all the columns taken in the aggregate. We shall therefore call such a collection of columns as is given in Table I, a *Life Table;* while we shall still keep the name *Mortality Table* for the l_x and d_x columns only. The name *Life Table* may also be used with extended meaning, so as to embrace all the tables, monetary and other, that are used in calculating life contingencies, and in which the rate of mortality is involved.

4. The fundamental column of the Life Table is the column of l_x. The first value in that column, l_0, called the radix, is the number of annual births in the imaginary population; and the succeeding numbers show how many persons, out of l_0 born alive, complete each year of age.

In the table the number of annual births is 127,283; and we observe that 100,000 live to complete the tenth year of their age; 89,685 live to complete the thirtieth; and so on. We also see that only 4 live to complete the century; and that, although 1 survives 101 years, all die before reaching the age 102. Age 102 is therefore the *limiting age* of the table, being the year of age on which some lives enter, but which none complete. To the limiting age, the Greek letter ω is assigned for symbol; and therefore $l_\omega=0$. Also, the difference between the limiting age, and the present age, is called the *complement of life;* so that, at age x, the complement of life is $\omega-x$; in the case of our table, $102-x$.

5. The column of d_x contains the differences between the numbers in the column of l_x; and shows how many, out of l_0 persons born alive, die in each year of their age. Thus, by the table, out of 127,283 persons born alive, 14,358 die before completing their first year; 691 survive to age 30, but die before reaching 31; and so on. The number, then, in column d opposite any age, x, is the number who complete that year of age, but die before completing the next; that is, the number in column d opposite age x, is the number who die in the $(x+1)$th year of age. As all born must die, it follows that the sum of all the numbers in column d is equal to l_0: also, the sum of the numbers in column d, from age x to the oldest age, is equal to l_x.

6. To find how many die aged between x and $x+n$, we may take the sum of the numbers in column d, from d_x to d_{x+n-1}, inclusive; but unless n be very small, it will be easier to obtain the result by means of the l column; because $l_x-l_{x+n}=d_x+d_{x+1}+\&c.+d_{x+n-1}$. Thus, by the table, the number of persons, out of 127,283 born alive, who die between ages 20 and 30, is $l_{20}-l_{30}=6376$.

7. Passing now to the column L, we have the population living in a stationary community. Such a community, sustained by l_0 annual births, will, on the supposition of uniform distribution of births and deaths, always contain $\frac{1}{2}(l_0+l_1)=\mathrm{L}_0$ children in the first year of their age; $\frac{1}{2}(l_1+l_2)=\mathrm{L}_1$ in the second year of their age; and so on. Thus, by the table, a population sustained by 127,283 annual births, will always contain 95,787 young persons aged between 20 and 21. The total population at all ages will be the sum of all the numbers in column L; and that is given in column T. By the table, the total population that would be supported by 127,283 annual births, is $6,082,031=\mathrm{T}_0$. The column T bears exactly the same relation to the column L, that the

column l bears to the column d: that is, T_x is the sum of the numbers in column L, from age x to the oldest age: therefore T_x is the total population, aged x and upwards, in the community. In the community of T_0 inhabitants, there must be l_0 deaths annually; because there are l_0 births; and, the population being stationary, the deaths must be equal in number to the births. Similarly, there must be l_x deaths annually of persons aged x and upwards; and $l_x - l_{x+n}$ deaths annually, of persons aged between x and $x+n$. Also, the number of inhabitants aged between x and $x+n$, is $T_x - T_{x+n}$; and, therefore, the proportion of deaths to population, between ages x and $x+n$, is $\frac{l_x - l_{x+n}}{T_x - T_{x+n}}$; and, for the whole community, the proportion of deaths to population is $\frac{l_0}{T_0}$. When n is unity, $\frac{l_x - l_{x+n}}{T_x - T_{x+n}} = \frac{d_x}{L_x} = m_x$, the central death rate at age x. By the table, the proportion of deaths to the population for the whole community is $\frac{127{,}283}{6{,}082{,}031} = {\cdot}020928$ or about 21 per thousand. The proportion for the population aged less than 50, is $\frac{127{,}283 - 72{,}795}{6{,}082{,}031 - 1{,}475{,}603}$ $= \frac{54{,}488}{4{,}606{,}428} = {\cdot}011829$, or not quite 12 per thousand, while the proportion for the population aged 50 and upwards, is $\frac{72{,}795}{1{,}475{,}603} = {\cdot}049332$, or over 49 per thousand. These figures illustrate the remarks made in Chapter iii, Art. 13. They show that if from any cause there is an unusual proportion of young persons in a community, the ratio of deaths to population will be diminished; but, as previously remarked, it does not follow that therefore the members of that community enjoy unusual longevity.

8. Many questions relating to populations may be answered by simple proportion. Thus:—What population will A births per annum support? If X be the required population, we have $X = \frac{A}{l_0} \times T_0$ For example:—In a town, 1,000 boys are born each year. What is the male population; and how many young men aged 18 are available annually for military service? In the country in which the town is situated, every young man of 18 must serve three years in the army, and then five years in the first reserve. What permanent strength does the town contribute to the army; and what to the first reserve, respectively? The male population

of the town is $\frac{1,000}{127,283} \times 6,082,031 = 47,784$. The number of young men of 18, annually available for military service, is $\frac{1000}{l_0} \times l_{18} = 762$. The permanent strength contributed to the army is $\frac{1000}{l_0} \times (T_{18} - T_{21}) = \frac{1,000}{127,283} \times 288,904 = 2,270$: and the permanent strength contributed to the first reserve is $\frac{1000}{l_0} \times (T_{21} - T_{26}) = \frac{1,000}{127,283} \times 469,937 = 3,692$.

9. As another example, let us take the following question. A large commercial establishment employs 500 clerks and shopmen. They are taken on as juniors at age 14, at a yearly salary of £20; increasing £5 per annum for the first six years; and thereafter £15 at the end of each three years of service, until a maximum salary of £200 per annum is reached; and at age 60 they are retired on a pension of £100 per annum. Assuming this arrangement to hold permanently, what will be the ultimate state of the staff in the following particulars; (*a*) How many juniors per annum will be required? (*b*) What will be the total annual sum payable in salaries? (*c*) How many clerks will be superannuated each year? (*d*) How many pensioners will there be permanently on the books; and what will be the annual sum payable in pensions?

$T_{14} - T_{60}$ $\frac{98540 \times 500}{3816160}$

(*a*) We find that the population according to the life table between ages 14 and 60 is 3,816,160, which is kept up by 98,540 annual entrants at age 14. Therefore, by simple proportion, to support a staff of 500, there must be 13 youths taken on. The number is more exactly 12·91, if we may admit of the fraction of a youth. The fraction means that, generally, 13 youths will be required every year; but that in about one year out of ten, it will be necessary to take on only 12.

(*b*) To find the total sum payable yearly in salaries, it will be convenient to divide the staff into two classes; viz.:—those entitled to yearly increments, and those who receive an addition only triennially. If the number employed were the same as the population shown by the table, the amount payable in salaries to the first class would be as follows: $20L_{14} + 25L_{15} + 30L_{16} + 35L_{17} + 40L_{18} + 45L_{19} = 20(T_{14} - T_{20}) + 5(T_{15} + T_{16} + \ldots + T_{19} - 5T_{20})$; but this amount must be reduced in the proportion of $\frac{500}{T_{14} - T_{60}} = \frac{500}{3,816,160} = \cdot000131$. Performing the calculation, we find the yearly salaries payable to members of this class amount to £2,483. Similarly, the salaries payable to members of the second class will amount to

$$\cdot000131\{50(T_{20}-T_{23})+65(T_{23}-T_{26})+\&c.+200(T_{50}-T_{60})\}$$
$$=\cdot000131\{50(T_{20}-T_{60})+15(T_{23}+T_{26}+\&c.+T_{50}-10T_{60})\}=£55{,}800.$$

The total annual salaries of the establishment therefore amount to £58,283.

(*c*) The clerks superannuated each year will number $\cdot000131\times l_{60}=8$; or more nearly 7·7; the fraction meaning that in about seven years out of ten, 8 clerks will be superannuated each year; and in the other three years, only 7 each year.

(*d*) The number of pensioners will be $\cdot000131\times T_{60}=106$; and as they will draw £100 each, the total amount of pensions per annum will be £10,600.

In the above calculations it has been assumed that employés remain with the firm until death or superannuation, and no account has been taken of resignations and dismissals. For practical purposes, therefore, the results are so far vitiated.

10. The following question illustrates the column of the table which is headed N'_x (described in Chapter iii, Art. 15), and the difference between it and the column headed T_x. A Society grants each year 100 deferred annuities, of £15 per annum each, on lives aged 28. The first payment of £15 is to be made in each case at age 60; that is, at the end of 32 years; and the payments are to be made thereafter at the end of every year throughout life. Each annuity is to be purchased by an annual contribution of £2·278, payable at the beginning of each year for 32 years, the last payment to take place at age 59; and in the event of the nominee* dying before age 60 no return is to be made. When the society reaches a stationary condition, (*a*), how many members will there be who contribute, and how many who receive annuities? (*b*), what annual sum will the society receive in contributions, and what annual sum will it pay out in annuities? and, (*c*), on the average, how many contributions will each person make who joins the society, and how many payments of annuity will he receive?

(*a*) The number of members on the books who contribute will be $\frac{100}{l_{28}}\times(T_{28}-T_{59})=2{,}661$. The number of members who will have ceased to contribute and who will have become entitled to receive £15 at the end of each year will be $\frac{100}{l_{28}} \times T_{59} = \cdot0010983 \times 872{,}170 = 958$. If by the question is meant, how many members will have actually begun to receive annuities, the answer must be $\frac{100}{l_{28}} \times T_{60} = 892$.

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* The person on whose life an annuity depends is called the *nominee.*

(*b*) This second part of the question might seem to be capable of answer by means of the reply to the first part. There are permanently 2,661 members contributing; and each of them pays per annum £2·278. Therefore it might be thought that the income of the society from this source must be £6,061. But this is not so. The members pay contributions at the *beginning* of each year of age, whereas the column L, from which is formed the column T, gives us the population living *throughout* the year. In each calendar year, the number of contributions the society will receive must be $\frac{100}{l_{28}} \times (l_{28} + l_{29} + \ldots + l_{59}) = \frac{100}{l_{28}} \times (N'_{27} - N'_{59})$ $= 2{,}744$; and, each being £2·278, the total annual income from this source must be £6,251. Contributions are paid each year by all who *enter on* the year of age; and therefore, in estimating the number receivable, we must make use of the column N′, formed by summation of the column l; whereas the membership consists of those who are *living in* each year of age; and to find the number of members we must avail ourselves of the column T, formed by summation of the column L. $N' = \Sigma$

In the same way, the number of payments of annuities made each year, is $\frac{100}{l_{28}} \times N'_{59} = 925$. This is seen to be approximately the mean between the number who have ceased contributing and the number who have actually received at least one payment of annuity. The number of payments of annuity made each year being 925, and each payment being £15, the total sum paid annually in annuities is £13,875.*

(*c*) If in any one year l_{28} members were to join the society, the total number of contributions to be paid by these entrants during their membership would be $(l_{28} + l_{29} + \&c. + l_{59}) = N'_{27} - N'_{59} = 2{,}498{,}691$; and this number being equally distributed among the entrants, gives an average number of 27·44 each. The proportion being supposed to be the same whatever the number of entrants may be, the persons who join the society will on the average pay 27·44 contributions each.

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* The income of the society will also be stationary at £13,875—this income being made up of £6,251 contributions, and £7,624 interest on invested funds.

In the same way the average number of payments of annuity received by persons joining the society will be $\frac{N'_{59}}{l_{28}}=9{\cdot}25$. The average number of payments of annuity received by members who actually draw at least one payment, will be $\frac{N'_{59}}{l_{60}}=14{\cdot}31$.

11. As a further illustration, we may take the following question: A widows' annuity society is recruited by the annual entrance of κ married couples; the husband being aged x, and the wife y. After n years from the commencement, how many widows will there be on the fund? and how many when the fund has existed long enough to have attained a stationary state?

If there were to begin with, l_x men and l_y women, forming $l_x \times l_y$ pairs; the number of pairs unbroken at the end of n years would be $l_{x+n} \times l_{y+n}$. Therefore the number of remaining couples of the κ who entered n years ago, will be $\kappa\frac{l_{x+n}l_{y+n}}{l_x l_y}=\kappa\,{}_np_{xy}$. In the same way the number of remaining couples of those who entered $n-1$ years ago, will be $\kappa\,{}_{n-1}p_{xy}$; and so on. Therefore the total number of married couples in the society will be $\kappa({}_1p_{xy}+{}_2p_{xy}+\text{\&c.}+{}_np_{xy})=\kappa|_ne_{xy}$. In the same way the number of men (husbands or widowers) in the society will be $\kappa|_ne_x$; and of women (wives or widows), $\kappa|_ne_y$. There being, therefore, $\kappa|_ne_y$ women, of whom $\kappa|_ne_{xy}$ remain married, the number of widows must be $\kappa(|_ne_y-|_ne_{xy})$.

If we increase indefinitely the value of n, the last expression becomes $\kappa(e_y-e_{xy})$; which is the number of widows on the books at the end of any year after the society has attained a stationary state.

We have considered that the entrants join the society at the beginning of the year. If they join at equal intervals throughout the year, the expressions will be $\kappa(|_n\mathring{e}_y-|_n\mathring{e}_{xy})$ and $\kappa(\mathring{e}_y-\mathring{e}_{xy})$ respectively.

In practical questions of this description, great difficulties are met with. Males and females are not subject to the same rates of mortality; and the number of couples who enter is small, and is not constant from year to year; and their ages vary greatly. There is usually, also, the probability of the re-marriage of widows to be taken into account. The actuary has, therefore, often to make very rough estimates, basing them, as far as possible, on data derived from the experience of the society the state of which he is investigating; or, if that be not available, or be insufficient, from that of the class to which the members of the society for the most part belong.

12. The problems so far considered have related to persons; and a *person* has been introduced as the unit. We pass now to another class of problems, where a *year of life* is the unit.

We have seen that, of l_x persons who attain the precise age x, l_{x+1} will complete a year of life in the first year; and d_x will live on the average half a year each; therefore the quantity of existence in the first year due to the l_x persons will be $l_{x+1}+\frac{1}{2}d_x=\text{L}_x$. Similarly for future years; therefore $\Sigma\text{L}_x=\text{T}_x$ will be the total future existence due to the l_x persons; giving $\frac{\text{T}_x}{l_x}=\mathring{e}_x$ years to each; and the average age at death of the l_x persons will be $x+\mathring{e}_x$ years. The existence within the next n years due to the l_x persons, is $\text{T}_x-\text{T}_{x+n}$; giving $\frac{\text{T}_x-\text{T}_{x+n}}{l_x}={}_{\lfloor n}\mathring{e}_x$ years for each. Of these years, $n\times l_{x+n}$ are due to those who complete age $x+n$; leaving $\text{T}_x-\text{T}_{x+n}-nl_{x+n}$ for those who die between age x and age $x+n$. But l_x-l_{x+n} persons die between these ages; therefore the average amount of existence between ages x and $x+n$, belonging to those who die between these ages, is $\frac{\text{T}_x-\text{T}_{x+n}-nl_{x+n}}{l_x-l_{x+n}}$, and their average age at death is $x+\frac{\text{T}_x-\text{T}_{x+n}-nl_{x+n}}{l_x-l_{x+n}}$. For example, if $x=20$ and $n=10$, we find ${}_{\lfloor 10}\mathring{e}_{20}=\frac{\text{T}_{20}-\text{T}_{30}}{l_{20}}=9{\cdot}680$. Also, the existence within the 10 years, due to all those who reach age 20, is $\text{T}_{20}-\text{T}_{30}=929{,}902$; the existence in the period due to those who survive it is $10l_{30}=896{,}850$; leaving 33,052 years to the $l_{20}-l_{30}=6{,}376$ persons who die in the 10 years; or 5·184 to each. The average age at death of those who die between 20 and 30, is therefore 25·184. Similarly the average age at death of those who die below 20 is $\frac{\text{T}_0-\text{T}_{20}-20l_{20}}{l_0-l_{20}}=3{\cdot}734$.

13. We have seen, Art. 7, that the ratio of deaths to population in a stationary community is $\frac{l_0}{\text{T}_0}$: also (Chap. iii, Art. 16), that the complete expectation of life is $\frac{\text{T}_0}{l_0}$. It therefore follows that the ratio of deaths to population is equal to the reciprocal of the complete expectation of life.

14. The average age at death of a stationary population, kept up by births alone, is $\frac{T_0}{l_0} = \mathring{e}_0$. What will be the effect n years hence on the average age at death, if there be an annual influx of $\frac{1}{\kappa} \times l_x$ persons aged x? The total original population being stationary, will at the end of n years, as at present, be T_0; and out of it there will be l_0 deaths in any year, say in the year from $n-\frac{1}{2}$ to $n+\frac{1}{2}$ years hence. The average age at death being $\frac{T_0}{l_0}$, we have the sum of the ages at death of all those who die equal to T_0. At the end of n years the total population, due to the immigrants, will be $\frac{1}{\kappa}(T_x - T_{x+n})$, which is the immigrant population living through the year $n-\frac{1}{2}$ to $n+\frac{1}{2}$. The deaths in that year among them will be $\frac{1}{\kappa}(l_x - l_{x+n})$, as may be seen from the following considerations. In the first year there enter $\frac{1}{\kappa} l_x$ persons who on the average are at risk for half a year, and the deaths among them are therefore $\frac{1}{\kappa} \cdot \frac{d_x}{2}$, or $\frac{1}{\kappa}(l_x - l_{x+\frac{1}{2}})$. After the first year the immigrant population aged from x to $x+1$ will be stationary, and equal to $\frac{1}{\kappa} L_x$; and in any year the deaths among them will be $\frac{1}{\kappa} d_x$. In the second year therefore the deaths of those aged from x to $x+1$ will be $\frac{1}{\kappa} d_x$, and the deaths of those aged from $x+1$ to $x+2$ will be $\frac{1}{\kappa} \cdot \frac{d_{x+1}}{2}$, being the deaths in a population of $\frac{1}{\kappa} L_{x+1}$ exposed to risk on the average for half a year. The total deaths in the second year will therefore be $\frac{1}{\kappa}(d_x + \frac{1}{2} d_{x+1})$, or $\frac{1}{\kappa}(l_x - l_{x+1\frac{1}{2}})$. Similarly in the third year the deaths in the immigrant population will be $\frac{1}{\kappa}(l_x - l_{x+2\frac{1}{2}})$, and generally the deaths in the nth year will be $\frac{1}{\kappa}(l_x - l_{x+n-\frac{1}{2}})$, and the deaths in the year from $n-\frac{1}{2}$ to $n+\frac{1}{2}$ will be $\frac{1}{\kappa}(l_x - l_{x+n})$ as above. The average age at death of these $\frac{1}{\kappa}(l_x - l_{x+n})$ persons who die will be $x + \frac{T_x - T_{x+n} - n l_{x+n}}{l_x - l_{x+n}}$: and the sum of their ages at death will be $\frac{1}{\kappa}(l_x - l_{x+n})\left\{x + \frac{T_x - T_{x+n} - n l_{x+n}}{l_x - l_{x+n}}\right\}$. Therefore the total sum of the ages at death of all those who die in the year in question in the whole community as modified by the immigration will be

$$T_0 + \frac{1}{\kappa}(l_x - l_{x+n})\left\{x + \frac{T_x - T_{x+n} - n l_{x+n}}{l_x - l_{x+n}}\right\};$$

and, dividing this total by $l_0 + \frac{1}{\kappa}(l_x - l_{x+n})$, the number of deaths, we have for the average age at death,

$$\left\{T_0+\frac{1}{\kappa}(l_x-l_{x+n})\left(x+\frac{T_x-T_{x+n}-nl_{x+n}}{l_x-l_{x+n}}\right)\right\}\div\left\{l_0+\frac{1}{\kappa}(l_x-l_{x+n})\right\}.$$

15. To take a numerical example:—If in a stationary community the number who reach age 20 be annually increased by immigration by twenty per-cent, what will be the effect ten years hence on the average age at death of the whole population?

The average age at death of a stationary community is $\frac{T_0}{l_0}=47{\cdot}784$: the number of annual deaths is $l_0=127{,}283$: and the sum of the ages at death is $T_0=6{,}082{,}031$. The number of annual immigrants aged 20 will be $\frac{1}{5}\times l_{20}=19{,}212$; and at the end of ten years the population due to immigration will be $\frac{1}{5}(T_{20}-T_{30})=185{,}980$, which is the immigrant population living through the year $9\frac{1}{2}$ to $10\frac{1}{2}$ years hence: and the deaths in this population in that year will be $\frac{1}{5}(l_{20}-l_{30})=1{,}275$. The total years lived above 20 by those who die in that year will be $\frac{1}{5}(T_{20}-T_{30}-10l_{30})=6{,}610$, giving 5·184 years on the average to each; so that their average age at death will be 25·184 years. The sum of their ages at death will therefore be $1{,}275\times25{\cdot}184=32{,}110$ years. The sum of the ages at death of all those who die in the year in question in the total population will thus be $6{,}082{,}031+32{,}110=6{,}114{,}141$; and the number of deaths being $127{,}283+1{,}275=128{,}558$, the average age at death will be 47·559 years.

It is therefore seen that the immigration has the effect at first of reducing the average age at death, and so, if left out of account, of apparently acting unfavourably on the longevity of the community. If we test in another way the effect of the immigration we shall meet with the opposite result. We have found the total number of deaths in the year in question to be 128,558; also the entire population is 6,268,011; therefore the number of deaths per thousand of population is 20·510. But in a normal population, undisturbed by immigration, we have seen the number of deaths per thousand to be 20·928. The apparent effect of the immigration is, therefore, to reduce the death rate; although it also reduces the average age at death.

If the immigration were to continue till the population should again reach a stationary state, the total population would be $T_0+\frac{1}{5}T_{20}=6{,}890{,}879$: the number of deaths annually would be $l_0+\frac{1}{5}l_{20}=146{,}495$; the sum of the ages at death would be $T_0+\frac{1}{5}l_{20}\left(20+\frac{T_{20}}{l_{20}}\right)=7{,}275{,}123$. Therefore the average age at death would be $\frac{7{,}275{,}123}{146{,}495}=49{\cdot}661$; and the number of deaths per thousand of the population would be $\frac{146{,}495}{6{,}890{,}879}\times1{,}000$ $=21{\cdot}259$.

The apparent effect of the immigration is therefore reversed: both the average age at death and the death rate are increased.

These figures teach us to be very cautious in dealing with statistics of populations. Unless we are acquainted with, and have made the correct allowances for, every fact, it is dangerous to venture on any deductions.

16. The foregoing investigation is purely theoretical. We have tacitly assumed that, although immigration takes place into the community, yet the number of births will remain constant—in fact, that none of the immigrants will have children; but such a supposition is evidently inadmissible. To make allowance for the change in the birth rate, due to immigration, would necessitate very elaborate calculations, for which, moreover, the data are insufficient.

17. We have seen that the total population aged x and upwards is T_x; also that $\mathring{e}_x = \frac{T_x}{l_x}$. Therefore $T_x = l_x \times \mathring{e}_x = l_x(\frac{1}{2} + e_x)$. Thus, if we have a table of either the complete or the curtate expectation of life, we can, without much difficulty, dispense with the column of T_x. Similarly, because $N'_x = l_x \times e_x = l_x(\mathring{e}_x - \frac{1}{2})$, we might dispense with the column of N'_x.

18. We shall now pass to questions relating to the population aged x and upwards.

We have seen that T_x represents the total future lifetime of the l_x persons who attain age x. In the same way, $\frac{1}{2}(T_x + T_{x+1})$ represents the total future lifetime of the $\frac{1}{2}(l_x + l_{x+1})$ persons who attain the age $x + \frac{1}{2}$. But for $\frac{1}{2}(l_x + l_{x+1})$ we may substitute L_x, because, as we have seen, $\frac{1}{2}(l_x + l_{x+1}) = L_x$, and, taken one with another, the population L_x living between ages x and $x+1$ is aged on the average $x + \frac{1}{2}$. Therefore $\frac{1}{2}(T_x + T_{x+1})$ represents the future lifetime of the population aged between x and $x+1$. In the same way we have the future lifetime of the population aged between $x+1$ and $x+2$, equal to $\frac{1}{2}(T_{x+1} + T_{x+2})$; and so on. Therefore the future lifetime of the population aged x and upwards will be $\frac{1}{2}(T_x + T_{x+1}) + \frac{1}{2}(T_{x+1} + T_{x+2}) + \&c. = \frac{1}{2}T_x + \Sigma T_{x+1}$. If, therefore, we construct a column, Y_x, where $Y_x = \frac{1}{2}T_x + \Sigma T_{x+1}$, it will give us, for any age x, the future lifetime of the population aged x and upwards. As that population is T_x, the complete expectation of life of the population aged x and upwards, is $\frac{Y_x}{T_x}$.

It will be observed that the column Y_x is formed from the column T_x exactly as the column T_x is formed from the column l_x. It has not been thought necessary to tabulate the column Y_x, because the practical uses that it serves are very limited.

19. The persons, L_x, aged between x and $x+1$, have on the average lived half a year each after age x, or $\frac{1}{2}L_x$ years in the aggregate. The persons, L_{x+1}, aged between $x+1$ and $x+2$, have in the same way lived $1\frac{1}{2}$ years each after age x, or $\frac{3}{2}L_{x+1}$ years in the aggregate; and so on. Therefore $\frac{1}{2}L_x+\frac{3}{2}L_{x+1}+\frac{5}{2}L_{x+2}+\&c.$ is the total number of years lived after age x by the population aged x and upwards.

$$\begin{aligned}\text{But}\qquad &\tfrac{1}{2}L_x+\tfrac{3}{2}L_{x+1}+\tfrac{5}{2}L_{x+2}+\&c.\\ &=\tfrac{1}{2}(L_x+L_{x+1}+L_{x+2}+\&c.)+L_{x+1}+2L_{x+2}+\&c.\\ &=\tfrac{1}{2}T_x+T_{x+1}+T_{x+2}+\&c.\\ &=Y_x.\end{aligned}$$

Therefore the total years lived after age x by the population aged x and upwards, is Y_x. But Y_x is also, as we have seen, the total future lifetime of the population aged x and upwards; and therefore the total future lifetime of that population is the same as its total past lifetime after age x. If we make $x=0$ we find that the total future lifetime of a stationary population is exactly equal to its total past lifetime: in other words, the population has completed exactly half its life.

20. Since Y_x is the sum of the ages above x of the population aged x and upwards, the average age of that population is $x+\frac{Y_x}{T_x}$; and, adding this to the future lifetime, we have $x+\frac{2Y_x}{T_x}$ as the average age at death of the population aged x and upwards.

21. The average age of the whole population is $\frac{Y_0}{T_0}$, and the average age at death of the same population will be $\frac{2Y_0}{T_0}$. This last is quite a different function from $\mathring{e}_0$, which, as we have seen, is the average age at death in the community. In the case of $\frac{2Y_0}{T_0}$, we have the average age at death of the *present members* of the community, whom we follow to the end of their lives, and keep apart; not admitting new comers. In the case of $\mathring{e}_0$ it is the community itself that we consider, and not the individual members of it, and it is always being recruited by the annual births.

Omit chap.

CHAPTER VI.

THE FORMULAS OF DE MOIVRE, GOMPERTZ, AND MAKEHAM, FOR THE LAW OF MORTALITY.

1. In the early days of the history of life contingencies, when the number of mortality tables in existence was not great, and when the monetary tables based upon them were very few, efforts were made to discover some mathematical law, by means of which monetary values could be easily calculated. The methods of computing tables then in vogue were clumsy, and the aids to calculation were not so numerous as now; and anything that would save time and labour was gladly welcomed. It was with this end in view that De Moivre studied the subject, and conceived his famous hypothesis of equal decrements to human life. He did not imagine that a decreasing arithmetical series correctly represented the number living according to any table of actual observations; but by carefully examining all the tables at his disposal, he came to the conclusion that to assume the values in the column l_x to diminish by constant first differences, would give monetary values sufficiently accurate for practical purposes.

2. De Moivre published his hypothesis in his "Treatise of Annuities on Lives", first edition, 1725. On page 4 of the Preface he says, "The "hypothesis consists in supposing that the number of lives existing at "any age is proportional to the number of years intercepted between the "age given and the extremity of old age"; and on page 8 of the treatise itself he remarks, "We may consider that whatever be that law which "is observed by nature in the perpetual decrements of human life, that "law must, conformably to all the other laws of nature, be such as to "proceed regularly, at least for some short intervals of time." Again, on page 19, he speaks of his hypothesis as assuming "a perpetual and equable decrement of probabilities of life, to begin from a term given."

3. From the first of the foregoing quotations it will be seen that the fundamental idea of De Moivre's hypothesis is, that the numbers in the column d_x are constant. The last quotation puts the same idea in a somewhat different form. By the "probabilities of life to begin from a term given", he means the probabilities that a person of a given age, say x, will live 1, 2, 3, &c., years. Now, if the number living at age x be n, and if the constant in the column d_x be t, the probabilities that (x) will live 1, 2, 3, &c., years are, $\frac{n-t}{n}$, $\frac{n-2t}{n}$, $\frac{n-3t}{n}$, &c., and their "perpetual equable decrement" is $\frac{t}{n}$.

4. Speaking further of his hypothesis, he said, "Another thing was "necessary to my calculation, which was to suppose the extent of life "confined to a certain period of time, which I supposed to be at 86", and in another place he mentioned that below age 12 his hypothesis would not hold. It thus appears that the celebrated hypothesis consists in supposing the column l_x to be a decreasing arithmetical progression; and that to make it fit the tables existing when it was first promulgated, it was confined to the period of life from age 12 to age 86; and that the number living at age 12 was assumed to be $86-12$, or 74, the numbers living at ages 13, 14, &c., being consequently 73, 72, &c.

5. A very cursory examination of even the mortality tables which De Moivre had before him, is sufficient to show that his hypothesis only in the most rough manner represents the law of human mortality. Since the days of De Moivre a great deal of attention has been paid to this interesting subject, and it has been investigated on more philosophical grounds; the end in view being rather to discover a law of nature, than to gain facilities for making numerical calculations; although, just as the search for the philosopher's stone, which has never been found, led to great and useful discoveries in chemistry, so the quest after the law of human mortality has resulted in formulas which, if not absolutely embodying that law, at least confer very great practical benefits.

6. It must not be thought that in assuming a law of mortality we assume an unalterable table of mortality. There may be a constant law, which, when applied under varying circumstances, will produce very diverse results. Thus, we have seen that De Moivre's law is, that the column l_x decreases in arithmetical progression; and by changing the values of the radix and the common difference, that law can be made to produce an infinite number of mortality tables. Similarly, we might assume that the column l_x consists of a geometrical progression, of which

we might vary the common ratio. In fact, by supposing a law of mortality, we do not assume certain rates of mortality; but we merely take for granted that a change of the circumstances in which a population may be placed will produce changes in the rates of mortality of such a nature, that these rates may always be displayed by mathematical expressions of the same form.

7. The late Benjamin Gompertz, in 1825, contributed a paper to the Royal Society, which marked an epoch in the history of our science. In the earlier sentences of that paper he discussed what would be the effects of supposing the numbers living at successive ages to be in geometrical progression; and he showed that, if such were the case, there would be no assignable limit to the duration of human life, and the values of annuities would be equal at all ages. In Article 4 he then went on to say, "It is possible that death may be the consequence of two generally "coexisting causes; the one, chance, without previous disposition to "death or deterioration; the other, a deterioration, or increased inability "to withstand destruction. If, for instance, there be a number of "diseases to which the young and old are equally liable, and likewise "which should be equally destructive whether the patient be young or "old, it is evident that the deaths among the young and old by such "diseases would be exactly in proportion of the number of young to the "old; provided those numbers were sufficiently great for chance to have "its play; and the intensity of mortality might then be said to be "constant; and were there no other diseases but such as those, life of "all ages would be of equal value, and the number of living and dying, "from a certain number living at a given earlier age, would decrease in "geometrical progression, as the age increased by equal intervals of time; "but if mankind be continually gaining seeds of indisposition, or in other "words, an increased liability to death (which appears not to be an "unlikely supposition with respect to a great part of life, though the "contrary appears to take place at certain periods), it would follow that "the number of living out of a given number of persons at a given age, "at equal successive increments of age, would decrease in a greater ratio "than the geometrical progression, and then the chances against the "knowledge of any one having arrived to certain defined terms of old "age might increase in a much faster progression, notwithstanding there "might still be no limit to the age of man."

8. Pursuing the subject further, Gompertz, in Art. 5 of his paper, proceeded to investigate what would be the effect of supposing "the

" average exhaustion of a man's power to avoid death to be such that at " the end of equal infinitely small intervals of time he lost equal portions " of his remaining power to oppose destruction which he had at the " commencement of these intervals."

[9] On the supposition that " chance without previous disposition to death or deterioration" causes all the deaths, the "intensity of mortality", or, as we say now, the "force of mortality", is constant; whereas, if equal portions of the power to oppose destruction be lost at the end of equal infinitely small intervals of time, or, in other words, if the probability of dying in the successive equal infinitely small intervals of time increase in geometrical progression; then the force of mortality will also increase in geometrical progression. The probability of dying in the interval t is $-\frac{l_{x+t}-l_x}{l_x}$, and when t is infinitely small this becomes $\frac{-dl_x}{l_x}$, and by formula 19 of Chap. ii this is equal to $\mu_x dx$, where dx is the infinitely small interval of time. If, therefore, $\frac{-dl_x}{l_x}$, the probability of death, increase in geometrical progression, then μ_x must also increase in geometrical progression, dx being by hypothesis constant. This idea is the basis of Gompertz's hypothesis as to the law of human mortality. In simple language it is the assumption, that the force of mortality increases in geometrical progression.

[10] From this datum we can easily find the form of l_x viewed as a function of x. We may write

$$\mu_x = \mathrm{B}c^x \quad . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

where B and c are constants determined for the mortality table in question, and quite independent of the age of the life. By Chap. ii, formula 24,

$$\mu_x = -\frac{d\log_e l_x}{dx}.$$

Whence

$$d\log_e l_x = -\mu_x dx$$

$$= -\mathrm{B}c^x dx$$

and

$$\log_e l_x = -\int \mathrm{B}c^x dx$$

$$= -\frac{\mathrm{B}c^x}{\log_e c} + \log_e k$$

where $\log_e k$ is the constant introduced by integration. Writing now $-\frac{B}{\log_e c} = \log_e g$, we have

$$\log_e l_x = \log_e k + c^x \log_e g$$

and

$$l_x = k(g)^{c^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where k, g and c are constants.

In equation 1, writing B in terms of the constants,

$$\mu_x = -(\log_e g \log_e c)c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

[11] In equation 2 we have Gompertz's formula expressive of the law of human mortality. On it he himself remarked, "this equation between "the number living and the age becomes deserving of attention not in "consequence of its hypothetical deduction, which in fact is congruous "with many natural effects, as for instance the exhaustion of the "receiver of an air pump by strokes repeated at equal intervals of time, "but it is deserving of attention because it appears corroborated during "a long portion of life by experience; as I derived the same equation "from various published tables of mortality during a long period of "man's life, which experience therefore proves that the hypothesis "approximates to the law of mortality during the same portions of life; "and in fact the hypothesis itself was derived from an analysis of the "experience here alluded to."

[12] In the examples which accompanied the first publication of the formula, Gompertz restricted it to the period from age 10 or 15, to age 55 or 60. It is not applicable to the period of childhood, and in order practically to use it in the construction of mortality tables, the constants must be changed at a point between ages 50 and 60.

[13] In a paper which appeared in the *Journal of the Institute of Actuaries* in January 1860, Makeham discussed Gompertz's formula for the law of human mortality, and introduced a modification which has the effect of making the formula apply, with wonderful closeness, to many mortality tables from about age 20 to the utmost limits of life. By Gompertz's formula, the logarithm of the probability that a life aged x will live t years, decreases geometrically with the increase in x. Thus

$$\log {}_t p_x = \log k + c^{x+t} \log g - \log k - c^x \log g$$

$$= c^x(c^t - 1)\log g$$

where the coefficient of c^x is negative and constant. Examining various mortality tables for the period of life from age 20 to age 80, Makeham showed that the logarithm of the probability of living t years can be more accurately represented if we add to the geometrical progression above given a constant negative term, the magnitude of which is determined by the length of the interval t. Writing for that constant term, $t \log s$, we have

$$\log {}_tp_x = t \log s + c^x(c^t - 1) \log g,$$

and passing backwards to the mortality table this becomes

$$\begin{aligned}\log {}_tp_x = \log k + (x+t) \log s + c^{x+t} \log g \\ - \log k - x \log s - c^x \log g.\end{aligned}$$

The portion of this expression with negative signs represents $\log l_x$, and from it we derive the equation

$$l_x = ks^x (g)^{c^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

[14] The same expression for l_x can be obtained in another way, and one which shows that, had Gompertz himself pushed his investigation a little further, he would have arrived at Makeham's formula. In the first quotation given above from his writings, he says that it is possible that death may be the consequence of two generally coexisting causes; the one chance without any previous disposition to death or deterioration; the other a deterioration or increased inability to withstand destruction; and that as regards the first of these causes, the intensity of mortality might be said to be constant; while as regards the second, it would be an increasing geometrical progression. In deducing his formula for l_x he considered only the second of these causes of death, and assumed the force of mortality to be a geometrical progression; but he evidently had in his mind the possibility of the two causes being combined, and, as a consequence, the force of mortality being composed of two parts, the one a constant, and the other a geometrical progression. On this hypothesis we may therefore write

$$\mu_x = \mathrm{A} + \mathrm{B}c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

from which equation we must discover the form taken by l_x.

[15] Following a similar argument to that adopted in Art. 10, we have

$$d \log_e l_x = -(\mathrm{A} + \mathrm{B}c^x)\, dx$$

and
$$\begin{aligned}\log_e l_x &= -\int (\mathrm{A}+\mathrm{B}c^x)\,dx \\ &= -\int \mathrm{A}\,dx - \int \mathrm{B}c^x\,dx \\ &= -\mathrm{A}x - \frac{\mathrm{B}c^x}{\log_e c} + \log_e k.\end{aligned}$$

If now we write $-\mathrm{A}=\log_e s$ and, as before, $-\dfrac{\mathrm{B}}{\log_e c}=\log_e g$ we have

$$\log_e l_x = \log_e k + x\log_e s + c^x \log_e g$$

and
$$l_x = ks^x(g)^{c^x}, \text{ agreeing with equation 4.}$$

[16] If in equation 5 we write for A and B respectively their values in terms of the constants, s, g, and c, we have

$$\mu_x = -\log_e s - (\log_e g \log_e c)c^x \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Here μ_x is positive, because, from the nature of the functions, both $\log_e s$ and $\log_e g$ are negative, while $\log_e c$ and c^x are positive.

[17] If in the expression for l_x we make $x=0$ we have

$$l_0 = kg$$

from which we see that kg is the radix of the mortality table. In the expression for l_x the constant g appears with an exponent which depends upon x, but the constant k appears simply as a coefficient. It therefore follows that the constant k is not of importance except as fixing the radix of the table. If we wish the actual values of l_x to approximate to those of a given mortality table, we must determine k with that end in view; but, seeing that only the relative values of l_x are of real importance, we may in practice give to k any value we please.

[18] From any four equidistant values of l_x we can determine the values of the four constants k, s, g, and c; or from the values of ${}_tp_x$, ${}_tp_{x+t}$, and ${}_tp_{x+2t}$, we can determine the values of the three constants s, g, and c. These two conditions are really the same, because ${}_tp_x = \dfrac{l_{x+t}}{l_x}$, ${}_tp_{x+t} = \dfrac{l_{x+2t}}{l_{x+t}}$, and ${}_tp_{x+2t} = \dfrac{l_{x+3t}}{l_{x+2t}}$; from which we see that if we have as data the four values, l_x, l_{x+t}, l_{x+2t} and l_{x+3t}, we have also the three values, ${}_tp_x$, ${}_tp_{x+t}$ and ${}_tp_{x+2t}$. The only difference between them is, that from the four equidistant values of l_x we can determine the absolute values of the other quantities in the column l; whereas, from the three values of ${}_tp_x$ we can determine only their relative values.

[19] By the following process the constants may be determined:—

$$
\begin{aligned}
&(\alpha) \quad \log l_x &&= \log k + x \log s + c^x \log g \\
& \quad\quad \log l_{x+t} &&= \log k + (x+t) \log s + c^{x+t} \log g \\
& \quad\quad \log l_{x+2t} &&= \log k + (x+2t) \log s + c^{x+2t} \log g \\
& \quad\quad \log l_{x+3t} &&= \log k + (x+3t) \log s + c^{x+3t} \log g
\end{aligned}
$$

Differencing now both sides of the above equations twice, we have

$$
\begin{aligned}
&(\beta) \quad \Delta \log l_x &&= t \log s + c^x (c^t - 1) \log g \\
& \quad\quad \Delta \log l_{x+t} &&= t \log s + c^{x+t} (c^t - 1) \log g \\
& \quad\quad \Delta \log l_{x+2t} &&= t \log s + c^{x+2t} (c^t - 1) \log g
\end{aligned}
$$

and

$$
\begin{aligned}
&(\gamma) \quad \Delta^2 \log l_x &&= c^x (c^t - 1)^2 \log g \\
& \quad\quad \Delta^2 \log l_{x+t} &&= c^{x+t} (c^t - 1)^2 \log g
\end{aligned}
$$

Dividing now each member of the second of these last two equations by the corresponding member of the first, and taking the logarithm of the result we have

$$(\delta) \quad \log \{\Delta^2 \log l_{x+t}\} - \log \{\Delta^2 \log l_x\} = t \log c,$$

whence we can at once find $\log c$, because the left-hand member of equation (δ) is given in the original data. Inserting now in equation (γ) the value of c found by equation (δ), we arrive at the value of $\log g$; and proceeding similarly in succession to the equations (β) and (α), we obtain the values of the other two constants.

[20] To take a numerical example from the mortality table given in Table I, let the four ages be 30, 45, 60, and 75, so that $t = 15$

Age	$\log l$	Δ	Δ^2	$\log(-\Delta^2)$
30	4·9527198			
		−·0610820		
45	4·8916378		−·0608684	$\bar{2}$·7843918
		−·1219504		
60	4·7696874		−·2394631	$\bar{1}$·3792386
		−·3614135		
75	4·4082739			

15) ·5948468

$\log c =$ ·0396565

$c^t - 1 =$ 2·934113

$\log(c^t - 1) =$ ·4674768

By equation (γ)

$$-\log g = \frac{-\Delta^2 \log l_{30}}{c^{30}(c^t-1)^2}$$

$$\log(-\log g) = \log(-\Delta^2 \log l_{30}) - 30\log c - 2\log(c^t-1)$$

$$\begin{array}{rl}
\log(-\Delta^2 \log l_{30}) = & \bar{2}\cdot7843918 \\
30\log c = 1\cdot1896936 & \\
2\log(c^t-1) = 0\cdot9349536 & \\
 & \underline{2\cdot1246472} \\
\log(-\log g) = & \underline{\bar{4}\cdot6597446} \\
-\log g = & \underline{\cdot0004568}
\end{array}$$

By equation (β)

$$\log s = \frac{1}{t}\{\Delta \log l_{30} + c^{30}(c^t-1)(-\log g)\}$$

$$\begin{array}{rl}
\log(-\log g) = & \bar{4}\cdot6597446 \\
\log(c^t-1) = & \cdot4674768 \\
30\log c = & \underline{1\cdot1896936} \\
 & \underline{\bar{2}\cdot3169150} \\
c^{30}(c^t-1)(-\log g) = & \cdot0207451 \\
\Delta \log l_{30} = - & \underline{\cdot0610820} \\
15\log s = - & \underline{\cdot0403369} \\
\log s = - & \underline{\cdot0026891}
\end{array}$$

By equation (α)

$$\log k = \log l_{30} - 30\log s + c^{30}(-\log g)$$

$$\begin{array}{rl}
\log(-\log g) = & \bar{4}\cdot6597446 \\
30\log c = & \underline{1\cdot1896936} \\
 & \underline{\bar{3}\cdot8494382} \\
c^{30}(-\log g) = & \cdot0070703 \\
-30\log s = & \cdot0806738 \\
\log l_{30} = & \underline{4\cdot9527198} \\
\log k = & \underline{5\cdot0404639}
\end{array}$$

In order to verify the work we may, from the constants now found, form the value of $\log l_{75}$, and, if this agrees with the value in the original data, the calculations are proved to be correct.

[21] The principles of the process above illustrated are simple, and if the computations be made with care, no difficulties will be met with. It may however be well to remark that precautions must be taken with regard to the algebraical signs. From their nature, $\log s$ and $\log g$ are negative; and they therefore change the signs of all the expressions into which they enter as factors. When this point is remembered the work is easy.

[22] If a mortality table already follow Makeham's law, then, no matter which ages may be selected as a basis of calculation, the resulting constants will be the same. Such is the case from age 28 upwards with the mortality table at the end of this volume. It has been constructed in a way presently to be explained, from the experience of twenty offices collected by the Institute of Actuaries; and Makeham's formula was used in the process. In its formation, logarithms to seven figures were used, but the radix was taken as 100,000 at age 10, and the values of l_x were cut down to the nearest corresponding integer. Therefore at no age above 10 are there more than five figures in l_x: at age 83 the figures are reduced to four: at age 91 to three: at age 96 to two: at age 99 to one: and at age 102 they vanish altogether. Had the numbers been taken out to seven places, corresponding to the seven figure logarithms used in their calculation, then the constants would have been exactly reproduced from the four values of l_x used in the example. But a number cut down to five figures will not give a logarithm correct to seven figures; and hence there are small differences between the original constants and those now found. The following is a comparison of the numbers we have used and the original data:—

Age.	l_x	$\log l_x$ used in Example.	Correct $\log l_x$
30	89685	4·9527198	4·9527227
45	77918	4·8916378	4·8916403
60	58842	4·7696874	4·7696892
75	25602	4·4082739	4·4082745

As a consequence of the differences in the sixth and seventh places in the values of $\log l_x$, there are the following slight differences in the constants.

Constant.	Value found in Example.	Correct Value.
$\log c$	+0·0396565	+0·0396569
$\log g$	−0·0004568	−0·0004568
$\log s$	−0·0026891	−0·0026893
$\log k$	+5·0404639	+5·0404723

[23] When Makeham's formula is to be applied to a mortality table which only approximately follows the law, then the values of the constants deduced will differ with the different ages and intervals at which the four values of l_x are taken. Thus, with the Carlisle Mortality which Milne prepared from the mortality of Carlisle, if four sets of ages be selected, namely 15, 35, 55, and 75; 20, 40, 60, and 80; 25, 45, 65, and 85; and 30, 50, 70, and 90; the resulting constants will be

	First Set.	Second Set.	Third Set.	Fourth Set.
$\log c$	·0364852	·0411317	·0404469	·0397194
$\log g$	− ·0007393	− ·0003083	− ·0003272	− ·0003654
$\log s$	− ·0029073	− ·0034071	− ·0038187	− ·0039126
$\log k$	4·8452254	4·8549296	4·8681803	4·8743597

[24] It may be desirable under such circumstances to combine the various sets of constants in such a way as to get an average, and so form a mortality table running more closely to the original than a table formed from any single set of constants would do. If l_1, l_2, l_3, and l_4 denote the values of l_x obtained from the first, second, third, and fourth sets of constants respectively; l_x representing the value formed by combining the constants, we may write

$$\log l_x = \tfrac{1}{4}(\log l_1 + \log l_2 + \log l_3 + \log l_4).$$

Adding, similarly, suffixes to the constants to distinguish the sets, and using the letters without suffixes for the final set; we have

$$\begin{aligned}\log l_x &= \tfrac{1}{4}(\log k_1 + \log k_2 + \log k_3 + \log k_4)\\ &\quad + \tfrac{1}{4}x(\log s_1 + \log s_2 + \log s_3 + \log s_4)\\ &\quad + \tfrac{1}{4}(c_1{}^x \log g_1 + c_2{}^x \log g_2 + c_3{}^x \log g_3 + c_4{}^x \log g_4)\\ &= \log k + x \log s\\ &\quad + \tfrac{1}{4}(c_1{}^x \log g_1 + c_2{}^x \log g_2 + c_3{}^x \log g_3 + c_4{}^x \log g_4)\end{aligned}$$

when we write $\log k = \tfrac{1}{4}(\log k_1 + \log k_2 + \log k_3 + \log k_4)$

and $\log s = \tfrac{1}{4}(\log s_1 + \log s_2 + \log s_3 + \log s_4)$

Forming a similar equation for any other age, $x+t$, we have now two equations from which the two unknown quantities $\log g$ and $\log c$ may be determined. In the case of the four sets of constants above mentioned for the Carlisle Table, the final constants are,

$$\begin{aligned}\log c &= \cdot0392061\\ \log g &= -\cdot0004262\\ \log s &= -\cdot0035114\\ \log k &= 4\cdot8606738\end{aligned}$$

[25] Other methods of forming average constants have been proposed. Woolhouse (*J.I.A.*, xv, 404) in applying Makeham's formula to the Institute of Actuaries' Table, Male and Female, prepared by each set of constants $\log l_x$, $\Delta \log l_x$ for interval t, and $\Delta^2 \log l_x$; and then for $\log c$ taking the mean of $\log c_1$, $\log c_2$, &c., and for $\log l_x$, $\Delta \log l_x$, and $\Delta^2 \log l_x$, the mean of the values of each of these functions formed from the different sets of constants, he calculated from these data $\log g$, $\log s$, and $\log k$.

[26] Prof. C. F. McCay (*J.I.A.*, xxii, 27) from the three values $\log {}_tp_x$, $\log {}_tp_{x+t}$, and $\log {}_tp_{x+2t}$ found the value of $\log c$: then, using that value, and dividing the original data into two portions, $\log {}_{\frac{3t}{2}}p_x$, and $\log {}_{\frac{3t}{2}}p_{x+\frac{3t}{2}}$, he calculated $\log g$ and $\log s$.

[27] G. King and G. F. Hardy (*J.I.A.*, xxii, 200) propounded a method of using all the values of $\log l_x$ from ages x to $x+4t-1$ inclusive, instead of only the four equidistant values; and this is the method that has been followed in preparing the mortality table at the end of this volume, as will be explained immediately.

[28] The chief advantages of Makeham's formula are that, without adjustment or change of constants, it represents with great faithfulness the law of mortality as shown by many of the most important mortality tables, from about age 20 or 25 to the end of life, and that at the same time it possesses a property by means of which joint life probabilities may be easily calculated.

We have

$$\begin{aligned}\log {}_tp_x &= \log l_{x+t} - \log l_x\\ &= \log k + (x+t)\log s + c^{x+t}\log g\\ &\quad - \log k - x\log s - c^x \log g\\ &= t\log s + c^x(c^t-1)\log g.\end{aligned}$$

Similarly, $\log {}_tp_y = t\log s + c^y(c^t-1)\log g.$

Therefore, $\log {}_tp_{xy} = 2t\log s + (c^x+c^y)(c^t-1)\log g.$

If, now, we find w, so that $c^w=\frac{1}{2}(c^x+c^y)$, then

$$\begin{aligned}\log {}_tp_{xy}&=2t\log s+2c^w(c^t-1)\log g\\&=2\log {}_tp_w\\&=\log {}_tp_{ww}.\end{aligned}$$

Therefore by Makeham's formula we can substitute two lives of equal age for any two lives (x) and (y); and the same principles apply whatever may be the number of lives. We shall return to this subject and discuss it more fully in Chap. xii.

[29] We proceed now to give full particulars of the preparation of the mortality table at the end of this volume. The table is based on the experience of the Healthy Male Lives assured with 20 British Life Offices, which was collected by the Institute of Actuaries and published in 1869. In the volume of *Mortality Experience* the manner in which the experience was collected and subsequently dealt with is explained in detail. Here it will be sufficient to give only the figures actually used in the preparation of the mortality table. These are contained in the following table:—

Age.	log l_x	Age.	log l_x	Age.	log l_x
10	4·00000	39	3·91967	68	3·63099
11	3·99655	40	·91503	69	·60628
12	·99655	41	·91072	70	·57894
13	·99478	42	·90615	71	·55390
14	·99327	43	·90147	72	·52602
15	·99149	44	·89685	73	·48994
16	·99053	45	·89169	74	·45432
17	·99053	46	·88629	75	·40598
18	·98869	47	·88082	76	·36295
19	·98604	48	·87463	77	·31403
20	·98297	49	·86847	78	·26403
21	·98046	50	·86179	79	·20713
22	·97741	51	·85456	80	·14362
23	·97469	52	·84693	81	·07776
24	·97132	53	·83947	82	·00212
25	·96833	54	·83194	83	2·92001
26	·96609	55	·82363	84	·81934
27	·96307	56	·81354	85	·73319
28	·96025	57	·80339	86	·62745
29	·95684	58	·79289	87	·52100
30	·95363	59	·78184	88	·41426
31	·95003	60	·77069	89	·26908
32	·94682	61	·75695	90	·17687
33	·94319	62	·74259	91	·06342
34	·93957	63	·72729	92	1·90069
35	·93578	64	·71075	93	·64105
36	·93219	65	·69294	94	·16392
37	·92833	66	·67359	95	·16392
38	·92416	67	·65281	96	·01779

[30] In order to make use of as large a portion as possible of these data we must find in terms of the constants the sum of a series of values of $\log l_x$. Since $\log l_x = \log k + x \log s + c^x \log g$, we have, making x successively x, $x+1$, $x+2$, &c., $x+t-1$, and taking the sum,

$$\begin{aligned}
\Sigma_x^{x+t-1} \log l_x &= t \log k + (x + \overline{x+1} + \text{\&c.} + \overline{x+t-1}) \log s \\
&\quad + (c^x + c^{x+1} + \ldots + c^{x+t-1}) \log g \\
&= t \log k + \frac{t}{2}(2x+t-1) \log s \\
&\quad + c^x \frac{(c^t-1)}{c-1} \log g
\end{aligned}$$

[31] By means of this relation we can now find the constants. Proceeding as in Art. 19, but substituting the sums of t values of $\log l_x$ for the single values, we have

$$(\alpha) \quad \Sigma_x^{x+t-1} \log l_x = t \log k + \frac{t}{2}(2x+t-1) \log s + \frac{c^x(c^t-1)}{c-1} \log g$$

$$\Sigma_{x+t}^{x+2t-1} \log l_x = t \log k + \frac{t}{2}(2x+3t-1) \log s + \frac{c^{x+t}(c^t-1)}{c-1} \log g$$

$$\Sigma_{x+2t}^{x+3t-1} \log l_x = t \log k + \frac{t}{2}(2x+5t-1) \log s + \frac{c^{x+2t}(c^t-1)}{c-1} \log g$$

$$\Sigma_{x+3t}^{x+4t-1} \log l_x = t \log k + \frac{t}{2}(2x+7t-1) \log s + \frac{c^{x+3t}(c^t-1)}{c-1} \log g.$$

Differencing now both sides of the above equations twice, we have

$$(\beta) \quad \Delta\Sigma_x^{x+t-1} \log l_x = t^2 \log s + \frac{c^x(c^t-1)^2}{c-1} \log g$$

$$\Delta\Sigma_{x+t}^{x+2t-1} \log l_x = t^2 \log s + \frac{c^{x+t}(c^t-1)^2}{c-1} \log g$$

$$\Delta\Sigma_{x+2t}^{x+3t-1} \log l_x = t^2 \log s + \frac{c^{x+2t}(c^t-1)^2}{c-1} \log g$$

and

$$(\gamma) \quad \Delta^2\Sigma_x^{x+t-1} \log l_x = \frac{c^x(c^t-1)^3}{c-1} \log g$$

$$\Delta^2\Sigma_{x+t}^{x+2t-1} \log l_x = \frac{c^{x+t}(c^t-1)^3}{c-1} \log g.$$

Dividing now each member of the second of these last two equations by the corresponding member of the first, and taking the logarithm of the results we have

$$(\delta)\quad \log\{\Delta^2\Sigma_{x+t}^{x+2t-1}\log l_x\}-\log\{\Delta^2\Sigma_x^{x+t-1}\log l_x\}=t\log c,$$

whence we can at once find $\log c$; and from the value thus found, working backwards by means of equations (γ), (β), and (α), we can find the other constants.

[32] In applying this method, care must be taken to avoid the two extremes of the table. Seeing that usually there is not much confidence to be placed in a table above age 90, it will as a rule be desirable to omit the data above that age; and as the law of mortality seldom appears to remain uniform for many years below age 20, the younger ages must be excluded.

[33] For the mortality table at the end of the volume various groupings were tried, and the best was found to be, four groups of eighteen years of life each, including the experience from age 17 to age 88. Under this arrangement we have

$$\Sigma_{17}^{34}\log l_x=18\log k+459\log s+\frac{c^{17}(c^{18}-1)}{c-1}\log g$$

$$\Delta\Sigma_{17}^{34}\log l_x=324\log s+\frac{c^{17}(c^{18}-1)^2}{c-1}\log g$$

$$\Delta^2\Sigma_{17}^{34}\log l_x=\frac{c^{17}(c^{18}-1)^3}{c-1}\log g.$$

Taking now actual numbers

Ages.	$\Sigma\log l_x$	Δ	Δ^2	$\log(-\Delta^2)$
17 to 34	71·39993	− 1·26440		
35 " 52	70·13553	− 2·90501	−1·64061	·2150054
53 " 70	67·23052	−11·39347	−8·48846	·9288289
71 " 88	55·83705			

$$18\log c=\cdot 7138235$$

$$\log c=\cdot 03965686$$

[34] In the calculations which follow, the quantity $\log(c-1)$ frequently appears as one of the terms, and if we work with seven-figure

logarithms we have $c=1{\cdot}095612$: whence $(c-1)={\cdot}095612$. There are thus only five significant figures in $(c-1)$; and we cannot get the logarithm true to seven figures. In order to secure greater accuracy we must find c to at least eight places of decimals, and this may be done by means of the short eight-figure table given at the end of Babbage's logarithms.

$$\begin{array}{ll}
\text{We have as above} & \log c = {\cdot}03965686 \\
\text{By Babbage} & \log 1{\cdot}03 = \underline{{\cdot}01283723} \\
 & \qquad\qquad \underline{\underline{{\cdot}02681963}}
\end{array}$$

Whence $(c \div 1{\cdot}03) = 1{\cdot}06370115$ and $c = \underline{\underline{1{\cdot}0956122}}$

This is sufficient for our purpose, but if we wished still greater accuracy we should find, by means of Dodson's Antilogarithmic Canon, $c=1{\cdot}095612204$.

[35] Proceeding now with the calculation of the constants, from equation (γ) we have

$$\log(-\log g) = \log(-\Delta^2) - 17\log c - 3\log(c^{18}-1) + \log(c-1)$$

$$\begin{aligned}
\log(-\Delta^2) &= {\cdot}2150054 \\
-17\log c &= \bar{1}{\cdot}3258334 \\
-3\log(c^{18}-1) &= \bar{2}{\cdot}1383536 \\
\log(c-1) &= \underline{\bar{2}{\cdot}9805133} \\
\log(-\log g) &= \underline{\underline{\bar{4}{\cdot}6597057}}
\end{aligned}
\qquad\qquad \log g = -\underline{\underline{{\cdot}0004568}}$$

From equation (β)

$$\log s = \frac{1}{324}\left\{\Delta + \frac{c^{17}(c^{18}-1)^2}{c-1}(-\log g)\right\}$$

$$\begin{aligned}
17\log c &= {\cdot}6741666 \\
2\log(c^{18}-1) &= 1{\cdot}2410976 \\
-\log(c-1) &= 1{\cdot}0194867 \\
\log(-\log g) &= \underline{\bar{4}{\cdot}6597057} \\
 & \quad \underline{\underline{\bar{1}{\cdot}5944566}}
\end{aligned}
\qquad
\begin{aligned}
\frac{c^{17}(c^{18}-1)^2}{c-1}(-\log g) &= {\cdot}3930579 \\
\Delta &= \underline{-1{\cdot}2644000} \\
324\log s &= -{\cdot}8713421 \\
\log s &= -\underline{\underline{{\cdot}002689327}}
\end{aligned}$$

From equation (a)

$$\log k=\frac{1}{18}\left\{\Sigma_{17}^{34}\log l_x+459(-\log s)+\frac{c^{17}(c^{18}-1)}{c-1}(-\log g)\right\}$$

$$\begin{array}{rl}
17\log c= & \cdot 6741666\\
\log(c^{18}-1)= & \cdot 6205488\\
-\log(c-1)= & 1\cdot 0194867\\
\log(-\log g)= & \bar{4}\cdot 6597057\\
\hline
 & \bar{2}\cdot 9739078\\
\hline
\end{array}$$

$$\begin{array}{rr}
\Sigma_{17}^{34}\log l_x= & 71\cdot 3999300\\
459(-\log s)= & 1\cdot 2344011\\
\dfrac{c^{17}(c^{18}-1)}{c-1}(-\log g)= & \cdot 0941690\\
\hline
18\log k= & 72\cdot 7285001\\
\hline
\log k= & 4\cdot 0404723\\
\hline
\end{array}$$

[36] In order to verify the work, we may now find by means of the constants the sum of $\log l_x$ from ages 71 to 88 inclusive. Thus,

$$\Sigma_{71}^{88}\log l_x=18\log k+1431\log s+\frac{c^{71}(c^{18}-1)}{c-1}\log g$$

$$\begin{array}{rl}
71\log c= & 2\cdot 8156371\\
\log(c^{18}-1)= & \cdot 6205488\\
-\log(c-1)= & 1\cdot 0194867\\
\log(-\log g)= & \bar{4}\cdot 6597057\\
\hline
 & 1\cdot 1153783\\
\hline
\end{array}$$

$$\begin{array}{rr}
\dfrac{c^{71}(c^{18}-1)}{c-1}\log g= & -13\cdot 04302\\
1431\log s= & -\ \ 3\cdot 84843\\
\hline
 & -16\cdot 89145\\
18\log k= & 72\cdot 72850\\
\hline
\Sigma_{71}^{88}\log l_x= & 55\cdot 83705\\
\hline
\end{array}$$

The value thus found agrees to the last place of decimals with the value in the original data, and this proves that the constants have been correctly calculated.

[37] For reference we now collect the constants together.

$$\begin{aligned}
\log c&=+0\cdot 03965686\\
\log g&=-0\cdot 0004568\\
\log s&=-0\cdot 002689327\\
\log k&=+4\cdot 0404723
\end{aligned}$$

We have entered $\log s$ to nine places of decimals, because sometimes it has to be multiplied by very large coefficients.

[38] In order roughly to test the success of the graduation of a table by a given set of constants, it is useful to form the values of l_x at intervals of, say, five years, and this is easily done by a continued process. We have already seen (Art. 13) that

$$(a)\quad \log {}_tp_x=t\log s+c^x(c^t-1)\log g$$

whence $-\Delta \log {}_tp_x = c^x(c^t-1)^2(-\log g)$

(β) $\log(-\Delta \log {}_tp_x) = x\log c + 2\log(c^t-1) + \log(-\log g)$

and (γ) $\Delta \log(-\Delta \log {}_tp_x) = t \log c.$

We start with the initial value of $\log(-\Delta \log {}_tp_x)$ which we calculate by means of equation (β); then adding continuously the constant difference, $t\log c$, given in equation (γ), we form the column of the function. In the adjacent column we place the natural numbers which are the values of $-\Delta\log {}_tp_x$; and these we deduct successively from the initial value in the next column, that value being $\log {}_tp_x$, formed by means of equation (α). Having thus constructed a table of $\log {}_tp_x$, we can, by constant addition to the initial value of $\log l_x$, form a table of $\log l_x$ for intervals of t years.

[39] Commencing at age 10, and taking intervals of five years, the initial values are, $\log l_{10}$, $\log {}_5p_{10}$, and $\log(-\Delta \log {}_5p_{10})$; and the following are the calculations to form them.

$$\log l_{10} = \log k + 10\log s + c^{10}\log g$$

$$\begin{array}{rr}
10\log c = & \cdot3965686 \\
\log(-\log g) = & \overline{4}\cdot6597057 \\
\hline
 & \overline{3}\cdot0562743 \\
\hline
c^{10}\log g = - & \cdot0011383 \\
10\log s = - & \cdot0268933 \\
\hline
- & \cdot0280316 \\
\hline
\end{array}$$

$$\begin{array}{rr}
\log k = & 4\cdot0404723 \\
 & \cdot0280316 \\
\hline
\log l_{10} = & 4\cdot0124407 \\
\hline
\end{array}$$

$$\log {}_5p_{10} = 5\log s + c^{10}(c^5-1)\log g$$

$$\begin{array}{rr}
10\log c = & \cdot3965686 \\
\log(c^5-1) = & \overline{1}\cdot7624115 \\
\log(-\log g) = & \overline{4}\cdot6597057 \\
\hline
 & \overline{4}\cdot8186858 \\
\hline
\end{array}$$

$$\begin{array}{rr}
5\log s = - & \cdot0134466 \\
c^{10}(c^5-1)\log g = - & \cdot0006587 \\
\hline
\log {}_5p_{10} = - & \cdot0141053 \\
\hline
= & \overline{1}\cdot9858947 \\
\hline
\end{array}$$

$$\log(-\Delta\log {}_5p_{10}) = 10\log c + 2\log(c^5-1) + \log(-\log g)$$

$$\begin{array}{rr}
10\log c = & \cdot3965686 \\
2\log(c^5-1) = & \overline{1}\cdot5248230 \\
\log(-\log g) = & \overline{4}\cdot6597057 \\
\hline
\log(-\Delta\log {}_5p_{10}) = & \overline{4}\cdot5810973 \\
\hline
\end{array}$$

The following are the remaining operations in tabular form, $\log l_x$ having been increased by unity:—

Age.	$\log(-\Delta\log{}_5p_x)$	$-\Delta\log{}_5p_x$	$\log{}_5p_x$	$\log l_x$	l_x Graduated.	l_x Original.	Age.
10	$\bar{4}$·5810973	·0003812	$\bar{1}$·9858947	5·0124407	102906	100000	10
15	·7793816	·0006017	·9855135	4·9983354	99619	98060	15
20	·9776659	·0009499	·9849118	·9838489	96350	96155	20
25	$\bar{3}$·1759502	·0014995	·9839619	·9687607	93059	92967	25
30	·3742345	·0023672	·9824624	·9527226	89685	89873	30
35	·5725188	·0037370	·9800952	·9351850	86137	86254	35
40	·7708031	·0058993	·9763582	·9152802	82277	82230	40
45	·9690874	·0093130	·9704589	·8916384	77918	77927	45
50	$\bar{2}$·1673717	·0147018	·9611459	·8620973	72795	72743	50
55	·3656560	·0232090	·9464441	·8232432	66564	66624	55
60	·5639403	·0366387	·9232351	·7696873	58842	58978	60
65	·7622246	·0578395	·8865964	·6929224	49308	49311	65
70	·9605089	·0913080	·8287569	·5795188	37977	37926	70
75	$\bar{1}$·1587932	·1441429	·7374489	·4082757	25602	25467	75
80	·3570775	·2275504	·5933060	·1457246	13987	13919	80
85	·5553618	·3592211	·3657556	3·7390306	5483	5410	85
90	...	...	·0065345	·1047862	1273	1503	90
95	...	...	...	2·1113207	129	146	95

[40] In the foregoing process any error there may be in the quantities forming $\Delta\log{}_5p_x$ accumulates by the continued addition by which $\log{}_5p_x$ is formed, and the accumulation is continued in increasing ratio by the additions of $\log{}_5p_x$ to form $\log l_x$. It is therefore necessary to work to more places of decimals than are to be retained, and if very minute accuracy be required care must be taken to work out some of the logarithms and antilogarithms to more than seven figures. For example working to seven figures only, $c^5=1{\cdot}578644$ and $(c^5-1)={\cdot}578644$. Here we have unavoidably only six figures although we began with seven, and therefore we cannot get $\log(c^5-1)$ true to seven figures. But this is one of the quantities involved, and the error in it in the seventh place affects the fifth place at the higher ages of the table. In the small illustrative table calculated above this is not of consequence, but absolute correctness should be aimed at in computing the final mortality table.

[41] Proceeding now to construct the mortality table, we shall find that, for the purpose, the most convenient function to begin with is $\log(-\Delta\log p_x)$. Thus

$$\log l_x=\log k-x(-\log s)-c^x(-\log g)$$

$$\Delta\log l_x=\log p_x=-(-\log s)-c^x(c-1)(-\log g)$$

$$\Delta\log p_x=-c^x(c-1)^2(-\log g)$$

$$\log(-\Delta\log p_x)=x\log c+2\log(c-1)+\log(-\log g)$$

We write here $(-\log s)$ and $(-\log g)$ for convenience in arithmetical computations, because $\log s$ and $\log g$ are essentially negative.

[42] The function $\log(-\Delta\log p_x)$ may be constructed by starting from an initial value, and adding continuously $\log c$. Then, taking out the natural numbers, we can, by continued addition, form $\log p_x$ and $\log l_x$.

[43] In order to attain the highest degree of accuracy possible in the work we should take $\log(-\log g)$ as found in the original computation. That is $\log(-\log g)=\bar{4}{\cdot}6597057$. If we took the value from Art. 37 we should lose in correctness.

[44] The original data of the Institute of Actuaries' Healthy Males Table do not go below age 10, and therefore that is the age with which we must commence. In order to form the initial values we have

$$\log(-\Delta\log p_{10}) = 10\log c + 2\log(c-1) + \log(-\log g)$$

$$\begin{array}{r@{\;=\;}l}
10\log c & {\cdot}3965686 \\
2\log(c-1) & \bar{3}{\cdot}9610266 \\
\log(-\log g) & \bar{4}{\cdot}6597057 \\ \cline{2-2}
\log(-\Delta\log p_{10}) & \bar{5}{\cdot}0173009 \\ \cline{2-2}
\end{array}$$

$$\log p_{10} = -(-\log s) - c^{10}(c-1)(-\log g)$$

$$\begin{array}{r@{\;=\;}l}
10\log c & {\cdot}3965686 \\
\log(c-1) & \bar{2}{\cdot}9805133 \\
\log(-\log g) & \bar{4}{\cdot}6597057 \\ \cline{2-2}
 & \bar{4}{\cdot}0367876 \\ \cline{2-2}
\end{array}$$

$$\begin{array}{r@{\;=\;}l}
c^{10}(c-1)(-\log g) & {\cdot}00010884 \\
(-\log s) & {\cdot}00268933 \\ \cline{2-2}
\log p_{10} & -{\cdot}00279817 \\
 & \bar{1}{\cdot}99720183 \\ \cline{2-2}
\end{array}$$

$$\log l_{10} = \log k - 10(-\log s) - c^{10}(-\log g)$$

$$\begin{array}{r@{\;=\;}l}
10\log c & {\cdot}3965686 \\
\log(-\log g) & \bar{4}{\cdot}6597057 \\ \cline{2-2}
 & \bar{3}{\cdot}0562743 \\ \cline{2-2}
\end{array}$$

$$\begin{array}{r@{\;=\;}l}
c^{10}(-\log g) & {\cdot}0011383 \\
10(-\log s) & {\cdot}0268933 \\ \cline{2-2}
 & {\cdot}0280316 \\
\log k & 4{\cdot}0404723 \\ \cline{2-2}
\log l_{10} & 4{\cdot}0124407 \\ \cline{2-2}
\end{array}$$

[45] The following table now gives a complete exhibition of the work, but in it $\log l_x$ has been increased by unity, so as to give five figures to l_x throughout the greater part of the table. In copying from it Table No. I, the last figure in l_x has in a very few cases been altered a unit.

x	$\log(-\Delta\log p_x)$	$-\Delta\log p_x$	$\log p_x$	$\log l_x$	l_x
10	$\bar{5}$·0173009	·0000104	$\bar{1}$·9972018	5·0124407	102906
11	·0569578	·0000114	·9971914	·0096425	102245
12	·0966146	·0000125	·9971800	·0068339	101586
13	·1362715	·0000137	·9971675	·0040139	100928
14	·1759283	·0000150	·9971538	·0011814	100272
15	·2155852	·0000164	·9971388	4·9983352	99619
16	·2552421	·0000180	·9971224	·9954740	98962
17	·2948989	·0000197	·9971044	·9925964	98311
18	·3345558	·0000216	·9970847	·9897008	97656
19	·3742126	·0000237	·9970631	·9867855	97004
20	·4138695	·0000259	·9970394	·9838486	96350
21	·4535264	·0000284	·9970135	·9808880	95695
22	·4931832	·0000311	·9969851	·9779015	95039
23	·5328401	·0000341	·9969540	·9748866	94382
24	·5724969	·0000374	·9969199	·9718406	93722
25	·6121538	·0000409	·9968825	·9687605	93059
26	·6518107	·0000449	·9968416	·9656430	92393
27	·6914675	·0000491	·9967967	·9624846	91723
28	·7311244	·0000538	·9967476	·9592813	91050
29	·7707812	·0000590	·9966938	·9560289	90371
30	·8104381	·0000646	·9966348	·9527227	89685
31	·8500950	·0000708	·9965702	·9493575	88994
32	·8897518	·0000776	·9964994	·9459277	88294
33	·9294087	·0000850	·9964218	·9424271	87585
34	·9690655	·0000931	·9963368	·9388489	86866
35	$\bar{4}$·0087224	·0001020	·9962437	·9351857	86137
36	·0483793	·0001118	·9961417	·9314294	85395
37	·0880361	·0001225	·9960299	·9275711	84639
38	·1276930	·0001342	·9959074	·9236010	83869
39	·1673498	·0001470	·9957732	·9195084	83083
40	·2070067	·0001611	·9956262	·9152816	82277
41	·2466636	·0001765	·9954651	·9109078	81454
42	·2863204	·0001933	·9952886	·9063729	80606
43	·3259773	·0002118	·9950953	·9016615	79737
44	·3656341	·0002321	·9948835	·8967568	78842
45	·4052910	·0002543	·9946514	·8916403	77918
46	·4449479	·0002786	·9943971	·8862917	76964
47	·4846047	·0003052	·9941185	·8806888	75978
48	·5242616	·0003344	·9938133	·8748073	74957
49	·5639184	·0003664	·9934789	·8686206	73896
50	·6035753	·0004014	·9931125	·8620995	72795
51	·6432322	·0004398	·9927111	·8552120	71649
52	·6828890	·0004818	·9922713	·8479231	70456
53	·7225459	·0005279	·9917895	·8401944	69213
54	·7622027	·0005784	·9912616	·8319839	67917
55	·8018596	·0006337	·9906832	·8232455	66566
56	·8415165	·0006943	·9900495	·8139287	65152
57	·8811733	·0007606	·9893552	·8039782	63677
58	·9208302	·0008334	·9885946	·7933334	62134
59	·9604870	·0009130	·9877612	·7819280	60524
60	$\bar{3}$·0001439	·0010003	·9868482	·7696892	58842
61	·0398008	·0010960	·9858479	·7565374	57087
62	·0794576	·0012008	·9847519	·7423853	55257
63	·1191145	·0013156	·9835511	·7271372	53351
64	·1587713	·0014414	·9822355	·7106883	51368
65	·1984282	·0015792	·9807941	·6929238	49308
66	·2380851	·0017302	·9792149	·6737179	47176
67	·2777419	·0018956	·9774847	·6529328	44971

x	$\log(-\Delta\log p_x)$	$-\Delta\log p_x$	$\log p_x$	$\log l_x$	l_x
68	$\bar{3}$·3173988	·0020768	$\bar{1}$·9755891	4·6304175	42699
69	·3570556	·0022754	·9735123	·6060066	40365
70	·3967125	·0024929	·9712369	·5795189	37977
71	·4363694	·0027313	·9687440	·5507558	35543
72	·4760262	·0029924	·9660127	·5194998	33075
73	·5156831	·0032786	·9630203	·4855125	30585
74	·5553399	·0035920	·9597417	·4485328	28089
75	·5949968	·0039355	·9561497	·4082745	25602
76	·6346537	·0043118	·9522142	·3644242	23143
77	·6743105	·0047240	·9479024	·3166384	20732
78	·7139674	·0051757	·9431784	·2645408	18388
79	·7536242	·0056705	·9380027	·2077192	16133
80	·7932811	·0062127	·9323322	·1457219	13987
81	·8329380	·0068067	·9261195	·0780541	11969
82	·8725948	·0074575	·9193128	·0041736	10096
83	·9122517	·0081706	·9118553	3·9234864	8385
84	·9519085	·0089518	·9036847	·8353417	6844
85	·9915654	·0098077	·8947329	·7390264	5483
86	$\bar{2}$·0312223	·0107454	·8849252	·6337593	4303
87	·0708791	·0117728	·8741798	·5186845	3301
88	·1105360	·0128984	·8624070	·3928643	2471
89	·1501928	·0141317	·8495086	·2552713	1800
90	·1898497	·0154828	·8353769	·1047799	1273
91	·2295066	·0169632	·8198941	2·9401568	871
92	·2691634	·0185850	·8029309	·7600509	576
93	·3088203	·0203620	·7843459	·5629818	366
94	·3484771	·0223088	·7639839	·3473277	222
95	·3881340	·0244418	·7416751	·1113116	129
96	·4277909	·0267788	·7172333	1·8529867	71
97	·4674477	·0293392	·6904545	·5702200	37
98	·5071046	·0321444	·6611153	·2606745	18
99	·5467614	·0352177	·6289709	0·9217898	8
100	·5864183	·0385850	·5937532	·5507607	4
101	·6260752	·0422742	·5551682	·1445139	1
102	·6657320	·0463161	·5128940	$\bar{1}$·6996821	
103	·7053889	·0507445	·4665779	·2125761	

[46] In Art. 29 we gave a table of $\log l_x$ according to the original data, and if we take out the natural numbers and compare them with the values of l_x prepared by the constants as above, we shall find that after about age 20 the two sets run very close to each other, the graduated values being sometimes greater and sometimes less than the original. Below age 20 the case is however different. The graduated values err always on the side of excess, and the divergence gradually becomes very marked. In fact, there is a change in the law of mortality near age 20, which is not taken account of in the formula, the rate of mortality being lower at the younger ages than that assumed in the constants. If we call "original series" the values of l_x supplied in the data, and "normal series" those constructed from the constants; we shall have, on

subtracting the values in the original series from the corresponding values in the normal series, another series which may be named "complementary series." The following table gives these three series for ages from 10 to 25.

Age.	Original Series.	Normal Series.	Complementary Series.
10	100000	102906	2906
11	99209	102245	3036
12	99209	101586	2377
13	98805	100928	2123
14	98462	100272	1810
15	98060	99619	1559
16	97843	98962	1119
17	97843	98311	468
18	97429	97656	227
19	96837	97004	167
20	96155	96350	195
21	95600	95695	95
22	94931	95039	108
23	94339	94382	43
24	93610	93722	112
25	92967	93059	92

[47] The tendency is clearly seen in the values of the complementary series to increase from age 25 to age 10, although they run somewhat irregularly on account of the roughness of the original data. In order to form a final table keeping close to the original series, and at the same time running smoothly from one value to another, we must graduate the complementary series, and then deduct the graduated values from the normal series. This operation can very conveniently be performed by a duplication of the formula by means of which the normal series itself was constructed. We may treat the complementary series as a mortality table; calculate constants for it; and then reconstruct it by means of these constants. It has been found by applying Makeham's formula to many mortality tables that the constant, c, varies but little from one table to another, and that $\log c$ is always very nearly ·04. We are therefore justified in assuming that in all parts of the same table, c remains constant, and for the complementary series we may use the same value of $\log c$ as was used for the normal series. We have therefore only three constants to find, those corresponding to k, g, and s; and to distinguish them from those of the normal series we shall designate them by Greek letters κ, γ, and σ. For convenience, also, in speaking of the various quantities, we shall write λ_x for the values in the normal series, and λ'_x

for those in the complementary series; and we shall retain the symbol l_x for the values in the final mortality table.

[48] We might find the constants κ, γ, and σ, by the aggregate method which was followed in finding k, g, and s; but that would not be convenient. By it, none of the values of λ'_x would be identical, unless by accident, with the ungraduated values; and it is desirable that the radix at age 10 should not be altered. Also, by the aggregate method we should not construct a complementary series which would join on smoothly to the normal series between ages 20 and 30. For the present purpose it will be better to take isolated values of λ'_x; and those that will give the most satisfactory results are λ'_{10}, λ'_{16} and λ'_{22}; only three values being required as there are but three constants to find. The process is that described in Art. 19, and the following are the operations in full.

$$(\alpha) \qquad \begin{aligned} \log\lambda'_{10} &= \log\kappa + 10\log\sigma + c^{10}\log\gamma \\ \log\lambda'_{16} &= \log\kappa + 16\log\sigma + c^{16}\log\gamma \\ \log\lambda'_{22} &= \log\kappa + 22\log\sigma + c^{22}\log\gamma \end{aligned}$$

$$(\beta) \qquad \begin{aligned} \Delta\log\lambda'_{10} &= 6\log\sigma + c^{10}(c^6-1)\log\gamma \\ \Delta\log\lambda'_{16} &= 6\log\sigma + c^{16}(c^6-1)\log\gamma \end{aligned}$$

$$(\gamma) \qquad \Delta^2\log\lambda'_{10} = c^{10}(c^6-1)^2\log\gamma\,;$$

whence

$$\log\gamma = \frac{\Delta^2\log\lambda'_{10}}{c^{10}(c^6-1)^2}$$

$$\log\sigma = \tfrac{1}{6}\{\Delta\log\lambda'_{10} - c^{10}(c^6-1)\log\gamma\}$$

$$\log\kappa = \log\lambda'_{10} - 10\log\sigma - c^{10}\log\gamma.$$

Also in numbers

		Δ	Δ^2
$\log\lambda'_{10}=$	3·46330	$-$·41447	$-$·60094
$\log\lambda'_{16}=$	3·04883	-1·01541	
$\log\lambda'_{22}=$	2·03342		

$$\begin{array}{rl} 6\log c = & {\cdot}2379412 \\ (c^6-1) = & {\cdot}72958 \\ \log(c^6-1) = & \bar{1}{\cdot}86307 \\ \\ \log\gamma = & \underline{-{\cdot}45302} \end{array} \qquad \begin{array}{rl} 10\log c = & {\cdot}39657 \\ 2\log(c^6-1) = & \underline{\bar{1}{\cdot}72614} \\ & {\cdot}12271 \\ \log\{-\Delta^2\log\lambda'_{10}\} = & \underline{\bar{1}{\cdot}77883} \\ \log\{-\log\gamma\} = & \underline{\bar{1}{\cdot}65612} \end{array}$$

$$\begin{array}{rr}
10\log c = & \cdot39657 \\
\log(c^6-1) = & \overline{1}\cdot86307 \\
\log(-\log\gamma) = & \overline{1}\cdot65612 \\
\hline
 & \overline{1}\cdot91576 \\
\hline
\end{array}$$

$$\begin{array}{rr}
10\log c = & \cdot39657 \\
\log(-\log\gamma) = & \overline{1}\cdot65612 \\
\hline
 & \cdot05269 \\
\hline
\end{array}$$

$$\begin{array}{rr}
-c^{10}(c^6-1)\log\gamma = & \cdot82368 \\
\Delta\log\lambda'_{10} = & -\cdot41447 \\
\hline
6\log\sigma = & \cdot40921 \\
\hline
\log\sigma = & +\cdot06820 \\
\hline
\end{array}$$

$$\begin{array}{rr}
c^{10}(-\log\gamma) = & 1\cdot1290 \\
\log\lambda'_{10} = & 3\cdot4633 \\
\hline
 & 4\cdot5923 \\
10\log\sigma = & \cdot6820 \\
\hline
\log\kappa = & 3\cdot9103 \\
\hline
\end{array}$$

[**49**] From the constants the values of λ' must now be found in exactly the same way as we found the corresponding values of the normal series in Art. **45.**

The initial values are as follows :—

$$\log(-\Delta\log p_{10}) = 10\log c + 2\log(c-1) + \log(-\log\gamma)$$

$$\begin{array}{rr}
10\log c = & \cdot39657 \\
2\log(c-1) = & \overline{3}\cdot96103 \\
\log(-\log\gamma) = & \overline{1}\cdot65612 \\
\hline
\log(-\Delta\log p_{10}) = & \overline{2}\cdot01372 \\
\hline
\end{array}$$

$$\log p_{10} = \log\sigma - c^{10}(c-1)(-\log\gamma)$$

$$\begin{array}{rr}
10\log c = & \cdot39657 \\
\log(c-1) = & \overline{2}\cdot98051 \\
\log(-\log\gamma) = & \overline{1}\cdot65612 \\
\hline
 & \overline{1}\cdot03320 \\
\hline
\end{array}
\qquad
\begin{array}{rr}
\log\sigma = & \cdot06820 \\
c^{10}(c-1)(-\log\gamma) = & \cdot10794 \\
\hline
\log p_{10} = & \overline{1}\cdot96026 \\
\hline
\end{array}$$

The value of $\log\lambda'_{10}$ being already given, it does not require to be recalculated.

[**50**] The following table displays the operations in full. The values of λ' are obtained from the logarithms by means of a four-figure card.

x	$\log(-\Delta\log p_x)$	$-\Delta\log p_x$	$\log p_x$	$\log\lambda'_x$	λ'_x
10	$\bar{2}$·01372	·01032	$\bar{1}$·96026	3·46330	2906
11	·05338	·01131	·94994	·42356	2652
12	·09303	·01239	·93863	·37350	2363
13	·13269	·01357	·92624	·31213	2052
14	·17235	·01487	·91267	·23837	1731
15	·21200	·01629	·89780	·15104	1416
16	·25166	·01785	·88151	·04884	1119
17	·29132	·01956	·86366	2·93035	852
18	·33097	·02143	·84410	·79401	622
19	·37063	·02348	·82267	·63811	435
20	·41029	·02572	·79919	·46078	289
21	·44995	·02818	·77347	·25997	182
22	·48960	·03088	·74529	·03344	108
23	·52926	·03383	·71441	1·77873	60
24	·56892	·03706	·68058	·49314	31
25	·60857	·04060	·64352	·17372	15
26	·64823	·04449	·60292	0·81724	7
27	·68789	·04874	·55843	·42016	3
28	·72754	·05340	·50969	$\bar{1}$·97859	1
29	·76720	·05851	·45629	·48828	

[51] In the following table the values of l_x in the final series are obtained.

Age.	Normal Series λ_x	Complementary Series λ'_x	Final Series l_x	Age.	Normal Series λ_x	Complementary Series λ'_x	Final Series l_x
10	102906	2906	100000	20	96350	289	96061
11	102245	2652	99593	21	95695	182	95513
12	101586	2363	99223	22	95039	108	94931
13	100928	2052	98876	23	94382	60	94322
14	100272	1731	98541	24	93722	31	93691
15	99619	1416	98203	25	93059	15	93044
16	98962	1119	97843	26	92393	7	92386
17	98311	852	97459	27	91723	3	91720
18	97656	622	97034	28	91050	1	91049
19	97004	435	96569	29	90371	0	90371

[52] In order to judge how far the construction of the table by Makeham's formula has been successful, we must compare the result with the original facts. The experience of the 20 offices from which the original facts are derived was given in the form "Exposed to Risk" and "Died" for each year of age; the "Exposed to Risk" being the number of male lives assured during one entire year at each age, and the "Died" being the actual number of deaths among those assured lives during that year. The number of deaths at age x, divided by the number exposed to risk, gives the q_x of the original table; and if the table constructed by Makeham's formula closely follow the original, then by multiplying the "Exposed to Risk" by the q_x of the final table, and thus obtaining the "Expected Deaths", we shall reproduce the "Actual Deaths" of

the original observations. Of course the Actual Deaths will differ from the Expected at individual ages, because of the roughness of the original table; but taking small groups of ages, the numbers should be approximately equal. In the following table the comparison is made. The "Exposed to Risk" and "Actual Deaths" have been taken from the Institute of Actuaries' Volume of "Mortality Experience" page 244, and the "Expected Deaths" have been obtained by multiplying the "Exposed to Risk" by the q_x of the final table, four-figure logarithms being used in the process. The difference between the Expected and the Actual Deaths is entered with its proper sign in the column headed "Error"; and in the last column is given opposite the age the accumulated error from age 10 to age x inclusive. This column best shows the accuracy of the table. It will be seen that at no point is there a difference of consequence, and that the difference between the total actual and total expected deaths is only 21. The final table therefore faithfully represents the original facts.

Age.	Exposed to Risk.	Actual Deaths.	Expected Deaths.	Error.	Accumulated Error.
10	379	3	2	− 1	− 1
11	434	0	2	+ 2	+ 1
12	491·5	2	2	0	+ 1
13	578	2	2	0	+ 1
14	731	3	3	0	+ 1
15	908	2	3	+ 1	+ 2
16	1129	0	4	+ 4	+ 6
17	1421	6	6	0	+ 6
18	1810·5	11	9	− 2	+ 4
19	2414	17	13	− 4	0
20	3293·5	19	19	0	0
21	4578·5	32	28	− 4	− 4
22	6397	40	41	+ 1	− 3
23	8534	66	57	− 9	− 12
24	10936	75	76	+ 1	− 11
25	13622·5	70	96	+ 26	+ 15
26	16339	113	118	+ 5	+ 20
27	19170·5	124	140	+ 16	+ 36
28	21837	171	163	− 8	+ 28
29	24588	181	187	+ 6	+ 34
30	27112·5	224	209	− 15	+ 19
31	29213	215	230	+ 15	+ 34
32	31232	260	251	− 9	+ 25
33	32969	274	271	− 3	+ 22
34	34535·5	300	290	− 10	+ 12
35	35818·5	295	309	+ 14	+ 26
36	36840·5	326	326	0	+ 26
37	37360	357	340	− 17	+ 9
38	37804·5	389	354	− 35	− 26
39	38112·5	405	369	− 36	− 62
40	38195	377	382	+ 5	− 57
41	37838	396	393	− 3	− 60
42	37258·5	399	403	+ 4	− 56

Age	Exposed to Risk.	Actual Deaths.	Expected Deaths.	Error.	Accumulated Error.
43	36534·5	387	410	+ 23	− 33
44	35693	421	418	− 3	− 36
45	34735·5	429	425	− 4	− 40
46	33660·5	421	431	+ 10	− 30
47	32502	460	437	− 23	− 53
48	31228	440	442	+ 2	− 51
49	30055·5	459	448	− 11	− 62
50	28855·5	476	454	− 22	− 84
51	27510·5	479	458	− 21	− 105
52	26208·5	446	462	+ 16	− 89
53	24785	426	464	+ 38	− 51
54	23426	444	467	+ 23	− 28
55	22170·5	509	471	− 38	− 66
56	20746	479	470	− 9	− 75
57	19377·5	463	469	+ 6	− 69
58	18116·5	455	470	+ 15	− 54
59	16890·5	428	469	+ 41	− 13
60	15672·5	488	468	− 20	− 33
61	14392·5	468	461	− 7	− 40
62	13261	459	458	− 1	− 41
63	12147·5	454	452	− 2	− 43
64	11021·5	443	442	− 1	− 44
65	9984·5	435	432	− 3	− 47
66	9009·5	421	421	0	− 47
67	8081	396	408	+ 12	− 35
68	7214	399	394	− 5	− 40
69	6375·5	389	377	− 12	− 52
70	5622	315	360	+ 45	− 7
71	4953	308	344	+ 36	+ 29
72	4378	349	330	− 19	+ 10
73	3771·5	297	308	+ 11	+ 21
74	3228	340	286	− 54	− 33
75	2693	254	259	+ 5	− 28
76	2253	240	235	− 5	− 33
77	1848·5	201	209	+ 8	− 25
78	1531	188	188	0	− 25
79	1257	171	167	− 4	− 29
80	995	140	144	+ 4	− 25
81	782	125	122	− 3	− 28
82	609·5	105	103	− 2	− 30
83	464	96	85	− 11	− 41
84	339	61	67	+ 6	− 35
85	254·5	55	55	0	− 35
86	184	40	43	+ 3	− 32
87	128·5	28	32	+ 4	− 28
88	91·5	26	25	− 1	− 29
89	57·5	11	17	+ 6	− 23
90	43·5	10	14	+ 4	− 19
91	32	10	11	+ 1	− 18
92	20	9	7	− 2	− 20
93	10·5	7	4	− 3	− 23
94	4	0	2	+ 2	− 21
95	3·5	1	2	+ 1	− 20
96	2	2	1	− 1	− 21
97	0	0	0	0	− 21
Totals	1199092·5	20517	20496	+ 422 − 443	− 21

[53] The following summary shows for quinquennial groups of ages the Actual and Expected Deaths, and the Error. It will be noticed that, throughout its whole length, the table constructed from Makeham's constants runs very close to the original facts.

Ages.	Actual Deaths.	Expected Deaths.	Error.
10 to 14	10	11	+ 1
15 " 19	36	35	− 1
20 " 24	232	221	−11
25 " 29	659	704	+45
30 " 34	1273	1251	−22
35 " 39	1772	1698	−74
40 " 44	1980	2006	+26
45 " 49	2209	2183	−26
50 " 54	2271	2305	+34
55 " 59	2334	2349	+15
60 " 64	2312	2281	−31
65 " 69	2040	2032	− 8
70 " 74	1609	1628	+19
75 " 79	1054	1058	+ 4
80 " 84	527	521	− 6
85 " 89	160	172	+12
90 " 94	36	38	+ 2
95 and over	3	3	0

[54] The Mortality Table from age 10 upwards has been constructed, in the way described above, from the Experience of 20 British Companies, collected by the Institute of Actuaries. It was desirable for the purposes of this volume to have a table commencing at age 0, but that experience did not supply sufficient data below age 10 on which to base a mortality table. There were only 470 lives assured at these young ages, among whom there were only 4 deaths, and therefore it became necessary to seek some other way of calculating the probabilities during the infantile period of life. The older portion of the table was based upon the experience of Healthy Males, and in order to complete the table recourse was had to the observations on male lives residing in the Healthy Districts of England. The table may therefore be taken to represent the mortality among "Healthy Males" from birth to extreme old age.

[55] A description of the construction of the "Healthy English" Table is given by Farr, in the *Philosophical Transactions*, 1859, reprinted in *J.I.A.* ix, p. 121, to which we refer the reader. Here we need only state that we have accepted without alteration the probabilities of life as derived by Farr, using his p_x from age 0 to age 9 inclusive,

and thus working backwards from $l_{10}=100,000$ to the radix of the table, 127,283 at age 0. These probabilities harmonize perfectly with those of the remainder of the table, so that there is no break in the continuity at age 10, the point of junction.

[56] It only remains in this chapter to explain the calculation of the values of μ_x. The mortality table may be divided into three sections, namely, first, from age 0 to age 9, in which a law of mortality is not assumed; second, from age 10 to age 28, in which a complementary series is superimposed on the normal series, both the series following Makeham's law; and third, from age 29 to the end of the table, in which only the normal series following Makeham's law prevails.

[57] In this third section we have by equation 6,

$$\mu_x = -\log_e s - (\log_e g \log_e c)c^x,$$

whence μ_x can be easily calculated.

[58] In the middle section, keeping to our former notation,

$$\begin{aligned} l_x &= ks^x(g)^{c^x} - \kappa\sigma^x(\gamma)^{c^x} \\ &= \lambda_x - \lambda'_x. \end{aligned}$$

But, Chap. ii, formula 19, $\mu_x = -\dfrac{1}{l_x}\dfrac{dl_x}{dx}$. Therefore at this part of the table

$$\begin{aligned} \mu_x &= -\frac{1}{l_x}\left\{\frac{d\lambda_x}{dx} - \frac{d\lambda'_x}{dx}\right\} \\ &= -\frac{1}{l_x}\{\lambda_x[-\log_e s - (\log_e g \log_e c)c^x] \\ &\qquad -\lambda'_x[-\log_e \sigma - (\log_e \gamma \log_e c)c^x]\} \end{aligned}$$

on the same principles by which formula 6 is derived.

[59] For the first section of the table (ages below 10) another course must be pursued, which will be explained later on. (*See* Arts. 67 and 68.)

[60] We now proceed to give in full the numerical calculations by which the values of μ_x were found.

[61] Beginning with the Normal Series to the end of the table, the working formula is

$$\mu_x = -\log_e s - (\log_e g \log_e c)c^x,$$

and by this we shall get the correct force of mortality for ages 29 and upwards, and a series of values which will assist us for ages 10 to 28.

[62] Since the modulus of the common system of logarithms is ·434294482, we have

$$\begin{aligned}\log_e s &= -\cdot 0061923\\ \log_e g &= -\cdot 0010518\\ \log_e c &= \cdot 0913133\end{aligned}$$

$$\begin{aligned}\log_{10}(-\log_e g) &= \bar{3}\cdot 0219332\\ \log_{10}\log_e c &= \bar{2}\cdot 9605340\\ 10\log_{10} c &= \phantom{\bar{2}}\cdot 3965686\\ &\ \bar{4}\cdot 3790358\end{aligned}$$

$$\begin{aligned}-(\log_e g \log_e c)c^{10} &= \cdot 0002394\\ -\log_e s &= \cdot 0061923\\ \mu_{10} &= \cdot 0064317\end{aligned}$$

This is the initial value. To form the differences

$$\mu_x = -\log_e s - (\log_e g \log_e c)c^x$$

$$\Delta\mu_x = -(\log_e g \log_e c)(c-1)c^x$$

$$\log_{10}\Delta\mu_x = \log_{10}(-\log_e g) + \log_{10}\log_e c + \log_{10}(c-1) + x\log_{10} c$$

$$\begin{aligned}\log_{10}(-\log_e g) &= \bar{3}\cdot 0219332\\ \log_{10}\log_e c &= \bar{2}\cdot 9605340\\ \log_{10}(c-1) &= \bar{2}\cdot 9805133\\ 10\log_{10} c &= \phantom{\bar{2}}\cdot 3965686\\ \log\Delta\mu_{10} &= \bar{5}\cdot 3595491\end{aligned}$$

Starting from this value and adding continuously $\log_{10} c$, we form the column $\log\Delta\mu_x$, from which we derive $\Delta\mu_x$ and μ_x.

[63] The following is a complete table of the operations.

x	$\log \Delta\mu_x$ Normal Series.	$\Delta\mu_x$ Normal Series.	μ_x Normal Series.
10	$\bar{5}$·3595491	·0000229	·0064317
11	·3992060	·0000251	·0064546
12	·4388628	·0000275	·0064797
13	·4785197	·0000301	·0065072
14	·5181765	·0000330	·0065373
15	·5578334	·0000361	·0065703
16	·5974903	·0000396	·0066064
17	·6371471	·0000434	·0066460
18	·6768040	·0000475	·0066894
19	·7164608	·0000521	·0067369
20	·7561177	·0000570	·0067890
21	·7957746	·0000625	·0068460
22	·8354314	·0000685	·0069085
23	·8750883	·0000750	·0069770
24	·9147451	·0000822	·0070520
25	·9544020	·0000900	·0071342
26	·9940589	·0000986	·0072242
27	$\bar{4}$·0337157	·0001081	·0073228
28	·0733726	·0001184	·0074309
29	·1130294	·0001297	·0075493
30	·1526863	·0001421	·0076790
31	·1923432	·0001557	·0078211
32	·2320000	·0001706	·0079768
33	·2716569	·0001869	·0081474
34	·3113137	·0002048	·0083343
35	·3509706	·0002244	·0085391
36	·3906275	·0002458	·0087635
37	·4302843	·0002693	·0090093
38	·4699412	·0002951	·0092786
39	·5095980	·0003233	·0095737
40	·5492549	·0003542	·0098970
41	·5889118	·0003881	·0102512
42	·6285686	·0004252	·0106393
43	·6682255	·0004658	·0110645
44	·7078823	·0005104	·0115303
45	·7475392	·0005592	·0120407
46	·7871961	·0006126	·0125999
47	·8268529	·0006712	·0132125
48	·8665098	·0007354	·0138837
49	·9061666	·0008057	·0146191
50	·9458235	·0008827	·0154248
51	·9854804	·0009671	·0163075
52	$\bar{3}$·0251372	·0010596	·0172746
53	·0647941	·0011609	·0183342
54	·1044509	·0012719	·0194951
55	·1441078	·0013935	·0207670
56	·1837647	·0015267	·0221605
57	·2234215	·0016727	·0236872
58	·2630784	·0018326	·0253599
59	·3027352	·0020079	·0271925
60	·3423921	·0021998	·0292004
61	·3820490	·0024102	·0314002
62	·4217058	·0026406	·0338104
63	·4613627	·0028931	·0364510
64	·5010195	·0031697	·0393441
65	·5406764	·0034728	·0425138
66	·5803333	·0038048	·0459866
67	·6199901	·0041686	·0497914

x	$\log \Delta\mu_x$ Normal Series.	$\Delta\mu_x$ Normal Series.	μ_x Normal Series.
68	$\bar{3}$·6596470	·0045672	·0539600
69	·6993038	·0050038	·0585272
70	·7389607	·0054823	·0635310
71	·7786176	·0060064	·0690133
72	·8182744	·0065807	·0750197
73	·8579313	·0072099	·0816004
74	·8975881	·0078993	·0888103
75	·9372450	·0086546	·0967096
76	·9769019	·0094820	·1053642
77	$\bar{2}$·0165587	·0103886	·1148462
78	·0562156	·0113819	·1252348
79	·0958724	·0124702	·1366167
80	·1355293	·0136625	·1490869
81	·1751862	·0149688	·1627494
82	·2148430	·0164000	·1777182
83	·2544999	·0179680	·1941182
84	·2941567	·0196859	·2120862
85	·3338136	·0215682	·2317721
86	·3734705	·0236304	·2533403
87	·4131273	·0258897	·2769707
88	·4527842	·0283651	·3028604
89	·4924410	·0310771	·3312255
90	·5320979	·0340485	·3623026
91	·5717548	·0373039	·3963511
92	·6114116	·0408706	·4336550
93	·6510685	·0447784	·4745256
94	·6907253	·0490597	·5193040
95	·7303822	·0537504	·5683637
96	·7700391	·0588896	·6221141
97	·8096959	·0645202	·6810037
98	·8493528	·0706891	·7455239
99	·8890096	·0774478	·8162130
100	·9286665	·0848528	·8936608
101	·9683234	·0929658	·9785136

[**64**] For the middle section of the mortality table we have already calculated one portion of the force of mortality, namely, $-\log_e s - (\log_e g \log_e c)\, c^x$, which we must multiply by λ_x. For the other portion of the force of mortality we must calculate $-\log_e \sigma - (\log_e \gamma \log_e c) c^x$. We have, dividing $\log_{10}\sigma$ and $\log_{10}\gamma$ respectively by the modulus

$$\log_e \sigma = + \cdot 1570363$$

$$\log_e \gamma = -1\cdot 0431171.$$

Calling μ'_x the force of decrement in the complementary series, which corresponds to μ_x, the force in the normal series, we have as the initial terms

$$\log_{10}\Delta\mu'_{10}=\log_{10}(-\log_e\gamma)+\log_{10}(\log_e c)+\log_{10}(c-1)+10\log_{10}c$$

$$\begin{aligned}
\log_{10}(-\log_e\gamma) &= 0\cdot0183331\\
\log_{10}(\log_e c) &= \bar{2}\cdot9605340\\
\log_{10}(c-1) &= \bar{2}\cdot9805133\\
10\log_{10}c &= \cdot3965686\\
\log(\Delta\mu'_{10}) &= \bar{2}\cdot3559490
\end{aligned}$$

Also $\mu'_{10}=-\log_e\sigma-(\log_e\gamma\,\log_e c)c^{10}$

$$\begin{aligned}
\log_{10}(-\log_e\gamma) &= 0\cdot0183331\\
\log_{10}(\log_e c) &= \bar{2}\cdot9605340\\
10\log_{10}c &= 0\cdot3965686\\
 & \quad \bar{1}\cdot3754357\\
 & \quad \cdot2373754\\
\log_e\sigma &= \cdot1570363\\
\mu'_{10} &= \cdot0803391
\end{aligned}$$

[65] The following is a complete statement of the construction of μ'_x:

x	$\log\Delta\mu'_x$ Complementary Series.	$\Delta\mu'_x$ Complementary Series.	μ'_x Complementary Series.
10	$\bar{2}$·3559490	·0226960	·0803391
11	·3956059	·0248660	·1030351
12	·4352627	·0272435	·1279011
13	·4749196	·0298483	·1551446
14	·5145764	·0327021	·1849929
15	·5542333	·0358289	·2176950
16	·5938902	·0392545	·2535239
17	·6335470	·0430077	·2927784
18	·6732039	·0471198	·3357861
19	·7128607	·0516250	·3829059
20	·7525176	·0565611	·4345309
21	·7921745	·0619690	·4910920
22	·8318313	·0678940	·5530610
23	·8714882	·0743855	·6209550
24	·9111450	·0814976	·6953405
25	·9508019	·0892898	·7768381
26	·9904588	·0978270	·8661279
27	$\bar{1}$·0301156	·1071804	·9639549
28	·0697725	·1174283	1·0711353

[66] We must now form the products $\lambda_x \mu_x$ and $\lambda'_x \mu'_x$; deduct the second from the first; and divide the difference by l_x. The result will be the values of μ_x of the final table. In the following tables these operations are given in full:

x	$\log \mu_x$ Normal Series.	$\log \lambda_x$	$\log (\lambda_x \times \mu_x)$	$\lambda_x \times \mu_x$
10	$\bar{3}$·8083258	5·0124407	2·8207665	661·8606
11	·8098693	·0096421	·8195114	659·9505
12	·8115549	·0068339	·8183888	658·2469
13	·8133942	·0040117	·8174059	656·7588
14	·8153984	·0011797	·8165781	655·5081
15	·8175852	4·9983422	·8159274	654·5268
16	·8199649	·9954685	·8154334	653·7826
17	·8225603	·9926021	·8151624	653·3748
18	·8253872	·9896989	·8150861	653·2601
19	·8284601	·9867896	·8152497	653·5062
20	·8318058	·9838517	·8156575	654·1201
21	·8354369	·9808892	·8163261	655·1279
22	·8393838	·9779019	·8172857	656·5771
23	·8436687	·9748892	·8185579	658·5032
24	·8483123	·9718415	·8201538	660·9275
25	·8533453	·9687584	·8221037	663·9016
26	·8587898	·9656391	·8244289	667·4656
27	·8646772	·9624877	·8271649	671·6838
28	·8710414	·9592800	·8303214	676·5835

x	$\log \mu'_x$ Complementary Series.	$\log \lambda'_x$	$\log (\lambda_x' \times \mu_x)$	$\lambda_x' \times \mu'_x$
10	$\bar{2}$·9049269	3·4632956	2·3682225	233·4654
11	$\bar{1}$·0129853	·4237372	·4367225	273.3522
12	·1068742	·3734637	·4803379	302·2302
13	·1907367	·3119657	·5027024	318·2017
14	·2671551	·2385479	·5057030	320·4077
15	·3378484	·1510633	·4889117	308·2561
16	·4040190	·0488301	·4528491	283·6933
17	·4665390	2·9304396	·3969786	249·4472
18	·5260627	·7937904	·3198531	208·8590
19	·5830921	·6384893	·2215814	166·5641
20	·6380207	·4608978	·0989185	125·5795
21	·6911628	·2600714	1·9512342	89·3787
22	·7427729	·0334238	·7761967	59·7306
23	·7930600	1·7781513	·5712113	37·2573
24	·8421975	·4913617	·3335592	21·5555
25	·8903306	·1760913	·0664219	11·6526
26	·9375821	0·8450980	0·7826801	6·0629
27	·9840567	·4771213	·4611780	2·8919
28	0·0298443	·0000000	·0298443	1·0711

x	Difference $(\lambda_x \times \mu_x - \lambda'_x \times \mu'_x)$	log Difference.	$\log l_x$	$\log \mu_x$	μ_x Final Table.
10	428·3952	2·6318446	5·0000000	$\bar{3}$·6318446	·0042840
11	386·5983	·5807820	4·9982245	·5890354	·0038818
12	356·0167	·5514703	·9966124	·5548579	·0035880
13	338·5571	·5296319	·9950953	·5345366	·0034240
14	335·1004	·5251749	·9936126	·5315623	·0034007
15	346·2707	·5394158	·9921248	·5472910	·0035261
16	370·0893	·5683065	·9905298	·5777767	·0037825
17	403·9276	·6063035	·9888220	·6174815	·0041446
18	444·4011	·6477752	·9869239	·6608513	·0045799
19	486·9421	·6874774	·9848377	·7026397	·0050424
20	528·5406	·7230784	·9825471	·7405313	·0055021
21	565·7492	·7526240	·9800625	·7725615	·0059233
22	596·8465	·7758627	·9774081	·7984546	·0062872
23	621·2459	·7932635	·9746130	·8186505	·0065864
24	639·3720	·8057537	·9716979	·8340558	·0068243
25	652·2490	·8144134	·9686884	·8457250	·0070101
26	661·4027	·8204660	·9656062	·8548598	·0071591
27	668·7919	·8252911	·9624735	·8628176	·0072915
28	675·5124	·8296334	·9592752	·8703582	·0074192

[67] Below the age 10, the mortality table does not follow any known law, and the values of μ_x, down to age 3 inclusive, were calculated by means of formula 20 of Chap. ii, while μ_2 was inserted by a second difference, working backwards from μ_5, μ_4, and μ_3. It seems hardly necessary to give in detail the arithmetical work.

[68] In order to form μ_0 and μ_1, recourse was had again to Makeham's formula. The value of the constant c was taken the same as for the rest of the table, and constants which we may style $\mathbf{g}$ and $\mathbf{s}$ were calculated from the values of l_0, l_1, and l_2. The following is the process:—

$$\begin{aligned} \log l_0 - \log l_1 &= \log \mathbf{s} + c^0(c-1)\log \mathbf{g} = -\cdot 05198 \\ \log l_1 - \log l_2 &= \log \mathbf{s} + c^1(c-1)\log \mathbf{g} = -\cdot 01551 \\ & \qquad\qquad c^0(c-1)^2 \log \mathbf{g} = +\cdot 03647 \end{aligned}$$

Whence
$$\log \mathbf{g} = \frac{\cdot 03647}{c^0(c-1)^2}$$

and
$$\log \mathbf{s} = -\cdot 05198 - c^0(c-1)\log \mathbf{g}$$

$$\begin{array}{rlrr} \log \cdot 03647 = & \bar{2}\cdot 56194 & & \\ 2\log(c-1) = & \underline{\bar{3}\cdot 96103} & & \\ \log\log \mathbf{g} = & \underline{\underline{0\cdot 60091}} & \log \mathbf{g} = & +3\cdot 98942 \\ \log\log \mathbf{g} = & 0\cdot 60091 & & \cdot 05198 \\ \log(c-1) = & \underline{\bar{2}\cdot 98051} & c^0(c-1)\log \mathbf{g} = & \underline{\cdot 38143} \\ = & \bar{1}\cdot 58142 & \log \mathbf{s} = & -\cdot 43341 \end{array}$$

Dividing $\log \mathbf{g}$ and $\log \mathbf{s}$ as found above, by the modulus ·434294 we have

$$\log_e \mathbf{g} = \quad 9{\cdot}18597$$

$$\log_e \mathbf{s} = -0{\cdot}99799$$

Then

$$\mu_0 = -\log_e \mathbf{s} - (\log_e \mathbf{g} \log_e c)c^0 = {\cdot}15920$$

$$\mu_1 = -\log_e \mathbf{s} - (\log_e \mathbf{g} \log_e c)c = {\cdot}07901$$

[**69**] The formulas of Gompertz and Makeham possess properties which are of the greatest service in the calculation of joint-life annuities, and these we shall discuss in Chap. xii.

l_x l_{x+n}

n

$\frac{v^n l_{x+n}}{l_x}$

CHAPTER VII.

Annuities and Assurances.

1. If there be a society consisting of l_x persons all of the same age, x, how much must each now subscribe, in order to secure 1 to himself on his attaining the age $x+n$?

At the end of n years, a sum equal to l_{x+n} will become payable, as there will then be only l_{x+n} survivors of the l_x persons at present constituting the society; and an investment of $v^n \times l_{x+n}$ made now, will provide that sum. Therefore the share to be now contributed by each of the l_x persons is $v^n l_{x+n} \div l_x$.

2. A sum to be received on a given life attaining a specified age is called an *Endowment*, and may be symbolized by ${}_n\mathrm{E}_x$; and we therefore have the equation

$$ {}_n\mathrm{E}_x = v^n l_{x+n} \div l_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

3. The endowment may be looked at from another point of view. The value of a sum, the receipt of which depends upon a contingency, is the present value of the sum taken as a certainty, multiplied by the probability of its being received; therefore the value of 1 payable in n years if (x) be then alive, is $v^n {}_np_x$. We can at once see the truth of this reasoning if we consider the case of l_x such sums. A person could afford to give $l_x \times v^n {}_np_x$ for the right to receive at the end of n years 1 for each of the survivors of the l_x nominees; because the sum to be certainly received would be l_{x+n}, and its present value $v^n l_{x+n}$; which is identical with $l_x v^n {}_np_x$. If therefore he can pay $l_x v^n {}_np_x$ for l_x endowments, it follows that the value of each of such endowments is $v^n {}_np_x$.

4. We have said that the *present value* of an endowment payable in n years if (x) be then alive is $v^n{}_np_x$; but in speaking of present value we must bear in mind the distinction between a sum certain, and a sum depending on a contingency. Where no contingency is involved, the present value is, that sum which, invested now, will at the end of the period named, have accumulated exactly to the sum due; but this definition is inapplicable to a contingent benefit. For instance, in the case of the endowment, if we accumulate its present value until the end of the n years, the amount will be ${}_np_x$, which is less than unity. If therefore there be only one such endowment granted, and if the nominee survive to receive it, his contribution accumulated will be insufficient to provide the amount which will then become payable. If, on the other hand, the nominee die before the end of the n years, there will be a sum in hand for which there will be no claimant. In speaking, therefore, of the present value of a single contingent benefit, we must always presuppose a sufficient number of such benefits to form an average; so that the contributions for those that never mature may be available to meet the deficiency in the contributions for those that actually become payable. We have been led to this remark in considering the simplest of all contingent benefits, endowments, but it applies equally to every other class—assurances, annuities, &c.

5. A *Life Annuity*, often called simply an *Annuity*, is a periodical payment depending on the continuance of a given life or combination of lives. The simplest annuity consists of 1 payable at the end of each year that (x) completes; and it is evident that the value of that annuity is the sum of the values of a series of endowments, payable at the end of 1, 2, 3, &c., years, if (x) be alive. If the first payment of the annuity is to be made at once, instead of at the end of the first year, the annuity is called an *Annuity-due*. Using the symbol a_x for the value of the annuity, we have, from equation 1,

$$a_x = \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \&c.}{l_x} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

6. With an ordinary mortality table there is no means of calculating exactly the value of an annuity, without computing in some way, either directly or indirectly, the value of each of the endowments of which it is composed. Unless we assume some suitable mathematical formula expressing l_x in terms of x, the series forming the numerator of the right-hand member of equation 2 cannot be summed without finding the value of each of the terms separately. If therefore the value of a single annuity be required, the operation of calculating it is very tedious; and with but little more labour a complete table for all higher ages than that of the nominee can be prepared. Thus

$$a_x = \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \&c.}{l_x}$$

$$= \frac{vl_{x+1}}{l_x} \times \frac{l_{x+1} + vl_{x+2} + v^2l_{x+3} + \&c.}{l_{x+1}}$$

$$= vp_x(1 + a_{x+1}) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Therefore, beginning with the oldest age in the table, and working backwards step by step, we can form a complete table of annuities, almost as easily as we can calculate the value of the annuity for the youngest age alone. In Chapter xxi we shall return to this subject.

7. By formula 2, $a_x = \frac{vl_{x+1} + v^2l_{x+2} + \&c.}{l_x}$. In the same way $a_{x+n} = \frac{vl_{x+n+1} + v^2l_{x+n+2} + \&c.}{l_{x+n}}$. It therefore appears that, in its present form, the expression does not assist us to find a_{x+n} when we have found a_x, because the successive values of l_x are not multiplied by the same powers of v in the two cases. But by a very simple algebraical artifice we may so modify the expression, as to make the calculations for the value of an annuity at one age available for the value of an annuity at any other age. If in formula 2 we multiply numerator and denominator of the second member by v^x, we do not change the value. We observe that the result is, that the index of v is now always the same as the suffix of l; and therefore, having formed the products of v^xl_x for every value of x, we can use them for the values of annuities for any ages whatsoever. The product v^xl_x we may represent by the symbol D_x, and in this notation

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \&c.}{D_x}$$

$$= \frac{N_x}{D_x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

where we write N_x for the sum of the values of D for ages $x+1$ and upwards.

8. It is convenient to arrange these preliminary values, D and N, side by side in columnar form, as is done in Table IV. The product

$v^x l_x$ may first be prepared for every age, and set out in a column; and then, commencing at the end of the table, we can work upwards and form successively the values of N.

9. The values in the D and N columns have no meaning whatever in themselves. They are only aids to enable us to find easily the values of annuities, and, as we shall see later on, of other benefits.

10. In constructing the function D, we multiply l by a power of v, the index of which is the same as the age; but any other power of v would do equally well, provided that all the values of the function be of identical form, and that an addition of one year to the age be accompanied by an increment of 1 in the index of the power of v. In the numerator of the right-hand member of equation 2 we see that the powers of v increase progressively with the age of the life; and this property must be retained in column D; but there is evidently no need for the index of the power of v to be the same as the age. It is only needful that when the division is effected, the powers of v in the numerator and denominator shall so far cancel each other, as to reduce the expression to the sum of the present values of 1 to be received at the end of 1, 2, 3, &c., years provided the nominee survive.

11. George Barrett was the first in Great Britain to apply the principle, which the following pages will prove to be most important, in the calculation of life contingencies. He constructed very extensive tables based upon it, which he appears to have commenced about the year 1786; but these tables were never printed. The manuscript of them is now in the library of the Institute of Actuaries. Francis Baily gave a description of Barrett's method, in a paper he read before the Royal Society in 1812, which was not inserted in the Philosophical Transactions; and he published a full account of it in 1813 in the Appendix to his *Doctrine of Life Annuities and Assurances.* Although to Barrett no doubt belongs the credit of an independent discovery, yet it appears that on the Continent he was forestalled by a German Professor, Johan Nicolaus Tetens, who gave the method to the world in a work on life annuities and reversions which he published in 1785.

12. Barrett, instead of multiplying l_x by v^x, multiplied by $(1+i)^{\omega-x}=v^x(1+i)^{\omega}$, where ω is the limiting age of the table; and the numbers so formed he placed in a column which he headed with the letter A. The column A he summed from the bottom upwards, and placed the results in a column headed B; inserting against age x in column B the sum of the numbers in column A from age x inclusive up to the oldest age in the table. Thus Barrett's A_x corresponds to our D_x, and his B_x corresponds to our N_{x-1}. By Barrett's tables $a_x = \frac{B_{x+1}}{A_x}$, while $\frac{B_x}{A_x} = 1 + a_x$. (N_x)

13. Professor Tetens constructed his first column exactly as we do our column D, and he headed it with the letter C; and his second column he formed exactly as Barrett did his, by summing the numbers in the first column from age x upwards.

14. Griffith Davies, in a tract which he published in 1825, gave to the tables the form which we at present use. He assigned to them the letters D and N, seemingly because N and D are the initial letters of the words *Numerator* and *Denominator* respectively, and the number in the N column forms the numerator of the expression for the value of the annuity, while the number in the D column forms the denominator. In order to make the expression for a_x symmetrical, he formed N_x by summing the column D from age $x+1$ upwards, and thus departed from the example of Tetens and Barrett; and the Institute of Actuaries in their *Life Tables* have adopted Davies's form. In life assurance transactions premiums are almost invariably made payable at the beginning of each year, and therefore are of the nature of an annuity-due, rather than of an annuity. By Barrett's tables the value of a premium of 1 per annum on a life aged x, is therefore $\frac{B_x}{A_x}$, while by Davies's tables the value is $\frac{N_{x-1}}{D_x}$. Since, in the practical application of the science of life contingencies, premiums are of much more frequent occurrence than annuities, Davies's alteration in the summation of the columns has the effect of destroying the symmetry of the great majority of the formulas, and is therefore of doubtful benefit; but, as it has been almost universally adopted, we shall also accept it. The actuary must however be careful before using a book of tables, to ascertain their form. Several writers, notably Farr and David Chisholm, have adhered to Barrett's summation while retaining Davies's symbol N. It is always easy by inspection to say which is the form of the tables. Looking at the values against the oldest age, $\omega-1$, if the table be according to Davies's system, $N_{\omega-1}$ is equal to nothing, while by Farr and Chisholm

$N_{\omega-1}=D_{\omega-1}$. Davies's form of the N column is sometimes conveniently distinguished as the *terminal form*, while that used by Farr and Chisholm has been called the *initial form*.

15. The form of the D column adopted by Tetens and Davies is a decided improvement on that of Barrett, because the numbers in it are sensibly smaller. v^x is always less than unity except when $x=0$; and therefore D_x, except for age 0, is always less than l_x; whereas $(1+i)^{\omega-x}$ is always greater than unity except when $x=\omega$, and therefore Barrett's A_x is always greater than l_x.

16. But few tables have been published in Barrett's form. We may however mention columns for the Carlisle Table at 3 per-cent interest, and for Babbage's Equitable Experience at $2\frac{1}{2}$ per-cent and 3 per-cent interest, given by Charles Babbage, F.R.S., in his work—*A Comparative View of the various Institutions for the Assurance of Lives*, published in 1826.

17. We have seen that the value of an endowment on (x), payable in n years, is $v^n {}_np_x$. The annuity being, as already observed, a series of endowments, its present value may be written

$$a_x=\Sigma v^n {}_np_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

18. In equation 21 of Chapter iii we have a similar expression for the curtate expectation of life, the difference being that, in the annuity the rate of interest is involved, while in the expectation it is not. In other words, the curtate expectation of life is the present value of an annuity on the assumption that capital yields no interest. It may therefore be inferred that, with the necessary modifications, all the formulas referring to expectations apply also to annuities; and we shall find, as we proceed, that this is so. Later on we shall find formula 5 of great assistance in simplifying expressions, especially those in connection with survivorship assurances.

19. In Art. 15 of Chapter iii we introduced a column headed N′ by which expectations of life might be calculated. We now see that, as explained in the foot-note to that article, the N′ column is the same as the ordinary N_x column, on the supposition that money is unproductive.

20. So far, we have discussed the annuity which is to be entered on at once, and is to continue for the whole remainder of life. But we may have *Deferred Annuities*, or *Temporary Annuities*, or *Deferred*

$\frac{v^n l_{x+n}}{l_x}$ $\frac{v^n \ldots}{l_x}$

Temporary Annuities called *Intercepted Annuities*. An annuity deferred n years, ${}_{n|}a_x$, is one which will be entered on at the end of n years, and the first payment of which will be made at the end of $n+1$ years if (x) be then alive. At the end of the n years the value of the annuity will be a_{x+n}; and we may therefore consider the deferred annuity to be equivalent to an endowment of a_{x+n} payable at the end of n years: thus we have the equation

$$ {}_{n|}a_x = v^n {}_np_x a_{x+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We may conveniently represent the deferred annuity by the symbol $a_{\overline{n}||x}$, meaning an annuity to (x) after the term of n years.

21. Seeing that the whole life annuity consists of an annuity on (x) for the next n years, together with an annuity on (x) deferred n years, we have the value of a temporary annuity

$$ {}_{|n}a_x = a_x - {}_{n|}a_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

22. An intercepted annuity being merely a deferred temporary annuity, we have immediately from the last two equations

n m

$$ {}_{n|m}a_x = {}_{n|}a_x - {}_{n+m|}a_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where the annuity is to be entered on at the end of n years, and then to continue for m years if (x) live so long.

23. If we have a table of whole-life annuities, and the ordinary column of l_x, formulas 6, 7, and 8, afford us the means of calculating without much trouble the values of deferred, temporary, and intercepted annuities; and, until commutation columns were invented, these formulas were employed. But with the aid of commutation columns the work can be very much abbreviated. We have seen that ${}_{n|}a_x = v^n {}_np_x a_{x+n}$. But the second member of this equation is

$$\frac{D_{x+n}}{D_x} \times \frac{N_{x+n}}{D_{x+n}} = \frac{N_{x+n}}{D_x}, \quad \text{since} \quad v^n {}_np_x = \frac{D_{x+n}}{D_x}, \quad \text{and} \quad a_{x+n} = \frac{N_{x+n}}{D_{x+n}};$$

therefore in commutation symbols

$\frac{v^n l_{x+n}}{l_x}$

deferred temporary

$$ {}_{n|}a_x = \frac{N_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

also temporary

$$ {}_{|n}a_x = \frac{N_x - N_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Intercepted

$$ {}_{n|m}a_x = \frac{N_{x+n} - N_{x+n+m}}{D_x} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$v^n \; {}_np_x$

24. If the annuity consist of but one payment, to be made at the end of the nth year if (x) be alive, it is an endowment, and equation 11 becomes

$$ {}_nE_x = \frac{D_{x+n}}{D_x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

Thus, with the help of the commutation columns, to find the value of a deferred annuity, or of an endowment, we have to perform only one division; and to find the value of a temporary annuity, or of an intercepted annuity, one subtraction and one division.

25. An erroneous notion not infrequently met with is that, in calculating the values of life annuities, the life may be represented by a term-certain equal to the expectation. It is easy to prove that the value of the life annuity is less than the value of an annuity-certain for the term of the curtate expectation.

Let $e_x = n + \delta$, where n is a whole number and δ a proper fraction: then

$$e_x = n + \delta = {}_1p_x + {}_2p_x + \&c. + {}_kp_x \quad ✓ . \; . \; . \; . \quad (A)$$

where $k = \omega - x$, or the complement of life. Also

$$a_x = v\,{}_1p_x + v^2\,{}_2p_x + \&c. + v^k\,{}_kp_x \quad ✓ . \; . \; . \; . \; . \quad (B)$$

and * $a_{\overline{n+\delta}|} = v + v^2 + \&c. + v^n + \delta v^{n+1}$.

The value of the life annuity will be greater or less than that of the annuity-certain according as

$$v\,{}_1p_x + v^2\,{}_2p_x + \&c. + v^k\,{}_kp_x > \text{or} < v + v^2 + \&c. + v^n + \delta v^{n+1} \quad . \; . \quad (C)$$

and therefore, according as

$$v^{n+2}\,{}_{n+2}p_x + v^{n+3}\,{}_{n+3}p_x + \&c. + v^k\,{}_kp_x > \text{ or } < v(1 - {}_1p_x) + v^2(1 - {}_2p_x) + \&c. + v^n(1 - {}_np_x) + v^{n+1}(\delta - {}_{n+1}p_x)$$

Now, since v is less than unity, the first member of this inequality is less than

$$v^{n+2}({}_{n+2}p_x + {}_{n+3}p_x + \&c. + {}_kp_x) \quad . \; . \; . \; . \quad (D)$$

---

* Here the annuity for $n + \delta$ years is taken as an annuity of n payments of 1 each, and one payment of δ to be made at the end of $n + 1$ years. This is done for convenience in the demonstration. The annuity of practice would have its last payment of δ made at the end of $n + \delta$ years, and the annuity of theory would have a last payment of $\frac{(1+i)^\delta - 1}{i}$ made at the end of $n + \delta$ years, and both of these would be of larger value than the annuity we have assumed.

and the second member is greater than

$$v^{n+1}(1-{}_1p_x+1-{}_2p_x+\&c.+1-{}_np_x+\delta-{}_{n+1}p_x)$$

$$>v^{n+1}(n+\delta-{}_1p_x-{}_2p_x-\&c.-{}_{n+1}p_x)$$

or from (A) $$>v^{n+1}({}_{n+2}p_x+{}_{n+3}p_x+\&c.+{}_kp_x)$$

But this last expression is greater than expression (D), and therefore, returning to (C), we observe that in all cases

$$v{}_1p_x+v^2{}_2p_x+\&c.+v^k{}_kp_x<v+v^2+\&c.+v^n+\delta v^{n+1}$$

that is, that the value of the life annuity is less than the value of an annuity-certain for the term of the curtate expectation of life. In symbols, $a_x<a_{\overline{e_x}|}$.

26. The reason for this result may be explained as follows:—The life annuity is, on the average, equivalent to an annuity-certain of which the payment at the end of the first year is ${}_1p_x$, and at the end of the second year ${}_2p_x$, &c., and the total payments of which aggregate ${}_1p_x+{}_2p_x+\&c.+{}_kp_x=n+\delta$. The annuity-certain is an annuity consisting of n payments of 1 each, and 1 payment of δ, and the total payments of which therefore also aggregate $n+\delta$. In each case therefore the total amount payable is on the average the same, but in the case of the life annuity the payments are spread over a longer period of time, and consequently are more affected by discount, and have a smaller present value.

27. We have shown mathematically that a_x is the present value of an annuity on the life of (x); that is, that if l_x annuities be granted on lives all aged x, each at the price a_x, the fund so formed, with its interest at the assumed rate, will be sufficient to provide the payments of the annuities as they fall due; and when the last life fails, the fund too will be exhausted. The mathematical reasoning is quite conclusive, but a numerical example may possibly assist to convey conviction to some minds. The principle is the same whatever age we assume for the lives, but the work will be shortened if we take an advanced age, say 90. The value at 3 per-cent interest of an annuity on a life aged 90 is 1·685961, and, by the Mortality Table, the number living at age 90 is 1273. The total fund at the outset to provide the annuities will therefore be 2146·228.

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δ

The following table shows the operation of the fund:—

(1) Year.	(2) Fund at commencement of Year.	(3) Interest earned in Year.	(4) Sum of (2) and (3).	(5) Amount paid to surviving Annuitants at end of Year.	(6) Fund at end of Year.
1	2,146·228	64·387	2,210·615	871	1,339·615
2	1,339·615	40·188	1,379·803	575	804·803
3	804·803	24·144	828·947	366	462·947
4	462·947	13·888	476·835	222	254·835
5	254·835	7·645	262·480	129	133·480
6	133·480	4·004	137·484	71	66·484
7	66·484	1·995	68·479	37	31·479
8	31·479	·944	32·423	19	13·423
9	13·423	·403	13·826	9	4·826
10	4·826	·145	4·971	4	·971
11	·971	·029	1·000	1	·000
12	...	...	...	...	...

The second column of the foregoing table shows the amount of the fund at the beginning of each year, and the third column the interest earned thereon in the year. In column 4 appears the amount of the fund at the end of the year just before the payment of the annuities then due. The fifth column gives the amount that has to be paid out at the end of each year in respect of the annuities, and the sixth column the remaining balance of the fund. It is seen that the fund decreases year by year, until at the end of eleven years it is completely exhausted, the last annuitant living eleven years to receive the annuity, but dying before the end of the twelfth year. In such an example, where great exactitude is required, it is necessary to pay attention to the decimal places, and we have therefore carried out the value of the initial annuity to six figures.

28. So far, we have considered benefits depending upon the *life* of the nominee, and we now proceed to another, but cognate, class which depend upon his *death*.

The one class we may briefly name *Annuities*, the other class *Assurances*. Assurances are very often called *Reversions.*

29. In dealing with assurances it is usual, unless the contrary be stated, to assume that the sum assured will become payable at the end of the year in which the contingency happens whereon the assurance depends; and for this purpose the years date from the moment at which the contract is entered into. They may conveniently be called

Assurance Years. Thus, if an assurance be effected on the life of (x), that is, if a contract be entered into for a sum payable on the death of (x), and if (x) die in the nth year from the date of the contract, the sum assured will become payable exactly at the end of the nth year. On the usual assumption that the deaths are equally distributed throughout each year of age, as many individuals will die in the first half of the year of death as in the second; and for the purposes of the calculations of an assurance office it is sufficient to assume that the deaths all take place in the middle of the year of age. Therefore, an assurance on the life of (x), is practically an assurance payable six months after his death.

30. It has been common in treatises on life contingencies to deal with annuities and assurances separately, but the two classes of benefits are so intimately connected that they ought always to be taken together. To illustrate this close relation we may at once find the value of the assurance in terms of the annuity, without independent mathematical demonstration. If there be an annuity on (x) payable at the end of each year on which he enters, $= v(1+a_x)$ and another annuity payable at the end of each year which he completes, $= a_x$ it is evident that the difference between the two is the value of 1 payable at the end of that year on which (x) enters but which he does not complete; that is, the ~~value~~ of an assurance of 1 payable at the end of the year of the death of (x), or in other words, the value of an assurance on (x). Now $v(1+a_x)$ is evidently the value of the first named annuity, and, deducting from this the value of the ordinary annuity, a_x, we have the value of the assurance, $v(1+a_x)-a_x$. To this value we shall assign the symbol A_x.

31. Having thus shown the close connection between A_x and a_x, we may now proceed to find A_x independently, although shortly we shall return to the connection already illustrated.

32. To find A_x.

If an office enter into l_x contracts for a unit payable at the end of the year of the death of each of the l_x individuals all aged x, the amount to be provided at the end of the first year, will be d_x; at the end of the second year, d_{x+1}; at the end of the third year, d_{x+2}; &c.; and the present value of all these amounts will be $vd_x+v^2d_{x+1}+v^3d_{x+2}+$&c. All of the assured, being of the same age, must subscribe equally; and the contribution of each must be

$$\frac{1}{l_x}(vd_x+v^2d_{x+1}+v^3d_{x+2}+\text{\&c.}).$$

That is,
$$A_x = \frac{vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \&c.}{l_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

33. We may again adopt the algebraical artifice of Art. 7, and multiply numerator and denominator of the last expression by v^x: the denominator then becomes D_x, and the numerator the sum of a series of similar terms of the form $v^{x+1}d_x$. Writing $C_x = v^{x+1}d_x$, we have

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \&c.}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$= \frac{M_x}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where we write M_x for the sum of all the values of C_x from age x to the oldest age in the table, inclusive.

The function C is not often required independently of M, and it is but seldom tabulated; but the function M is, in principle, one of the most important that appear in the commutation table, although, as mentioned in Art. 38, it can be dispensed with in practice without serious inconvenience.

34. The sum, whether single or periodical, which is payable in consideration of a benefit, is usually called the *Premium*. Thus A_x is the *Single Premium* for an assurance of 1 on the life of (x).

35. We have seen from equation 12 that the value of 1 payable at the end of the nth year if (x) survive that year, is $\frac{D_{x+n}}{D_x}$; and we have called the benefit an Endowment. In the same way, the value of 1 payable at the end of the nth year if (x) die in that year, is $\frac{C_{x+n-1}}{D_x}$. A few writers have called this benefit an *Endowment Assurance*, but we shall retain the name Endowment Assurance for an assurance payable at the end of n years if (x) be then alive, or at the end of the year of the death of (x) if that event happen within the n years.

36. The column of M_x bears the same relation to assurances that the column of N_x bears to annuities. The formulas for the two classes of benefits are therefore very similar. We may have deferred, temporary, and intercepted assurances. Using the symbols ${}_{n|}A_x$, ${}_{|n}A_x$, and ${}_{n|m}A_x$ respectively, for these benefits, we write,

$v^x l_{x+1} = D_{x+1}$

$v^{x+1} d_x = C_x$

Deferred
$$ {}_{n|}A_x = \frac{M_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$v^n {}_np_x \left(v \frac{d_{x+n}}{l_x} + \right.$

Temp
$$ {}_{|n}A_x = \frac{M_x - M_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

v^{x+n}

Intercept (Def-Temp)
$$ {}_{n|m}A_x = \frac{M_{x+n} - M_{x+n+m}}{D_x} \quad . \quad . \quad . \quad . \quad (18)$$

These last three formulas may be arrived at in another manner. In the case of the deferred assurance, if the life survive the n years, the value of the benefit will then be A_{x+n}. Its present value must therefore be A_{x+n} discounted for n years and multiplied by the chance that (x) will survive that period, that is, $v^n {}_np_x A_{x+n}$, which is easily reducible to $\frac{M_{x+n}}{D_x}$. The temporary assurance is the whole-life assurance less the deferred assurance; or, in symbols, ${}_{|n}A_x = A_x - {}_{n|}A_x$; also, ${}_{n|m}A_x = {}_{n|}A_x - {}_{n+m|}A_x$. By introducing commutation symbols, these last two equations become at once equations 17 and 18.

37. We said, Art. 3, that the value of a sum, the receipt of which depends upon a contingency, is the present value of the sum taken as a certainty, multiplied by the probability of its being received; and from this principle we deduced the equation $a_x = \Sigma v^n {}_np_x$. The present value of the assurance may be considered as the aggregate of the present values of a series of sums payable in the event of (x) dying in the first year, or in the second year, or in the third year, and so on. The present values of these sums, taken as certainties, are v, v^2, v^3, &c., respectively; and the probabilities of their being received are respectively q_x, ${}_{1|}q_x$, ${}_{2|}q_x$, &c. We may therefore write the equation

$$A_x = \Sigma v^n {}_{n-1|}q_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

When we give to ${}_{n-1|}q_x$ its values in terms of p, equation 19 becomes $A_x = \Sigma v^n ({}_{n-1}p_x - {}_np_x)$. By a simple application of equation 5, we find that $\Sigma v^n {}_{n-1}p_x = v(1 + a_x)$: also $\Sigma v^n {}_np_x = a_x$. Therefore equation 19 is transformed into $A_x = v(1 + a_x) - a_x$, the expression already found in Art. 30.

38. We have seen that D_x is equal to $v^x l_x$, and that C_x is equal to $v^{x+1} d_x$; whence, writing d_x in terms of l_x and l_{x+1}, $C_x = v^{x+1}(l_x - l_{x+1}) = vD_x - D_{x+1}$.

The relation $C_x = vD_x - D_{x+1}$ expresses in its simplest form the connection between the assurance and the annuity. From it, seeing that $\Sigma C_x = M_x$ and $\Sigma D_x = N_{x-1}$, we have the relation

$$M_x = vN_{x-1} - N_x \tag{20}$$

and this is the same relation, expressed in commutation symbols, which we found in Art. 30, because M_x may be held to represent the assurance, N_{x-1} the annuity-due, and N_x the ordinary annuity. By means of equation 20, the column of M_x may without much inconvenience be dispensed with when we have a column of N_x.

39. By algebraical transformations, a number of equations may be deduced connecting A_x and a_x. The student can work them out for himself. It will be sufficient here if we merely write them down. Starting with the equation found in Art. 30, we have

$$A_x = v(1+a_x) - a_x \quad \checkmark \tag{21}$$

$$= \frac{1-ia_x}{1+i} \quad \checkmark \tag{22}$$

$$= v(1-ia_x) \tag{23}$$

$$= v - (1-v)a_x \tag{24}$$

$$= 1 - d(1+a_x) \quad \checkmark \tag{25}$$

$$= \frac{a_\infty - a_x}{1+a_\infty} \tag{26}$$

40. Although all the equations are algebraically identical, yet for the most part each has its own meaning, the study of which is instructive. In Art. 30 we have already seen the meaning of equation 21. We will now take the others in their order.

✓ **41.** For equation 22.—A unit paid down now will produce i at the end of each year which (x) survives, together with $1+i$ at the end of the year of his death. The value of the interest during the life of (x) is therefore ia_x; and, deducting this from the original unit, we have $1-ia_x$ as the value of an assurance of $(1+i)$ on the life of (x); and by the simple proportion we find the value of the assurance of 1 to be $\frac{1-ia_x}{1+i}$.

42. Again, for equation 23.—If the assurance were due for certain at the end of the first year the amount to be paid to purchase it would

$\times(1+i)$

be v, but it will not be due till the end of the year in which (x) dies; therefore via_x, the value during the life of (x) of the interest on v the purchase-money, must be deducted, that is, $A_x = v - via_x = v(1 - ia_x)$.

43. Again, for equation 25.—If the unit were payable at once, its value would be 1; but as it is not payable till the end of the year of the death of (x), we must deduct the value of the interest. The value at the beginning of the year of the interest for the year on 1 is d; and therefore the present value of the interest for each year on which (x) enters is $d(1+a_x)$; deducting which, we have $A_x = 1 - d(1+a_x)$.

44. Finally, for equation 26.—Suppose that x is in possession of a perpetuity of 1 per annum. That which will remain of this after the death of (x) is evidently now worth $a_\infty - a_x$. But at the end of the year of the death of (x) its value will be $1 + a_\infty$, because his representatives will be put in possession of the 1 then due under the perpetuity, and the remainder of the perpetuity will be worth a_∞. Hence $a_\infty - a_x$ is the present value of an assurance of $1 + a_\infty$; and by simple proportion, $A_x = \dfrac{a_\infty - a_x}{1 + a_\infty}$.

45. From the foregoing formulas we can write a_x in terms of A_x:

Thus ✓

$$a_x = \frac{1 - (1+i)A_x}{i} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (27)$$

$$= \frac{v - A_x}{d} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

$$= \frac{1 - A_x}{d} - 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

$$= a_\infty - (1 + a_\infty)A_x \quad . \; . \; . \; . \; . \; . \; . \quad (30)$$

46. An instructive comparison may be instituted between formulas 21 and 22. The argument of formula 22 is based upon a single unit unchanged during the whole of life. For every year survived the annuitant draws only the interest, so that at the end of the year in which he dies the unit is entire; and, similarly, at the end of any year in which the interest ceases to be drawn, the unit is entire.

The argument of formula 21 is based upon a unit exchanged from year to year, but intercepted only in that year in which the life fails. At the end of each year entered upon, 1 is received; and at the end of each year survived, 1 is paid; so that only in the year in which the life fails will the 1 become free for the payment of the assurance.

47. The effect of this distinction is, that if in these formulas we substitute, for the annuities and annuities-due respectively, the corresponding annuities and annuities-due of n payments only, we obtain entirely different results.

$\frac{1-i\,a_{x\overline{n}|}}{1+i}=v-vi\,a_{x\overline{n}|}$

In the case of formula 22, $\frac{1-ia_x}{1+i}$ becomes $\frac{1-i\,{}_{|n}a_x}{1+i}=v-vi\,{}_{|n}a_x$, which, by analogy, from the argument of equation 23, we see to be the value of an assurance payable if (x) die within n years, or payable at the end of $n+1$ years whether (x) be then alive or dead.

In the case of formula 21, because $1+{}_{|n-1}a_x$ is the annuity-due of n payments, $v(1+a_x)-a_x$ becomes $v(1+{}_{|n-1}a_x)-{}_{|n}a_x$, which is the present value of 1 payable at the end of the year in which the life shall fail, provided only that that event shall happen within n years.

In other words, $\frac{1-i\,{}_{|n}a_x}{1+i}$ is the single premium for an endowment assurance payable at the end of $n+1$ years or at previous death; and $v(1+{}_{|n-1}a_x)-{}_{|n}a_x$ is the single premium for an assurance, temporary for n years.

48. By suitable modifications the formulas connecting A_x and a_x may be made to apply to deferred and to increasing benefits. If $\mathbf{a}_x$ represent an annuity-due on the life of (x), and ${}_{n|}\mathbf{a}_x$ an annuity-due on the life of (x) deferred n years, that is, an annuity first payment at the end of n years if (x) be then alive, formula 21 may be written $\mathrm{A}_x=v\mathbf{a}_x-a_x$. Substituting now deferred annuities for the immediate annuities, we have ${}_{n|}\mathrm{A}_x=v\,{}_{n|}\mathbf{a}_x-{}_{n|}a_x$. The correctness of this last result becomes apparent if we write the expression in commutation symbols, and remember the relation given in equation 20.

Now an assurance on (x), increasing 1 per annum, may be looked upon as the sum of an immediate assurance of 1, and of a series of assurances, each of 1, deferred 1, 2, 3, &c., years. If we represent such an assurance by the symbol $(\mathrm{IA})_x$, and correspondingly the values of an increasing annuity-due and an increasing annuity by $(\mathrm{I}\mathbf{a})_x$ and $(\mathrm{I}a)_x$ respectively, evidently we shall have the relation

$$(\mathrm{IA})_x=v(\mathrm{I}\mathbf{a})_x-(\mathrm{I}a)_x.$$

Since it can be easily seen that $(\mathrm{I}\mathbf{a})_x=1+a_x+(\mathrm{I}a)_x$, the last equation may be put in the form $(\mathrm{IA})_x=v\{1+a_x+(\mathrm{I}a)_x\}-(\mathrm{I}a)_x$. We thus see that, having the value of the increasing annuity, we can at once find the value of the corresponding increasing assurance, and *vice versâ*.

$Ax - v\ddot{a}_x - a_x$

$A_{x\overline{n}|} = v\ddot{a}_{x\overline{n}|} - a_{x\overline{n}|}$

49. So far we have treated of annuities and assurances on one life only, but all the arguments and demonstrations equally apply to benefits depending on the joint duration of any number of lives. If $a_{wxyz\ldots(m)}$ be the symbol for an annuity on m joint lives, and $A_{wxyz\ldots(m)}$ that for the corresponding assurance, we must in equations 5 and 19 substitute ${}_np_{wxyz\ldots(m)}$ and ${}_{n-1|}q_{wxyz\ldots(m)}$ respectively, for ${}_np_x$ and ${}_{n-1|}q_x$. The formulas then become

$$a_{wxyz\ldots(m)}=\Sigma v^n{}_np_{wxyz\ldots(m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and

$$A_{wxyz\ldots(m)}=\Sigma v^n{}_{n-1|}q_{wxyz\ldots(m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Moreover, equations 21 to 30 hold for joint lives as well as for single lives. In fact, the principle remains the same, whether we have one life or many, provided only that the annuity be 1 per annum, and that the assurance be also 1, and payable one year after the last payment of the annuity.

50. A term-certain may frequently be substituted for a life. Thus an annuity payable for the next n years if (x) live so long, may be considered as an annuity payable during the joint duration of the life of (x) and of the term of n years certain; and it may be expressed in symbols ${}_{|n}a_x=a_{x\overline{n}|}$. Similarly, an assurance payable if (x) die within n years, or payable at the end of n years if he be then alive, may be looked upon as a joint-life assurance, payable on the failure of the life of (x) or on the failure of the term of n years certain, whichever may first happen; and may be written $A_{x\overline{n}|}$. The symbols $a_{x\overline{n}|}$ and $A_{x\overline{n}|}$ very conveniently express the benefits, and at the same time they graphically display important principles. We shall therefore as a rule employ them in preference to the alternative symbols ${}_{|n}a_x$, and ${}_{|n}\text{Æ}_x$, which may also be used.

51. In speaking of the connection between the annuity and the assurance, we remarked in Art. 49 that one of the conditions that must exist in order that equations 21 to 30 may hold is, that the assurance must be payable one year after the last payment of the annuity. It is therefore evident that the annuity corresponding to $A_{x\overline{n}|}$ is $a_{x\overline{n-1}|}$ and not $a_{x\overline{n}|}$. We write, instead of equation 21,

$$A_{x\overline{n}|}=v(1+a_{x\overline{n-1}|})-a_{x\overline{n-1}|} \quad . \quad . \quad . \quad . \quad (33)$$

and the other equations 22 to 30 may be similarly modified.

52. Commutation columns may be prepared for joint lives as well as for single lives, and in theory there is no limit to the number of lives that may be involved; but when the number of lives is greater than two, the extent of complete tables would necessarily be enormous, and in practice they have never been undertaken.

53. Commutation columns for two joint lives have been suggested on two different plans. The form adopted by Griffith Davies is

$$\mathrm{D}_{xy}=v^{x}l_{x}l_{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where (x) is the older of the two lives. The power of v is here made to depend only on the age of (x), although the lives are symmetrically involved; and where ordinary benefits only are under consideration, no inconvenience is experienced. But if the order of survivorship of the lives be in question, as for instance in the case of contingent assurances (to be discussed later on), we shall find that Davies's form of Joint-Life Commutation columns involves us in complexity; two cases requiring solution, namely, for $x>y$, and $x<y$, and we might almost add a third, namely, for $x=y$. In order to obviate this inconvenience, De Morgan proposed another form of commutation column, wherein the power of v would depend equally on the ages of (x) and (y); thus,

$$\mathrm{D}_{xy}=v^{\frac{x+y}{2}}l_{x}l_{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

It will be observed that in both Davies's and De Morgan's forms the power of v increases by unity for each addition of one year to the ages of both (x) and (y); and this is in accordance with the principle explained in Art. 10. In De Morgan's form, if (x) be taken one year older, while (y) remains the same as before, we shall have $\mathrm{D}_{x+1:y}=v^{\frac{x+y}{2}+\frac{1}{2}}l_{x+1}l_{y}$, while by Davies's form $\mathrm{D}_{x+1:y}=v^{x+1}l_{x+1}l_{y}$.

54. In the same way we may have $\mathrm{C}_{xy}=v^{x+1}(l_{x}l_{y}-l_{x+1}l_{y+1})$, or $\mathrm{C}_{xy}=v^{\frac{x+y}{2}+1}(l_{x}l_{y}-l_{x+1}l_{y+1})$, according as we choose Davies's or De Morgan's form of column. Columns N_{xy} and M_{xy} may be formed from D_{xy} and C_{xy} respectively, in precisely the same manner as N_{x} and M_{x} are formed from D_{x} and C_{x} respectively. Owing to the necessarily voluminous nature of joint-life tables, it is not usual to calculate both N_{xy} and M_{xy}; but for assurances use is made of the relation $\mathrm{M}_{xy}=v\mathrm{N}_{x-1:y-1}-\mathrm{N}_{xy}$, which holds equally for joint lives and single lives. The same symbols have always been used for joint-life commutation columns whether they are in Davies's or De Morgan's form. Care must be exercised not to fall into error on this account.

55. If (x) wish to assure his life for 1, he may do so by paying a single premium of A_x, but it may be inconvenient to him at once to part with so large a sum, and he may prefer to pay an annual equivalent. The sum A_x may thus be looked upon as the purchase-money of an annuity-due, because it is customary with assurance offices to make the annual premiums payable at the commencement and not at the end of each year. Since $1+a_x$ is the purchase-money for an annuity-due of 1 per annum~~, unity~~ will purchase an annuity-due of $\frac{1}{1+a_x}$ per annum, and A_x will purchase an annuity-due of $\frac{A_x}{1+a_x}$ per annum, which is therefore the annual premium corresponding to the single premium A_x. Using the symbol P_x, we have as the first equation for the annual premium

$$P_x = \frac{A_x}{1+a_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

56. It is instructive to view the same question from different standpoints, and if possible to discover the principles which underlie the mathematical formulas. The student can hardly be expected to retain in his memory complicated collections of symbols, but if he thoroughly understand the meaning of the formulas, he will find no difficulty in reproducing them at any moment when he requires them. We shall therefore now give further study to expressions for the annual premium.

57. When considering the relation between A_x and a_x, we found that sometimes by merely substituting an equivalent algebraical form, we effected a very great change in the meaning of the formula. The same remark applies to P_x.

58. If in equation 36 we give to a_x its value in terms of A_x obtained from equation 29, we shall have

$$P_x = \frac{dA_x}{1-A_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

an algebraical equivalent of equation 36 but which yet has an interpretation of its own. If (x), in assuring his life, be unwilling to pay the single premium at once, but agree to pay interest on it at the beginning of each year during his life, and to allow the single premium to be deducted from the sum assured at his death, the interest he must pay annually in advance will be dA_x, and for this his representatives will receive $1-A_x$. Therefore dA_x is the annual premium for an assurance of $1-A_x$; and, by simple proportion, the annual premium for an assurance of unity is $\frac{dA_x}{1-A_x}$.

59. This last argument may be repeated in a somewhat modified shape. If (x) do not wish to pay down a single premium, but prefer to make arrangements for his representatives to pay it at his death, he must agree to pay upon it interest in advance, that is dA_x, annually. But as the single premium is not to be paid till his death he must provide security for it by assuring it at a single premium, which is $A^2{}_x$. If this second single premium is not to be paid down now, (x) must at the beginning of each year pay $dA^2{}_x$ interest on it, and also pay a single premium of $A^3{}_x$ to assure it. Similarly if this third single premium is not to be paid down now, (x) must at the beginning of each year pay $dA^3{}_x$ interest on it, and a single premium of $A^4{}_x$ to assure it. A similar arrangement must be made for the fourth single premium if it is not to be paid at once, and so on for the fifth, and sixth, &c., without limit. The total interest that (x) will have to pay yearly in advance will therefore be $d(A_x+A^2{}_x+A^3{}_x+\&c.)$; and he must also pay a single premium of A^∞_x, which however is zero. The expression in the brackets is a rapidly converging infinite geometrical series, the sum of which is $\frac{A_x}{1-A_x}$. Therefore the total interest that (x) will have to pay annually in advance is $\frac{dA_x}{1-A_x}$; and for this his representatives will receive 1 at his death, as well as the sum of $(A_x+A^2{}_x+A^3{}_x+\&c.)$, which however does not benefit them, because they must, according to arrangement, at once pay it back to the office. Thus it is proved that $\frac{dA_x}{1-A_x}$ is the annual premium for the assurance of 1.

60. Yet another slight modification of the argument may enable the student to grasp the idea which we have been trying to bring within his reach. If (x) be unwilling to pay for his assurance by a single premium now, and wish to defer the payment till the end of the year of his death, the single premium on which interest must be charged must evidently be that single premium which will provide for the return of itself along with the sum assured, because at the end of the year of the death of (x) the assurance office must pay the sum assured, and also secure to itself the single premium on which until now it has received only interest. If $\mathcal{A}_x$ represent such a single premium, its value is easily found. The

benefit assured is $1+A_x$, and the present value of the same is $\mathrm{A}_x(1+A_x)$, which from the nature of the case must be equal to A_x. Whence $A_x=\frac{\mathrm{A}_x}{1-\mathrm{A}_x}$. The interest that (x) must pay at the beginning of each year is therefore $\frac{d\mathrm{A}_x}{1-\mathrm{A}_x}$, the same as before.

61. If in equation 36 we substitute for A_x its value in terms of a_x as given in equation 25, we have

$$\mathrm{P}_x=\frac{1}{1+a_x}-d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

and this formula may be interpreted as follows. The annual equivalent payable in advance during the life of (x) for 1 paid down now, is $\frac{1}{1+a_x}$; and this includes the annual interest, d, on the unit during the life of (x), together with an annual sum to provide for the repayment of the unit at his death. But since in the case of the assurance the unit is not to be paid down at once, but only at the end of the year of the death of (x), therefore there is no interest payable during the life of (x), and we must deduct the annual interest from the annual equivalent just found. The remainder, $\frac{1}{1+a_x}-d$, is P_x, the annual equivalent payable in advance for 1 due at the end of the year of the death of (x).

62. Another, and perhaps a better interpretation of the formula is as follows:—

Two persons hold annuities of 1 per annum, and in each case the first annual payment is due at the end of one year. One of these annuities is payable in perpetuity, and the other is a life annuity. Both annuitants desire their annuities to be converted into others payable at the beginning of the year, and agree for that object to sacrifice a portion of their income. Now, as in the case of the life annuitant no payment would originally have been due for the year of death, so he must give up the larger annual sum, because under the new arrangement he obtains the greater advantage, and this additional sacrifice must be the annual premium to provide for the additional 1 which he will obtain for the year of his death, and which previous to the proposed alteration would not have been payable. In each case the annual payments must be reduced in the proportion of the value of an annuity to the value of an annuity-due. In the case of the perpetuity, the ratio is $\frac{a_\infty}{1+a_\infty}$; and the annual sacrifice of income is $1-\frac{a_\infty}{1+a_\infty}=d$. In the case of the life annuity, the ratio is $\frac{a_x}{1+a_x}$, and the annual sacrifice of income is $1-\frac{a_x}{1+a_x}=\frac{1}{1+a_x}$. As already observed, the difference, $\frac{1}{1+a_x}-d$, is P_x, the annual premium.

63. We have seen, (equation 36), that to obtain the annual premium we must divide the single premium by the annuity-due; and by expressing the single premium in terms of the annuity as given in equation 25 we obtained equation 38. The expression for the annual premium will take different forms according as we use one or other of the alternative forms for A_x given in Art. 39.

For example, taking equation 21, we have $P_x=v-\frac{a_x}{1+a_x}$, and taking equation 25 we have $P_x=\frac{1}{1+a_x}-d$ already deduced. It is interesting to note the effects of substituting term annuities of n payments, for the whole-life annuities and annuities-due in these two identical expressions for P_x. The first expression becomes $v-\frac{a_{x\overline{n}|}}{1+a_{x\overline{n-1}|}}$; and by Art. 47 this is equal to $\frac{A^{1}_{x\overline{n}|}}{1+a_{x\overline{n-1}|}}$, where $A^{1}_{x\overline{n}|}$ is used as a symbol for the single premium for a temporary assurance on (x). Therefore the result is the annual premium for the temporary assurance, which we may write $P^{1}_{x\overline{n}|}$. The second expression becomes $\frac{1}{1+a_{x\overline{n-1}|}}-d=\frac{1}{1+a_{x\overline{n-1}|}}-1+v=\frac{v(1+a_{x\overline{n-1}|})-a_{x\overline{n-1}|}}{1+a_{x\overline{n-1}|}}=\frac{A_{x\overline{n}|}}{1+a_{x\overline{n-1}|}}$ by equation 33; and this is the annual premium of n payments for an endowment assurance payable at the end of n years or at previous death. This we may write $P_{x\overline{n}|}$.

64. Algebraically transforming equation 38, we have an expression for the annuity in terms of the annual premium

$$a_x=\frac{1}{P_x+d}-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

the meaning of which may be explained as follows:—If the annual sum payable under an annuity-due were P_x+d, the value of the annuity-due would be 1, as the purchaser would secure throughout his life the interest payable in advance, d, on his purchase-money, and by the premium P_x

the return of the capital to his representatives at the end of the year of his death. For an annuity-due of 1 per annum he can therefore afford to give $\frac{1}{P_x+d}$, and for an ordinary annuity, $\frac{1}{P_x+d}-1$.

65. In Art. 27 we showed by a numerical illustration that a fund formed by investing the value of l_x annuities at the assumed rate, will provide for the payment of the annuities as they fall due; and that, when the last life fails, the fund too will be exhausted. The two following tables supply a similar illustration for assurance premiums, the first treating of single, and the second of annual premiums.

66. In the case of the first, it is supposed that 1273 persons aged 90 assure their lives for 1 each at a single premium of ·921768. The fund of 1173·4107 receives an annual increment of interest at 3 per-cent, and at the end of each year 1 has to be paid out for each death that has occurred in the year. It will be seen that the fund steadily diminishes, until at the end of 12 years it has become completely exhausted.

67. The second table is very similar, but perhaps more instructive. Besides the increment of interest, the fund is increased by an annual income from premium, each survivor having to pay in a premium of ·34318 at the beginning of each year. In this case it will be observed that the fund does not diminish from the beginning. On the contrary, it shows a tendency at the commencement to increase: in fact, if we were to prepare a table for a young age at entry, say 20, we should find that for many years the fund would continue to grow, and only when the surviving lives became advanced in age would the fund begin to diminish. This elementary illustration shows that it is not a proof that an insurance company is prosperous when its investments are accumulating. Even with growing funds it may be steadily advancing towards insolvency, because, although there may be an increase of assets, it may be less than that required under normal conditions such as are assumed in the table. On the other hand the assurance fund of an old company may be dwindling, and yet the company may in reality be accumulating surplus wealth.

Sufficiency of Single Premium.

Single Premium = ·921768.

(1) Year.	(2) Fund at commencement of Year.	(3) Interest earned in Year.	(4) Sum of (2) and (3).	(5) Claims.	(6) Fund at end of Year.
1	1,173·4107	35·2023	1,208·6130	402	806·6130
2	806·6130	24·1984	830·8114	296	534·8114
3	534·8114	16·0443	550·8557	209	341·8557
4	341·8557	10·2557	352·1114	144	208·1114
5	208·1114	6·2434	214·3548	93	121·3548
6	121·3548	3·6408	124·9956	58	66·9956
7	66·9956	2·0099	69·0055	34	35·0055
8	35·0055	1·0501	36·0556	18	18·0556
9	18·0556	·5417	18·5973	10	8·5973
10	8·5973	·2579	8·8552	5	3·8552
11	3·8552	·1157	3·9709	3	·9709
12	·9709	·0291	1·0000	1	·0000

Sufficiency of Annual Premium.

Annual Premium = ·34318.

(1) Year.	(2) Fund at commencement of Year.	(3) Premium paid at commencement of Year.	(4) Sum of (2) and (3).	(5) Interest earned in Year.	(6) Sum of (4) and (5).	(7) Claims.	(8) Fund at end of Year.
1	000·0000	436·8681	436·8681	13·1060	449·9741	402	47·9741
2	47·9741	298·9098	346·8839	10·4065	357·2904	296	61·2904
3	61·2904	197·3285	258·6189	7·7586	266·3775	209	57·3775
4	57·3775	125·6039	182·9814	5·4894	188·4708	144	44·4708
5	44·4708	76·1860	120·6568	3·6198	124·2766	93	31·2766
6	31·2766	44·2702	75·5468	2·2664	77·8132	58	19·8132
7	19·8132	24·3658	44·1790	1·3254	45·5044	34	11·5044
8	11·5044	12·6977	24·2021	·7262	24·9283	18	6·9283
9	6·9283	6·5204	13·4487	·4036	13·8523	10	3·8523
10	3·8523	3·0886	6·9409	·2083	7·1492	5	2·1492
11	2·1492	1·3727	3·5219	·1057	3·6276	3	·6276
12	·6276	·3432	·9708	·0292	1·0000	1	·0000

68. In Art. 60 we proved that the annual premium is merely interest on that single premium which is such as to provide for the return of itself along with the sum assured at the end of the year of death. It will be well here to display this fact numerically. Calculating for age 90 the special single premium by means of the formula $\mathit{A}_x = \frac{\mathrm{A}_x}{1-\mathrm{A}_x}$, we have $A_{90}=11{\cdot}783$ and $dA_{90}={\cdot}3432$, which we have seen to be the annual premium.

69. In Art. 55 we found the annual premium by looking at the single premium as the purchase-money of an annuity. But it is unnecessary to consider the annuity as being for the whole of life. By taking a temporary annuity-due of t payments only, we shall find the annual premium payable for t years and then to cease, although the benefit of the assurance is still to continue. Such a premium is frequently called a *Premium by Limited Payments*, or sometimes simply a *Limited Premium*, and assigning to it the symbol ${}_tP_x$, we have the equation

$$ {}_tP_x=\frac{A_x}{1+a_{x\overline{t-1}|}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

This last premium must not be confused with the annual premium for an assurance temporary for t years, which is $P^1_{x\overline{t}|}=\dfrac{A^1_{x\overline{t}|}}{1+a_{x\overline{t-1}|}}$ already found in Art. 63.

70. It is evident that we can vary indefinitely the way in which the annual premium shall be paid, simply by finding the value of the corresponding annuity. This question will be more fully discussed in Chapter xvi on Varying Benefits.

71. In Art. 2 we found ${}_tE_x$, the value of an endowment payable at the end of t years if x be then alive, or, in other words, the single premium for the endowment. As this benefit is an assurance payable on the failure of the term-certain of t years if (x) be then alive, we may, in analogy to the notation used in Chapter iv for survivorship probabilities, attach to it the graphic symbol $A_{x\overset{1}{\overline{t}|}}$. The annual premium $P_{x\overset{1}{\overline{t}|}}$ for this benefit is the annuity-due of t payments which the value of the benefit will purchase; that is

$$P_{x\overset{1}{\overline{t}|}}=\frac{A_{x\overset{1}{\overline{t}|}}}{1+a_{x\overline{t-1}|}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

72. Returning now to the temporary assurance, we found, Art. 47, that $A^1_{x\overline{t}|}=v(1+a_{x\overline{t-1}|})-a_{x\overline{t}|}$, and, Art. 63, $P^1_{x\overline{t}|}=v-\dfrac{a_{x\overline{t}|}}{1+a_{x\overline{t-1}|}}$.

Neither of these formulas is very convenient for the purposes of computation when only an ordinary table of annuities is available. Each formula involves both $a_{x\overline{t-1}|}$ and $a_{x\overline{t}|}$, and these functions have to be independently calculated. But we can, by writing $a_{x\overline{t-1}|}=a_{x\overline{t}|}-v^t{}_tp_x$, make the formulas more workable, although in appearance more complex. We shall have

$$\begin{aligned}\overset{1}{A}_{x\overline{t}|}&=v(1+a_{x\overline{t-1}|})-a_{x\overline{t}|}\\&=v(1+a_{x\overline{t}|}-v^t{}_tp_x)-a_{x\overline{t}|}\\&=v(1-v^t{}_tp_x)-(1-v)a_{x\overline{t}|}\quad\ldots\ldots\quad(42)\end{aligned}$$

and

$$\begin{aligned}\overset{1}{P}_{x\overline{t}|}&=\frac{\overset{1}{A}_{x\overline{t}|}}{1+a_{x\overline{t-1}|}}\\&=\frac{v(1-v^t{}_tp_x)-(1-v)a_{x\overline{t}|}}{1+a_{x\overline{t-1}|}}\\&=\frac{1-v^t{}_tp_x-(1-v)(1-v^t{}_tp_x+a_{x\overline{t}|})}{1-v^t{}_tp_x+a_{x\overline{t}|}}\\&=\frac{1-v^t{}_tp_x}{1-v^t{}_tp_x+a_{x\overline{t}|}}-d\quad\ldots\ldots\ldots\ldots\quad(43)\end{aligned}$$

In both cases we have reduced the expressions to terms involving only $v^t{}_tp_x$ and $a_{x\overline{t}|}$; and since $v^t{}_tp_x$ has to be found in calculating $a_{x\overline{t}|}$, because $a_{x\overline{t}|}=a_x-v^t{}_tp_xa_{x+t}$, the work is reduced to a minimum.

73. In Art. 63 we incidentally found an expression for the annual premium for an endowment assurance. We found that

$$\boxed{P_{x\overline{n}|}=\frac{1}{1+a_{x\overline{n-1}|}}-d}\quad\ldots\ldots\quad(44)$$

This formula may be interpreted as follows in a way very similar to that applied in Art. 61 to the corresponding formula for P_x.

The annual equivalent payable in advance during the next n years if (x) survive, for 1 paid down now, is $\dfrac{1}{1+a_{x\overline{n-1}|}}$, and this includes the annual interest, d, on the unit during the currency of the annuity, together with an annual sum to provide for the repayment of the unit when the annuity ceases, that is, at the death of (x) or at the end of n years. But, since in the case of the endowment assurance the unit is not paid down until the annuity ceases, the annual interest is not required, and we must deduct d from the annual equivalent just found. The remainder $\dfrac{1}{1+a_{x\overline{n-1}|}}-d$ is the annual sum to provide for the unit when the annuity ceases, that is $\mathrm{P}_{x\overline{n}|}$, the annual premium for the endowment assurance.

74. By assigning to $\mathrm{A}_{x\overset{1}{\overline{n}|}}$ in equation 41 its value given in equation 1, we have $\dfrac{v^n{}_np_x}{1+a_{x\overline{n-1}|}}$ as the annual premium for the endowment. Since an endowment assurance is simply a temporary assurance together with an endowment, therefore to find the premium for a temporary assurance from the premium for an endowment assurance, we have merely to deduct the premium for the endowment. That is,

$$\begin{aligned}\mathrm{P}^{1}_{x\overline{n}|} &= \mathrm{P}_{x\overline{n}|} - \mathrm{P}_{x\overset{1}{\overline{n}|}}\\ &= \frac{1}{1+a_{x\overline{n-1}|}} - d - \frac{v^n{}_np_x}{1+a_{x\overline{n-1}|}}\\ &= \frac{1-v^n{}_np_x}{1+a_{x\overline{n}|}-v^n{}_np_x} - d\end{aligned}$$

the same result as found in formula 43.

75. From Art. 49, and equations 31 and 32 there given, it follows that

$$\begin{aligned}\mathrm{P}_{wxyz\ldots(m)} &= \frac{\mathrm{A}_{wxyz\ldots(m)}}{1+a_{wxyz\ldots(m)}}\\ &= \frac{1}{1+a_{wxyz\ldots(m)}} - d \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)\end{aligned}$$

In fact, as already remarked, all formulas applicable to single lives, hold equally for joint lives.

76. Commutation tables afford great assistance in the calculation of premiums. We have already given formulas in commutation symbols for the single premiums for endowments, annuities, and assurances; and it now remains to find formulas for the annual premiums.

Since ${}_n\mathrm{E}_x = \dfrac{\mathrm{D}_{x+n}}{\mathrm{D}_x}$, and $1+a_{x\overline{n-1}|} = \dfrac{\mathrm{N}_{x-1}-\mathrm{N}_{x+n-1}}{\mathrm{D}_x}$, it follows that

$$\mathrm{P}_{x\overset{1}{\overline{n}|}} = \frac{\mathrm{D}_{x+n}}{\mathrm{N}_{x-1}-\mathrm{N}_{x+n-1}} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

It will be observed that D_x being in both numerator and denominator, disappears from the final formula. This is always the case in annual premiums (except when in part of the expression one life only is involved and in another part two); and therefore in the operations D_x may be neglected.

77. For a deferred annuity, when a premium is to be paid at the beginning of each year until the annuity is entered on, we have for the annual premium, which we shall denote by $P_{n|}a_x$,

$$P_{n|}a_x = \frac{N_{x+n}}{N_{x-1} - N_{x+n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

78. Here we may remark that, when we prefix the letter P without suffix, to the symbol for a benefit, we mean the annual premium for the benefit. Thus $PA_{\overset{1}{x}\overline{n}|} = P_{\overset{1}{x}\overline{n}|}$, and $P_nE_x = P_{x\overset{1}{\overline{n}|}}$. It may frequently happen that we shall find it convenient to have an alternative symbol for a function; and for some functions, such as $P_{n|}a_x$, it is not easy to devise a suitable single symbol in harmony with the scheme of notation.

79. Because $A_x = \dfrac{M_x}{D_x}$, and $(1 + a_x) = \dfrac{N_{x-1}}{D_x}$, therefore

$$P_x = \frac{M_x}{N_{x-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

80. This is a very useful expression where both M and N columns are tabulated; but it often happens that the M column is not given, and therefore in this and other cases it is useful to have a second formula by means of which we may dispense with M. By writing, in equation 38, $1 + a_x$ in commutation symbols, we have

$$P_x = \frac{D_x}{N_{x-1}} - d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Similarly

$$P_{xy} = \frac{D_{xy}}{N_{x-1:y-1}} - d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$_nP_x = \frac{M_x}{N_{x-1} - N_{x+n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$P_{x\overset{1}{\overline{n}|}} = \dfrac{D_{x+n}}{N_{x-1} - N_{x+n-1}}$

$$P_{\overset{1}{x}\overline{n}|} = \frac{M_x - M_{x+n}}{N_{x-1} - N_{x+n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$= \frac{D_x - D_{x+n}}{N_{x-1} - N_{x+n-1}} - d \quad . \quad . \quad . \quad . \quad . \quad (53)$$

$$= v - \frac{N_x - N_{x+n}}{N_{x-1} - N_{x+n-1}} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1}} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$= \frac{D_x}{N_{x-1} - N_{x+n-1}} - d \quad . \quad . \quad . \quad . \quad . \quad (56)$$

The last formula for the annual premium for an endowment assurance is a very convenient one, as it not only avoids the use of the M column, but it requires fewer values to be extracted from the book of tables than does formula 55.

81. To find $a_{\overset{r}{\overline{wxyz\ldots(m)}}}$, the value of an annuity to continue as long as at least r lives out of m lives survive.

The value of the n^{th} payment of the annuity is $v^n{}_np_{\overset{r}{\overline{wxyz\ldots(m)}}}$, and the value of the annuity is $\Sigma v^n{}_np_{\overset{r}{\overline{wxyz\ldots(m)}}}$. But, Chap. ii, formula 15,

$$v^n{}_np_{\overset{r}{\overline{wxyz\ldots(m)}}}=v^n\left\{\mathrm{Z}^r-r\mathrm{Z}^{r+1}+\frac{r(r+1)}{\underline{|2}}\mathrm{Z}^{r+2}-\&c.\right\}$$

where each power of Z represents the sum of certain probabilities of the form ${}_np_{wxyz\ldots(t)}$. It therefore appears that to obtain $\Sigma v^n{}_np_{\overset{r}{\overline{wxyz\ldots(m)}}}$ we must take the sum, for all values of n, of v^n multiplied into the corresponding value of each of the probabilities included in each power of Z. But by formula 31, $\Sigma v^n{}_np_{wxyz\ldots(t)}=a_{wxyz\ldots(t)}$; and therefore, when we take the sum for n of the discounted probabilities included in a power of Z, we have simply to replace the probabilities by the corresponding joint-life annuities. Extending in this sense the meaning of Z, so that, when dealing with annuities, Z^r signifies the sum of the values of the annuities on r joint lives for all the combinations of r lives that can be made out of m lives, we have

$$a_{\overset{r}{\overline{wxyz\ldots(m)}}}=\mathrm{Z}^r-r\mathrm{Z}^{r+1}+\frac{r(r+1)}{2}\mathrm{Z}^{r+2}-\&c. \quad . \quad . \quad . \quad (57)$$

$$=\frac{\mathrm{Z}^r}{(1+\mathrm{Z})^r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

in accordance with Chap. ii, formula 16, and Chap. iii, formula 30.

It will be noticed that this investigation as regards annuities is identical with the investigation as regards expectations of life (Chap. iii, Art. 29). We have only to substitute the symbol a for the symbol e.

82. An interesting case occurs when we seek the value of an annuity on the last survivor of m lives of equal ages. The annuities included under each power of Z become joint-life annuities on lives of equal ages, and under Z^n there are (m, n) such annuities, say $\dfrac{\underline{|m}}{\underline{|n}\;\underline{|m-n}}$, each on n joint lives. Making this substitution in formula 57, and taking $r=1$, and assuming the common age of the lives to be x, we have

$$a_{\overline{xxx\,..\,(m)}} = ma_x - \frac{m(m-1)}{2}\,a_{xx} + \frac{m(m-1)(m-2)}{\underline{|3}}\,a_{xxx} - \&c. \;. \quad (59)$$

83. We have already seen in other investigations that a term certain may be substituted for a life. Thus, by formula 57, $a_{\overline{x\overline{n}|}} = a_{\overline{n}|} + a_x - a_{x\overline{n}|}$. This is the value of an annuity granted for a term certain, n, and then to continue as long as a life now aged x may survive.

84. As an example of the foregoing formulas, let it be required to determine, when $m=4$, the values of $a_{\overset{\quad 1}{\overline{wxyz}}}$, $a_{\overset{\quad 2}{\overline{wxyz}}}$, $a_{\overset{\quad 3}{\overline{wxyz}}}$, and $a_{\overset{\quad 4}{\overline{wxyz}}}$.

$$\begin{aligned}
a_{\overset{\quad 1}{\overline{wxyz}}} &= Z - Z^2 + Z^3 - Z^4 \\
&= (a_w + a_x + a_y + a_z) \\
&\quad - (a_{wx} + a_{wy} + a_{wz} + a_{xy} + a_{xz} + a_{yz}) \\
&\quad + (a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz}) \\
&\quad - a_{wxyz}.
\end{aligned}$$

$$\begin{aligned}
a_{\overset{\quad 2}{\overline{wxyz}}} &= Z^2 - 2Z^3 + 3Z^4 \\
&= a_{wx} + a_{wy} + a_{wz} + a_{xy} + a_{xz} + a_{yz} \\
&\quad - 2(a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz}) \\
&\quad + 3a_{wxyz}
\end{aligned}$$

$$\begin{aligned}
a_{\overset{\quad 3}{\overline{wxyz}}} &= Z^3 - 3Z^4 \\
&= a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz} \\
&\quad - 3a_{wxyz}
\end{aligned}$$

$$a_{\overset{\quad 4}{\overline{wxyz}}} = Z^4 = a_{wxyz}.$$

85. To find $a_{\overset{\qquad [r]}{\overline{wxyz\ldots(m)}}}$, the value of an annuity payable only while exactly r lives out of m lives survive, we must use formulas 13 and 14 of Chap. ii, exactly as we did in Art. 31 of Chap. iii. We shall have

$$a_{\overset{\qquad [r]}{\overline{wxyz\ldots(m)}}} = Z^r - (r+1)Z^{r+1} + \frac{(r+1)(r+2)}{\underline{|2}}\,Z^{r+2} - \&c. \;. \quad (60)$$

$$= \frac{Z^r}{(1+Z)^{r+1}} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (61)$$

86. As an example, let it be required to determine, when $m=4$, the values of $a_{\overset{\quad [1]}{\overline{wxyz}}}$, $a_{\overset{\quad [2]}{\overline{wxyz}}}$, $a_{\overset{\quad [3]}{\overline{wxyz}}}$, and $a_{\overset{\quad [4]}{\overline{wxyz}}}$. We can at once write down the values sought.

$$
\begin{aligned}
a^{[1]}_{\overline{wxyz}} = {} & a_w + a_x + a_y + a_z \\
& -2(a_{wx} + a_{wy} + a_{wz} + a_{xy} + a_{xz} + a_{yz}) \\
& +3(a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz}) \\
& -4a_{wxyz}
\end{aligned}
$$

$$
\begin{aligned}
a^{[2]}_{\overline{wxyz}} = {} & a_{wx} + a_{wy} + a_{wz} + a_{xy} + a_{xz} + a_{yz} \\
& -3(a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz}) \\
& +6a_{wxyz}
\end{aligned}
$$

$$
\begin{aligned}
a^{[3]}_{\overline{wxyz}} = {} & a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz} \\
& -4a_{wxyz}
\end{aligned}
$$

$$
a^{[4]}_{\overline{wxyz}} = a_{wxyz}
$$

87. The annuity payable while exactly r lives out of m lives are in being can be found in another way. It is the difference between the values of an annuity payable as long as at least r lives survive, and an annuity payable as long as at least $r+1$ lives survive. That is,

$$
a^{[r]}_{\overline{wxyz..(m)}} = a^{r}_{\overline{wxyz...(m)}} - a^{r+1}_{\overline{wxyz...(m)}} \quad . \quad . \quad . \quad . \quad (62)
$$

To take an example,

$$
\begin{aligned}
a^{[2]}_{\overline{wxyz}} & = a^{2}_{\overline{wxyz}} - a^{3}_{\overline{wxyz}} \\
& = Z^2 - 2Z^3 + 3Z^4 - Z^3 + 3Z^4 \\
& = Z^2 - 3Z^3 + 6Z^4 \\
& = a_{wx} + a_{wy} + a_{wz} + a_{xy} + a_{xz} + a_{yz} \\
& \quad -3(a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz}) \\
& \quad +6a_{wxyz}
\end{aligned}
$$

which is the value already found for the same annuity.

88. The argument of Art. 30 clearly applies to annuities and assurances on the last r survivors of m lives; and we may therefore write without further proof

$$
A^{r}_{\overline{wxyz...(m)}} = v(1 + a^{r}_{\overline{wxyz...(m)}}) - a^{r}_{\overline{wxyz...(m)}} \quad . \quad . \quad . \quad (63)
$$

$$
P^{r}_{\overline{wxyz...(m)}} = \frac{1}{1 + a^{r}_{\overline{wxyz...(m)}}} - d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)
$$

89. Formulas 57 and 60 for $a_{\overline{wxyz\ldots(m)}}^{\,r}$, and $a_{\overline{wxyz\ldots(m)}}^{\,[r]}$, respectively, apply to temporary or deferred annuities. In the powers of Z we have only to substitute temporary or deferred annuities, as the case may be, for the whole of life annuities.

90. We must here guard ourselves against a misconception with regard to deferred annuities. We have seen, formula 6, that ${}_{n|}a_x = v^n\,{}_np_x a_{x+n}$, and the same argument which gives this formula also leads to the equation

$${}_{n|}a_{wxyz\ldots(m)} = v^n\,{}_np_{wxyz\ldots(m)}\,a_{w+n:x+n:\ldots(m)}.$$

But we cannot correctly write

$${}_{n|}a_{\overline{wxyz\ldots(m)}}^{\,r} = v^n\,{}_np_{\overline{wyxz\ldots(m)}}^{\,r}\,a_{\overline{w+n:x+n:\ldots(m)}}^{\,r}$$

for the deferred annuity on the last r survivors of m lives will become payable even if all the m lives do not survive the n years, provided only that at least r of them so survive; but in the third factor of the right hand member of the last equation it is assumed that all the m lives are alive after the n years, and the formula is therefore wrong.

91. The formulas for temporary and deferred assurances on the last r survivors of m lives are very similar to those already found for corresponding benefits on single lives, only we must take care, as in the case of the deferred annuity discussed in the last article, which of the alternative formulas we adopt. Unless it is specially stipulated for in the question, we must not use a formula which presupposes that all the m lives must survive the term named.

Thus we shall have

$${}_{|n}A_{\overline{wxyz\ldots(m)}}^{\,r} = v(1 + {}_{|n-1}a_{\overline{wxyz\ldots(m)}}^{\,r}) - {}_{|n}a_{\overline{wxyz\ldots(m)}}^{\,r} \quad . \quad . \quad . \qquad (65)$$

and

$$P_{|n}A_{\overline{wxyz\ldots(m)}}^{\,r} = \frac{{}_{|n}A_{\overline{wxyz\ldots(m)}}^{\,r}}{1 + {}_{|n-1}a_{\overline{wxyz\ldots(m)}}^{\,r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66)$$

also

$${}_{n|}A_{\overline{wxyz\ldots(m)}}^{\,r} = A_{\overline{wxyz\ldots(m)}}^{\,r} - {}_{|n}A_{\overline{wxyz\ldots(m)}}^{\,r} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (67)$$

but we must not write

$${}_{n|}A_{\overline{wxyz\ldots(m)}}^{\,r} = v^n\,{}_np_{\overline{wxyz\ldots(m)}}^{\,r}\,A_{\overline{w+n:x+n.\ldots(m)}}^{\,r}.$$

92. It may be well to point out that the annuity, or the assurance, on the last r survivors of m lives, is the most general form possible of the benefit. If we make $m=r$ we have joint-life benefits, and if we make both m and r equal to unity we have single-life benefits.

93. The question has sometimes been asked, to what amount, on the average, will an annuity-due accumulate if it be forborne, and improved at interest during the existence of a given life; and different writers have given answers very much at variance with each other. This is because the question has been looked at from different stand-points: in fact, two questions have been answered instead of one. First, we may suppose each of l_x persons of the same age, x, entitled to an annuity-due of 1 per annum, and agreeing among themselves to throw all these annuities into a common fund, and to allow the representatives of each to withdraw at his death his share of the fund; it being stipulated that, since the annuities are of equal value at the outset, all shall share alike, whenever they may die; and in pursuance of this arrangement we may ask, what is the share to be withdrawn at each death? The argument here is very simple. Since an annual sum of P_x, paid in advance, will secure 1 at the end of the year of the death of (x), it necessarily follows that the annuity-due of 1 per annum will secure a benefit of $\frac{1}{P_x}$; and this is clearly the share to be paid on the death of each member of the hypothetical society above described. In this case those who die early leave to their representatives more than the accumulations of their own annuities; and this excess is made good out of the contributions of those who die late. All receive equally, irrespective of the amounts they may have paid in.

94. Secondly, we may suppose the annuities thrown into a common fund, as above described; but instead of all receiving equally, we may imagine the representatives of each withdrawing only the actual accumulations of his particular annuity. In this case the representatives of those who die early would withdraw very little, while the representatives of those who die late would withdraw a great deal; but nevertheless we may ask what is, on the average, the sum withdrawn at each death. To answer the question, an algebraical investigation is necessary. There are at the outset l_x members of the society, of whom d_x die in the first year, d_{x+1} in the second year, and so on. The following statement shows the number of deaths each year, and the total amounts in consequence withdrawn.

(1) Year.	(2) Number of Deaths.	(3) Total Amount withdrawn.
1	d_x	$(l_x - l_{x+1})(1+i)$
2	d_{x+1}	$(l_{x+1}-l_{x+2})\{(1+i)+(1+i)^2\}$
3	d_{x+2}	$(l_{x+2}-l_{x+3})\{(1+i)+(1+i)^2+(1+i)^3\}$
4	d_{x+3}	$(l_{x+3}-l_{x+4})\{(1+i)+(1+i)^2+(1+i)^3+(1+i)^4\}$
&c.	&c.	&c. &c.

If we carry on the above table to the end of life, and then sum the columns, we shall have l_x for the total number of deaths, being the sum of column 2. Also it will be noticed that the majority of the terms in the third column mutually cancel each other; and that the sum of those that remain is, $l_x(1+i)+l_{x+1}(1+i)^2+l_{x+2}(1+i)^3+$&c. This is the total amount that will be paid out of the fund; and dividing by l_x, the number of members, we have the answer to the question, namely:—the sum on the average withdrawn at each death.

If we multiply numerator and denominator of the resulting expression by $(1+i)^x$, it becomes

$$\frac{(1+i)\{l_x(1+i)^x+l_{x+1}(1+i)^{x+1}+l_{x+2}(1+i)^{x+2}+\&c.\}}{l_x(1+i)^x}$$

95. It will be seen that the expression now is analogous to that in commutation symbols for the present value of an annuity, but that we have substituted throughout $(1+i)$ for v. In fact, as we are dealing with amounts of annuities, we must here *accumulate* at compound interest; whereas, while we were dealing with the present values of annuities, it was necessary to *discount* at compound interest. The expression being in commutation form, it would be easy to prepare suitable commutation tables, but they would serve little, if any, practical use.

96. If money yield interest, the answer to the second question is always greater numerically than the answer to the first. The reason is not far to seek. In the first case, those who die early withdraw more than their own contributions, and therefore less is left in the fund to bear interest, than in the second case. The total amount to be distributed is therefore less in the first case than in the second, and the share of each member is necessarily less also.

97. It is perhaps worthy of remark that, on two extreme suppositions, the two questions have the same answer. If money yield no interest, P_x in the first answer becomes $\frac{1}{1+e_x}$, and the share of each member of the society becomes $1+e_x$, which it is easy to see is also the share of each member under the second supposition, when we write $i=0$.

Again, if money yield interest, but if the rate of mortality become zero, that is if we substitute a term certain, say n years, for the life of (x), $\frac{1}{P_x}$ becomes the amount of an annuity-certain-due in n years, that is $(1+i)\frac{(1+i)^n-1}{i}$, and this is the value of the expression given in the answer in the second case, when $l_x=l_{x+1}=l_{x+2}=$&c.

98. A case occurs sometimes in practice, where an annuity is to accumulate at one rate of interest during the existence of a given life, and it is desired to find its present value at another rate of interest. We may look upon the annuity as the interest upon a fund invested, and the question thus becomes; to find the present value of a fund invested at compound interest, and accumulating during the existence of a given status. Let us assume that the rate of accumulation is j, and let the rate at which the benefit is to be valued be i: let us also assume an annuity-due as before. We may look upon the annuity-due as the interest in advance upon a fund of $(1+j)\div j$. That sum will produce an annuity-due of 1 for n years, and will itself remain intact at the end of the period. Therefore an assurance consisting of the accumulations of an annuity-due of 1 forborne, is equivalent to an assurance consisting of a sum of $(1+j)\div j$ and its accumulations, less an assurance of a sum of $(1+j)\div j$ itself. We therefore have, if $\mathcal{A}$ be the value at rate i, of an assurance of the amount of an annuity-due of 1,

$$\mathcal{A}=\Sigma\frac{1+j}{j}(1+j)^n\times(1+i)^{-n}{}_{n-1|}q_x-\frac{1+j}{j}A_x$$

$$=\frac{1+j}{j}\left\{\Sigma\left(\frac{1+j}{1+i}\right)^n{}_{n-1|}q_x-A_x\right\},$$

where A_x is taken at rate i. If now we take $\frac{1+j}{1+i}=\frac{1}{1+J}$, so that $J=\frac{i-j}{1+j}$, we have

$$\mathcal{A}=\frac{1+j}{j}\left\{\Sigma(1+J)^{-n}{}_{n-1|}q_x-\mathrm{A}_x\right\}$$

$$=\frac{1+j}{j}(\mathrm{A}'_x-\mathrm{A}_x)\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (68)$$

where A'_x is taken at rate J.

If $j=i$ then $J=0$, and $\mathrm{A}'_x=1$, and we have

$$\mathrm{A}=\frac{1+i}{i}(1-\mathrm{A}_x)$$

$$=\frac{1-\mathrm{A}_x}{d}$$

$$=1+a_x,$$

as it ought to be. If we take $j=\cdot03$ and $i=\cdot04$, then it will be found that the rate at which to calculate A'_x is nearly ·00975.

If $j>i$ then J becomes negative; that is, A'_x must be calculated at a negative rate of interest.

99. It is sometimes required in practice to find the value of an annuity payable during the joint lives of (x) and (y), and for t years after the death of (y) should (x) live so long, the symbol for which benefit is $a_{x:\overline{y(\overline{t|})}}$.

The annuity may be divided into two parts; (1), the portion during the first t years; and (2), that after t years. The value of the first portion is $a_{x\overline{t|}}$. The nth payment after t years will be made if (x) live $n+t$ years and (y) at least n years, and the value of this payment therefore is $v^{n+t}{}_{n+t}p_x\times{}_np_y$, or $v^t{}_tp_x\times v^n{}_np_{x+t:y}$, or $\dfrac{\mathrm{D}_{x+t}}{\mathrm{D}_x}v^n{}_np_{x+t:y}$. In order to find the value of the second portion of the annuity, we must take the sum of the last expression for every integral value of n from unity upwards, which is $\dfrac{\mathrm{D}_{x+t}}{\mathrm{D}_x}\times a_{x+t:y}$. For the whole annuity we therefore have

$$a_{x.\overline{y(\overline{t|})}}=a_{x\overline{t|}}+\frac{\mathrm{D}_{x+t}}{\mathrm{D}_x}a_{x+t:y}$$

$$=a_x-\frac{\mathrm{D}_{x+t}}{\mathrm{D}_x}(a_{x+t}-a_{x+t:y})\quad . \; . \; . \quad (69)$$

This formula is analogous to No. 16 of Chapter iv.

100. So far we have investigated the processes by which annuities, assurances, and premiums, are derived from the Mortality Table. The inverse problem sometimes presents itself in practice. Having given a table of the values of a benefit for all ages, to ascertain the mortality table on which it is based. As we have seen, Chap. iii, Art. 18, the question resolves itself into ascertaining from the data the value of p_x. We may briefly here solve this problem for the three cases in which the data are respectively a_x, A_x, and P_x. Taking these in their order:—

101. Having given a complete table of a_x, to deduce the mortality table. We have by formula 3

$$a_x = vp_x(1+a_{x+1})$$

whence
$$p_x = \frac{(1+i)a_x}{1+a_{x+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

By means of this formula, beginning at the oldest age in the table and working downwards to the youngest age, we can find all the values of p_x. It may be remarked that, if the table of annuities is not complete, but extends only from one limit of age to another, say from s to t, we can find the values of p_x for that section of the mortality table to which the tabulated values of a_x extend. Thus, if we have a table of a_x for ages from 20 to 60, we can find all the values of p_x from p_{20} to p_{59}. The most convenient way of passing to a table of p_x from a table of a_x will be by logarithms. Placing in one column $\log a_x$ and in the next $\log (1+a_{x+1})$ we should in a third column place the differences between the numbers in the first two columns. Then, increasing each of these differences by $\log (1+i)$, we should have at once $\log p_x$.

102. Having given a complete table of A_x, to deduce the mortality table.

It is evident that

$$A_x = v(1-p_x) + vp_xA_{x+1}$$
$$= v - vp_x(1-A_{x+1})$$

whence
$$p_x = \frac{1-(1+i)A_x}{1-A_{x+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

By a suitable arrangement of columns it would not be difficult practically to apply the last formula, but it would be less laborious to make use of the relation existing between a_x and A_x so as to construct a table of a_x from the given table of A_x, and then to proceed as explained in the last preceding article. In Chapter viii on Conversion Tables, it will be shown how a_x may be derived very easily from A_x.

103. Having given a complete table of P_x, to deduce the mortality table.

Since
$$P_x = \frac{1}{1+a_x} - d, \text{ and}$$

$$a_x = vp_x(1+a_{x+1}) = \frac{vp_x}{P_{x+1}+d}$$

it follows that
$$P_x + d = \frac{1}{1+\dfrac{vp_x}{P_{x+1}+d}} = \frac{P_{x+1}+d}{vp_x+P_{x+1}+d}$$

whence, by a suitable modification,

$$p_x = (1+i)(P_{x+1}+d)\left\{\frac{1}{P_x+d} - 1\right\} \quad . \quad . \quad . \quad . \quad (72)$$

Formula 72 cannot be reduced to any simpler shape, and to employ it would be extremely tedious. By far the best way in practice would be to pass to a_x from P_x by means of Conversion Tables explained in Chap. viii, and then to proceed as in Article 101.

104. Although De Moivre's hypothesis as to the law of mortality is now quite superseded, yet it is an improving exercise to make use of it to deduce the values of annuities and assurances, and we shall here find these for single lives, and for two joint lives.

105. To find the value of a_x on De Moivre's hypothesis of equal decrements of life.

Let n represent the complement of life at age x, that is, let $n = \omega - x = 86 - x$, according to the limit of life adopted by De Moivre. Then $l_x = n$, $l_{x+1} = n-1$, $l_{x+2} = n-2$, &c. Therefore

$$a_x = \frac{1}{n}\{(n-1)v + (n-2)v^2 + \ldots\ldots + v^{n-1}\}$$

$$va_x = \frac{1}{n}\{\qquad\qquad (n-1)v^2 + \ldots\ldots + 2v^{n-1} + v^n\}$$

Therefore $(1-v)a_x$, or

$$via_x = \frac{1}{n}\{(n-1)v - v^2 - \ldots\ldots - v^{n-1} - v^n\}$$

$$= \frac{1}{n}(nv - a_{\overline{n|}})$$

and
$$a_x = \frac{n-(1+i)a_{\overline{n|}}}{ni} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

106. We found, Chap. iii, Art. 9, that by De Moivre's hypothesis $\mathring{e}_x=\frac{n}{2}$ and therefore that $n=2\mathring{e}_x$. Making use, in formula 73, of this value for n we have

$$a_x=\frac{2\mathring{e}_x-(1+i)a_{\overline{2\mathring{e}_x}|}}{2i\mathring{e}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

It was this last formula No. 74 that the older writers frequently employed for calculating the values of life annuities, even by tables not based upon De Moivre's hypothesis. The primitive mortality tables had been constructed on faulty principles, and as it was from an inspection of them that De Moivre deduced his hypothesis, formula 74 gave results not widely different from those obtained by calculating the annuity directly from the mortality table. Thus, by the Northampton Table, $\mathring{e}_x$ at age 10 is 39·78, and working out formula 74 at 4 per-cent interest we find that $a_{10}=17{\cdot}192$, the true value by the Northampton Table being 17·525.

107. To find the value of A_x by De Moivre's hypothesis.

We have $$A_x=\frac{vd_x+v^2d_{x+1}+\&c.}{l_x},$$

but by De Moivre's hypothesis $d_x=d_{x+1}=d_{x+2}=\&c.=1$, and $l_x=n$.

Therefore $$A_x=\frac{a_{\overline{n}|}}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Had we commenced with formula 75, and from it passed to the value of a_x by means of the relation $a_x=\frac{1-A_x}{d}-1$, the analysis would have been simpler.

108. To find the value of A_{xy} by De Moivre's hypothesis.

Let x be the age of the older life, and y that of the younger; and let the complement of life of (x) be n, and that of (y), m: then $m>n$.

Now

$$\begin{aligned}A_{xy}&=\frac{(l_{xy}-{}^1l_{xy})v+({}^1l_{xy}-{}^2l_{xy})v^2+\&c.}{l_{xy}}\\&=\frac{1}{nm}\{[nm-(n-1)(m-1)]v+[(n-1)(m-1)-(n-2)(m-2)]v^2\\&\qquad+\&c.+[(n-n+1)(m-n+1)-(n-n)(m-n)]v^n\}\end{aligned}$$

But, taking the general term of this expression,

$$(n-r)(m-r)-(n-r-1)(m-r-1)=n+m-2r-1\,;$$

and giving to r every value from 0 to $n-1$, we have

$$\begin{aligned} \mathrm{A}_{xy} &= \frac{1}{nm}\{(n+m)(v+v^2+\ \ldots\ +\&c.+v^n) \\ &\qquad\qquad -(v+3v^2+5v^3+\&c.+\overline{2n-1}.v^n)\} \\ &= \frac{1}{nm}\{(n+m)a_{\overline{n}|}-(v+3v^2+5v^3+\&c.+\overline{2n-1}.v^n)\}. \end{aligned}$$

It now remains for us to assign a value to the increasing annuity, which may be represented for the moment by the symbol a'. We have

$$\begin{aligned} a' &= v+3v^2+5v^3+\ \ldots\ldots\ +\overline{2n-1}v^n \\ va' &= \qquad v^2+3v^3+\ \ldots\ldots\ +\overline{2n-3}v^n+\overline{2n-1}v^{n+1} \end{aligned}$$

Therefore $(1-v)a'$, or

$$\begin{aligned} via' &= v+2v^2+2v^3+\ \ldots\ldots\ +2v^n-\overline{2n-1}v^{n+1} \\ &= 2v(v+v^2+\ \ldots\ldots\ +v^n)-2nv^{n+1}+v-v^{n+1} \\ &= 2va_{\overline{n}|}-2nv\,v^n+v(1-v^n) \end{aligned}$$

and $$a' = 2\frac{a_{\overline{n}|}-nv^n}{i}+a_{\overline{n}|}.$$

Substituting this value in the expression for A_{xy}, we have

$$\mathrm{A}_{xy} = \frac{1}{nm}\left\{(n+m-1)a_{\overline{n}|}-2\frac{a_{\overline{n}|}-nv^n}{i}\right\} \quad . \quad . \quad . \quad (76)$$

If we wish to find a_{xy} by De Moivre's hypothesis, we may proceed directly as we did in finding a_x; but the work will be easier, and the derived expression will be simpler, if we first find A_{xy} by formula 76, and from A_{xy} pass to a_{xy}.

109. Writing $n=2\mathring{e}_x$, and $m=2\mathring{e}_y$, we shall have

$$\mathrm{A}_{xy} = \frac{1}{4\mathring{e}_x\mathring{e}_y}\left\{(2\mathring{e}_x+2\mathring{e}_y-1)a_{\overline{2\mathring{e}_x}|}-2\frac{a_{\overline{2\mathring{e}_x}|}-2\mathring{e}_x v^{2\mathring{e}_x}}{i}\right\} \quad . \quad . \quad (77)$$

Formula 77 may be used to calculate A_{xy} and a_{xy} by such a table as the Northampton Table in the same way as we used formula 74 to calculate a_x.

It need hardly be said that, with modern and more correct mortality tables, formulas 74 and 77 give results very far removed from the truth.

110. It will be convenient to close this chapter with the dem stration of four formulas for the value of an annuity on a compound status. They will be useful to us when we are dealing with reversionary annuities.

111. Employing the letters *a*, *b*, *c*, to denote the lives belonging to one status, and the letters *x*, *y*, *z*, to denote those belonging to another status, we have, where $a_{(abc)(xyz)}$ represents the value of an annuity payable so long as the joint lives *abc* are jointly in existence with the joint lives *xyz*,

$$a_{(abc)(xyz)}=a_{abcxyz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

This relation is evidently true, because by the statement of the case, the annuity in $a_{(abc)(xyz)}$ will cease on the failure of any one of the lives in either status.

112. By formula 57,

$$a_{\overline{xyz}}=a_x+a_y+a_z-a_{xy}-a_{xz}-a_{yz}+a_{xyz}.$$

If now the receipt of the annuity be further conditioned on the joint existence of other lives, *a*, *b*, *c*, each probability involved in each payment of the annuity $a_{\overline{xyz}}$ must be multiplied by the corresponding p_{abc}.

We therefore have

$$\begin{aligned} a_{abc:\overline{xyz}}&=\Sigma v^n{}_np_{abc\,n}p_{\overline{xyz}} \\ &=a_{abcx}+a_{abcy}+a_{abcz}-a_{abcxy}-a_{abcxz}-a_{abcyz}+a_{abcxyz} \quad . \quad (79) \end{aligned}$$

113. Again,

$$\begin{aligned} a_{\overline{abc}:\overline{xyz}}&=\Sigma v^n{}_np_{\overline{abc}\,n}p_{\overline{xyz}} \\ &=\Sigma v^n\{1-(1-{}_np_a)(1-{}_np_b)(1-{}_np_c)\}\{1-(1-{}_np_x)(1-{}_np_y)(1-{}_np_z)\} \\ &=\Sigma v^n[\{1-(1-{}_np_a)(1-{}_np_b)(1-{}_np_c)\} \\ &\quad+\{1-(1-{}_np_x)(1-{}_np_y)(1-{}_np_z)\} \\ &\quad-\{1-(1-{}_np_a)(1-{}_np_b)(1-{}_np_c)(1-{}_np_x)(1-{}_np_y)(1-{}_np_z)\}] \\ &=a_{\overline{abc}}+a_{\overline{xyz}}-a_{\overline{abcxyz}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80) \end{aligned}$$

Formula 80 is evidently true, for if 1 be receivable annually as long as one of the lives *a*, *b*, *c*, survives, and another 1 be receivable as long as one of the lives *x*, *y*, *z*, survives, and if at the same time 1 be payable as long as one of the lives *a*, *b*, *c*, *x*, *y*, *z*, survives, there remains an annuity of 1 receivable as long as one of the lives *a*, *b*, *c*, survives jointly with one of the lives *x*, *y*, *z*.

114. Lastly

$$a_{\overline{(abc)(xyz)}} = a_{abc} + a_{xyz} - a_{abcxyz} \quad . \quad . \quad . \quad . \quad . \quad (81)$$

115. All the four formulas 78 to 81 are quite general. Any number of lives from one upwards may be included in each status, and a term certain may be substituted for a life. Thus for example

$$a_{\overline{(a\overline{n|})(x\overline{m|})}} = {}_{|n}a_a + {}_{|m}a_x - {}_{|n}a_{ax} \text{ if } n < m,$$

or

$$= {}_{|n}a_a + {}_{|m}a_x - {}_{|m}a_{ax} \text{ if } m < n,$$

Taking the first case where $m > n$, this is an annuity payable for m years if (x) survive, or payable for n years if (a) survive, (x) having died within the n years.

CHAPTER VIII.

CONVERSION TABLES FOR SINGLE AND ANNUAL ASSURANCE PREMIUMS.

1. We have seen, Chap. vii, formula 25, that

$$A=1-d(1+a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

so that if we know the value of a, we can at once find the value of A at a given rate of interest, even if we are ignorant of the age of the life, and of the table of mortality on which the value of a is based. In fact, the relation between the assurance and the annuity is quite independent of the rate of mortality.

Again, Chap. vii, formula 38, we had the equation

$$P=\frac{1}{1+a}-d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

showing that P and a are also connected in such a way, that the rate of mortality is not involved between them, and that, knowing the value of one of them and the rate of interest, we can, by a mere arithmetical operation, ascertain the value of the other.

2. The foregoing important principles were made use of by the late William Orchard to construct tables (published in 1856) for finding by inspection A and P, having given a; and those tables, being so useful to the actuary, deserve a full explanation in these pages. In practical operations Orchard's Conversion Tables save to the computer a great deal of labour, and by a simple reference to them a tedious calculation may often be avoided. They have an additional interest for us, since, by

closely investigating the principles on which they are constructed, a great deal of light will be thrown upon our subject.

3. Tables XXXIX and XL at the end of this volume are Orchard's Conversion Tables in a modified and condensed shape; and the following extracts from Orchard's own book show the form which he himself adopted. The extracts are from the tables at 4 per-cent interest.

Value of £1 Annuity.	SINGLE PREMIUM FOR ASSURANCE OF £100 AT 4 PER-CENT.										Diff.	
	0	1	2	3	4	5	6	7	8	9		
10·0	57·692	57·654	57·615	57·577	57·538	57·500	57·462	57·423	57·385	57·346	1	4
·1	57·308	57·269	57·231	57·192	57·154	57·115	57·077	57·038	57·000	56·962	2	8
·2	56·923	56·885	56·846	56·808	56·769	56·731	56·692	56·654	56·615	56·577	3	12
·3	56·538	56·500	56·462	56·423	56·385	56·346	56·308	56·269	56·231	56·192	4	15
·4	56·154	56·115	56·077	56·038	56·000	55·962	55·923	55·885	55·846	55·808	5	19
											6	23
·5	55·769	55·731	55·692	55·654	55·615	55·577	55·538	55·500	55·462	55·423	7	27
·6	55·385	55·346	55·308	55·269	55·231	55·192	55·154	55·115	55·077	55·038	8	31
·7	55·000	54·962	54·923	54·885	54·846	54·808	54·769	54·731	54·692	54·654	9	35
·8	54·615	54·577	54·538	54·500	54·462	54·423	54·385	54·346	54·308	54·269		
·9	54·231	54·192	54·154	54·115	54·077	54·038	54·000	53·962	53·923	53·885		
11·0	53·846	53·808	53·769	53·731	53·692	53·654	53·615	53·577	53·538	53·500		
·1	53·462	53·423	53·385	53·346	53·308	53·269	53·231	53·192	53·154	53·115		
·2	53·077	53·038	53·000	52·962	52·923	52·885	52·846	52·808	52·769	52·731		
*	*	*	*	*	*	*	*	*	*	*		

Value of £1 Annuity.	ANNUAL PREMIUM FOR ASSURANCE OF £100 AT 4 PER-CENT.										Diff.
	0	1	2	3	4	5	6	7	8	9	
10·0	5·245	5·237	5·228	5·220	5·212	5·204	5·195	5·187	5·179	5·171	·8
·1	5·163	5·155	5·147	5·139	5·131	5·122	5·114	5·106	5·098	5·090	·8
·2	5·082	5·074	5·067	5·059	5·051	5·043	5·035	5·027	5·019	5·011	·8
·3	5·003	4·996	4·988	4·980	4·972	4·964	4·957	4·949	4·941	4·933	·8
·4	4·926	4·918	4·910	4·903	4·895	4·887	4·880	4·872	4·865	4·857	·8
·5	4·850	4·842	4·834	4·827	4·819	4·812	4·804	4·797	4·789	4·782	·7
·6	4·775	4·767	4·760	4·752	4·745	4·738	4·730	4·723	4·715	4·708	·7
·7	4·701	4·694	4·686	4·679	4·672	4·664	4·657	4·650	4·643	4·636	·7
·8	4·628	4·621	4·614	4·607	4·600	4·593	4·586	4·578	4·571	4·564	·7
·9	4·557	4·550	4·543	4·536	4·529	4·522	4·515	4·508	4·501	4·494	·7
11·0	4·487	4·480	4·473	4·466	4·459	4·453	4·446	4·439	4·432	4·425	·7
·1	4·418	4·411	4·405	4·398	4·391	4·384	4·378	4·371	4·364	4·357	·7
·2	4·351	4·344	4·337	4·330	4·324	4·317	4·310	4·304	4·297	4·291	·7
*	*	*	*	*	*	*	*	*	*	*	

4. The equation $A=1-d(1+a)$ is of the first degree; and from this it follows that the values of the assurance, A, corresponding to equidifferent values of the annuity, a, will also be equidifferent, and consequently will be formed by the simple continued addition to, or subtraction from, an initial value, of a constant quantity. Thus, let a become $a+\Delta a$, and let the corresponding value of A be $A+\Delta A$: then, from the fundamental equation,

$$A+\Delta A=1-d(1+a+\Delta a) \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

and therefore

$$\Delta A=-d\times\Delta a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and this is the constant difference of the series of values of A, corresponding to a series of values of a whose constant difference is Δa.

Orchard took Δa equal to ·01, and his ΔA therefore is equal to $-{\cdot}01\times d$; by the continuous subtraction of which quantity, taken positively, from the assurance value corresponding to annuity 0, the tables for the different rates of interest were respectively formed. The initial value of A for $a=0$ we get from equation 1: it is $1-d=v$: and the same equation supplies a verification formula to check the work at convenient stages.

5. The following is a type of the construction of the 4 per-cent table. The initial value of A is $v={\cdot}96153846$, and the constant difference, for a constant difference of ·01 in a, is ${\cdot}01\times d={\cdot}01\times{\cdot}03846154={\cdot}0003846$

Value of Annuity.	Value of Assurance.
0·00	·9615385 ·0003846
0·01	·9611539 ·0003846
0·02	·9607693 ·0003846
0·03	·9603847 ·0003846
0·04 &c.	·9600001 &c.

6. In the above example we have each time written down the value of ΔA before performing the subtraction, but this course is needless, and is productive of trouble. The value of ΔA should be written at the foot of a card, which is moved down as the subtractions are performed.

7. The last figure of ΔA is only approximately true, and the error in it is continuously multiplied as the work proceeds, so that a correction must be introduced to counteract the accumulation of error. Thus, in the above example, by multiplying *d* to seven decimal places by ·01, we merely insert two more zeros after the decimal point, and when we cut down the product to seven figures we omit 15, or rather 15·4, in the eighth and ninth decimal places. If therefore we were to perform one hundred subtractions without correction, the sixth and seventh figures of the result would be wrong to the extent of 15·4. If, however, fifteen times in the course of the hundred subtractions we deduct ·0003847 instead of ·0003846, the remaining error will be infinitesimal. The corrected deduction should be made in this case at about the fourth operation, and thereafter at about every sixth or seventh succeeding operation.

8. If only five places of figures are to be retained in the final results, as is the case in Orchard's volume, it will be necessary to perform the calculations to at least six places, but perfect accuracy will be more certainly assured if we work to seven figures. In transferring the values from the working sheet, they must be cut down to the required dimensions, and the last of the retained figures must be corrected for the figures that are rejected; whenever the rejected figures are equal to, or greater than 50, the last retained figure being increased by unity. This operation of correction will be performed, as it were automatically, if we increase the first rejected figure of the initial value by 5. The result will be, that on the working sheet the first rejected figure of all the values will be 5 too great, but this error will not affect the last retained figure in any of the values when the first rejected figure is less than 5; and when it is 5 or greater, the last retained figure will be increased by unity. This, as we have seen, is what is required.

9. In Orchard's book the table of proportional parts corresponds to a difference of ·001 in the value of *a*, and it therefore consists merely of the first nine multiples of $\cdot 001 \times d$.

10. In Table No. XXXIX at the end of this volume, A is given by inspection for only integral values of *a*, and therefore all the decimal places in *a* must be proportioned for. The table of differences contains three sections, respectively for the first, second, and third decimal places in *a*. It is true that the second and third sections might have been omitted, because the figures contained in them are the same as in the first section, being only removed further from the decimal point; but it is believed that facility and accuracy of operation will be better secured by giving all three sections.

11. The value of A decreases as the value of *a* increases, and it follows that the proportional parts of A are subtractive. When in Table No. XXXIX we have found the value of A, corresponding to the integers in *a*, we must *deduct* the correction for the decimal places. The operation of subtraction is always more irksome, and more liable to error, than the operation of addition, and should therefore be avoided when possible. In the present case we can render the correction additive by entering the table with the next higher value of *a*, and then proportioning for the arithmetical complement of the decimal places. In the following numerical example the assurance is found by each of these processes.

Example.—Required the single premium corresponding to annuity-value 10·642 at 4 per-cent interest.

Single Premium for 10 =	·57692	P.P. for	·6	·02308
Deduct P.P.	·02470	„	·04	·00154
	·55222	„	·002	·00008
				·02470
Or, Single Premium for 11 =	·53846	P.P. for	·3	·01154
Add P.P.	·01377	„	·05	·00192
	·55223	„	·008	·00031
				·01377

12. It will be noticed that in the above example the two processes give results differing by unity in the last place. This is because the last figures in Table No. XXXIX and in the table of differences are only approximately true. The discrepancy, which can never exceed unity in the last place, is of no consequence in practice. It will frequently occur, both with our contracted table, and with Orchard's more extensive one.

13. The extract from Orchard's book given on page 148 will enable us to solve the above example by means of his table. Thus

	Single Premium for 10·64,		·55231
	Deduct P.P. for	·002	·00008
			·55223
Or,	Single Premium for 10·65,		·55192
	Add P.P. for	·008	·00031
			·55223

14. Annuity-values are frequently employed to three decimal places only. It may be well to point out, however, that the fourth decimal place, especially where the higher rates of interest are concerned, may affect the fifth decimal place of the single premium. Thus, at 6 per-cent, the tabular difference corresponding to ·1 being ·00566, that corresponding to ·0005 (the limit of the error in the annuity-value when the fourth decimal place is rejected) will be $\frac{\cdot 00566 \times \cdot 0005}{\cdot 1} = \cdot 0000283$. It thus appears that when the rate of interest is 6 per-cent, the rejection of the fourth decimal place in the annuity-value will give rise to an error in that of the single premium for assurance, which may equal but cannot exceed ·00003. From what precedes it will be seen that, with the tabulated differences, it is perfectly easy to proportion for the fourth decimal place of the annuity.

15. By equation 2 of this chapter, $P = \frac{1}{1+a} - d$. The variable a here occurring in the denominator, the difference of this expression is of a more complex form than that of the expression for the single premium, and the table does not admit of formation by the continual addition or subtraction of a constant quantity. The formation nevertheless can be very easily effected.

Thus, when a becomes $a+\Delta a$, P becomes $P+\Delta P$, and we have by equation 2

$$P+\Delta P = \frac{1}{1+a+\Delta a} - d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whence, by subtraction,

$$\Delta P = \frac{1}{1+a+\Delta a} - \frac{1}{1+a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which it appears that the quantities by the continual addition of which to the initial value the successive values of P will be formed, are the differences of the series of reciprocals belonging to the successive values of $1+a$, and these differences can be readily obtained from any table of reciprocals.

16. With regard to this series of differences, it will also be noted that they are independent of the rate of interest. When formed for use with one rate therefore, they can be applied to any other.

17. From equation 2 it will be seen that the initial value of P is v, corresponding to $a=0$: also from equation 6 that the differences are subtractive, because the reciprocal of $1+a+\Delta a$ is less than the reciprocal of $1+a$.

18. Orchard tabulated to five places of decimals the value of P for every value of a to two places of decimals, and gave proportional parts for the third place of a. Therefore, in his tables, $\Delta a = \cdot 01$.

The following is a type, at 4 per-cent interest, of the method on which his table was constructed.

a	$\frac{1}{1+a}$	$\Delta\frac{1}{1+a}=\Delta P$	P
0·00	1·000000	·009901	·961538
·01	·990099	·009707	·951637
·02	·980392	·009518	·941930
·03	·970874	·009335	·932412
·04	·961539	·009158	·923077
·05	·952381	·008985	·913919
·06	·943396	·008817	·904934
·07	·934579	·008653	·896117
·08	·925926	·008495	·887464
·09	·917431	·008340	·878969
·10	·909091	...	·870629
&c.	&c.	&c.	&c.

The headings sufficiently explain the first three columns. The first value in the fourth column is v, and the succeeding values are formed by continuously subtracting the difference in the third column. Seeing that in this case we do not continuously subtract a constant quantity, there is no accumulation of error, and therefore a correction for this cause is not required; also, it will be found to be abundantly sufficient for purposes of accuracy to calculate the table to only one place more of figures than are to be finally retained. If we follow the artifice of Art. 8, and increase by 5 the rejected figure of the initial value, we shall not require to correct the last retained figure of the subsequent values in cutting them down to their final dimension.

19. The differences for the third place of decimals in the annuity being very small, they can be inserted by inspection, and a special method of calculating them is not required. The fourth decimal place in the annuity affects only very rarely the fifth decimal place in the annual premium.

20. Table No. XL at the end of this volume gives P by inspection for only integral values of a, and interpolation must be resorted to for three decimal places of the annuity. The values in the table of differences are therefore very considerable, and they must be prepared independently of P. The following is a numerical illustration of the construction of the table. The first column gives the values of a to one place of decimals, starting from $a=1$; and the second, the reciprocals of $1+a$. In col. 3 the column of $\frac{1}{1+a}$ is differenced, and these differences form the differences of P for increments of ·1 in the value of the annuity. Seeing that in the table of proportional parts accompanying Table No. XL the proportional parts are given against each integral value of the annuity, for the increments ·1, ·2, ·3, &c., up to ·9, we have, in the fourth column of the numerical illustration, summed in groups of ten the differences in the third column. The first nine values in each group are the proportional parts tabulated in Table XL, while the tenth—also carried out in col. 5—is ΔP for integral increments of a; by the successive subtraction of which from the value corresponding to $a=1$, we construct the successive values of P given in Table XL.

21. From this illustration of the method of constructing the annual premium conversion table, it will be seen that, as already stated, the differences of P are independent of the rate of interest, and that therefore the table of proportional parts serves for all rates. In constructing the table of P at any determined rate of interest, we have merely to start with the proper initial value, and from it continuously deduct the differences found as in col. 5.

(1) a	(2) $\frac{1}{1+a}$	(3) $\Delta\frac{1}{1+a}$	(4) Prop. Parts.	(5) ΔP	(6) P at 4 °/o Interest.
1·0	·500000	·023809	·023809	...	·461538
·1	·476191	·021646	·045455		
·2	·454545	·019762	·065217		
·3	·434783	·018116	·083333		
·4	·416667	·016667	·100000		
1·5	·400000	·015385	·115385		
·6	·384615	·014245	·129630		
·7	·370370	·013227	·142857		
·8	·357143	·012315	·155172		
·9	·344828	·011495	·166667		
				·166667	
2·0	·333333	·010752	·010752	...	·294871
·1	·322581	·010081	·020833		
·2	·312500	·009470	·030303		
·3	·303030	·008912	·039215		
·4	·294118	·008404	·047619		
2·5	·285714	·007936	·055555		
·6	·277778	·007508	·063063		
·7	·270270	·007112	·070175		
·8	·263158	·006748	·076923		
·9	·256410	·006410	·083333		
				·083333	
3·0	·250000	·006098	·006098	...	·211538
·1	·243902	·005807	·011905		
·2	·238095	·005537	·017442		
·3	·232558				
	&c.	&c.	&c.		&c.

22. One or two examples of the use of the table will assist in understanding it.

Example 1.—Required the annual premium corresponding to annuity-value 10·642, at 4 per-cent interest.

Ann. Prem. for 10·,	·05245	P. Pt. for ·6,	·00470	P. Pt. for ·1,	·00074
Deduct P. Pt.	·00501	,, ·042	·00031	·042 reversed	240
P=	·04744		·00501		30
					1
					31

It will be observed in the above example that in order to obtain the proportional part for the second and third decimal places of the annuity, we must find, for $a = 10·6$, the differences of P for an increment of ·1 in a. That is done by subtracting the value in the table of proportional parts under 10·6 from that under 10·7. The result of the subtraction must then be multiplied by the second and third decimal places of a.

The extract from Orchard's book given on p. 148 enables us to solve the example by means of his table. Thus:—

Annual Premium for 10·64,		·04745
Deduct P. Pt. for	·002	1
	P=	·04744

Or,

Annual Premium for 10·65,		·04738
Add P. Pt. for	·008	6
	P=	·04744

Orchard's table being of one hundred times the extent of ours, gives the required result with less labour: also it can be used in either of the two ways explained in Art. 11, while ours cannot be.

Example 2.—Required the annual premium corresponding to annuity-value 6·079 at 5 per-cent interest.

Annual Premium for 6·,	·09524	P. Pt. for ·1	·00201
Deduct P. Pt.	159	079 reversed	970
P=	·09365		141
			18
			·00159

Example 3.—Required the annual premium corresponding to the annuity of the last example, but at $3\frac{1}{2}$ per-cent interest.

The proportional part will be the same, and need not be worked out again. We have:—

$$\begin{array}{rr} \text{Annual Premium for } 6\cdot, & \cdot 10904 \\ \text{Deduct P. Pt.} & 159 \\ \hline & \cdot 10745 \end{array}$$

23. In the second and third examples we have worked out P at two rates of interest for the same value of a, in order to illustrate an important principle by means of which the power of Annual Premium Conversion Tables is very much increased.

24. If $P_{(i)}$, $a_{(i)}$ and $d_{(i)}$ represent the annual premium, the annuity-value, and the rate of discount, at rate of interest i; and $P_{(j)}$, $a_{(j)}$, and $d_{(j)}$, the same functions at rate of interest j, we shall have

$$P_{(i)} = \frac{1}{1+a_{(i)}} - d_{(i)}, \quad \text{and}$$

$$P_{(j)} = \frac{1}{1+a_{(j)}} - d_{(j)}.$$

Now if we so change the rates of mortality that $a_{(i)} = a_{(j)}$, we shall have $P_{(i)} - P_{(j)} = d_{(j)} - d_{(i)}$:

that is, $$P_{(j)} = P_{(i)} - (d_{(j)} - d_{(i)})$$

Whence, if we have found the annual premium corresponding to an annuity-value at one rate of interest, we can find the annual premium corresponding to *the same annuity-value* at any other rate of interest, by simply deducting the difference between the rates of discount.

Thus,
$$\begin{array}{r} d_{(5)} = \cdot 04762 \\ d_{(3\frac{1}{2})} = \cdot 03382 \\ \hline d_{(5)} - d_{(3\frac{1}{2})} = \cdot 01380 \end{array}$$

Also, Example 3,
$$\begin{array}{r} P_{(3\frac{1}{2})} = \cdot 10745 \\ \text{Deduct } (d_{(5)} - d_{(3\frac{1}{2})}) = \cdot 01380 \\ \hline P_{(5)} = \cdot 09365 \end{array}$$

which is the value found for $P_{(5)}$ in example 2.

25. From these considerations it appears that, in a Conversion Table, it is not really necessary to tabulate P at more than one rate of interest, because we can with very little trouble pass to any other rate.

26. In fact, we may go further, and say that a table of reciprocals is almost as useful for finding the annual premium from the annuity as ordinary Conversion Tables; and we may even add that in some respects it is more useful, because it will give the results true to more places of decimals. We have only to enter the table of reciprocals with $1+a$, and from the tabular result deduct d.

Thus, to solve again example 3,

$$\frac{1}{1+a}=\frac{1}{1+6{\cdot}079}=\cdot141263$$

$$d_{(3\frac{1}{2})}=\underline{\cdot033816}$$

$$P_{(3\frac{1}{2})}=\underline{\underline{\cdot107447}} \text{ as before.}$$

27. In the three examples, our table gives the value of P correct in the last place; but in using the beginning of the table great caution must be exercised, as with that portion an error may easily occur affecting the last two decimal places in P. However, this is not of much consequence, because it will be found that for values of a greater than 4 the table is trustworthy, and in practice smaller values of a do not often occur.

28. In actual business transactions it not unfrequently happens that the actuary knows at what rate of annual premium a given life may be assured, and he wishes to obtain the value of an annuity—called in such connection a *Life Interest*—on the same life. Thus,—to take an example which comes within the range of the extract from Orchard's table given on p. 148—a life can be assured at an annual premium of £4. 12*s*. 3*d*. per-cent$=4{\cdot}6125$. What is the value of an annuity at 4 per-cent depending on that life? We find in the table against $P=4{\cdot}614$, that $a=10{\cdot}820$, and against $P=4{\cdot}607$, that $a=10{\cdot}830$. Therefore an increase of ·007 in the premium causes a decrease of ·010 in the annuity, and consequently an increase of ·0055 in the premium will cause a decrease of ·008 in the value of the annuity. The required annuity-value is therefore $10{\cdot}830-{\cdot}008=10{\cdot}822$.

29. A table of reciprocals would answer the question equally readily. Thus:—

$$P = \cdot 046125$$

$$\text{Add } d_{(4)} = \cdot 038462$$

$$P + d = \frac{1}{1+a} = \cdot 084587$$

Therefore $1 + a = 11 \cdot 82215$

and $a = 10 \cdot 82215$

30. The small Table No. XL answers the question without much difficulty. The first premium in the table, above the given premium ·04613, is ·05245 corresponding to annuity 10. Between the premiums there is a difference of ·00632, and making a first entry in the table of proportional parts, we find that the premium next above the given premium, and corresponding to $a = 10 \cdot 8$, is $\cdot 05245 - \cdot 00616 = \cdot 04629$, still ·00016 in excess of the given premium. Taking ·00616 (the number under "$\Delta a = \cdot 8$") from ·00688 (the number under "$\Delta a = \cdot 9$") we find that the decrease ·00072 in the premium causes an increment in the annuity of ·1. Therefore a decrease of ·00016 in the premium will cause an increment in the annuity of ·022, giving for the total annuity 10·822.

31. In the examples of Arts. 28 to 30, a is found from P by an inverse use of the table; but by aid of the principles already displayed in this chapter, it is easy to construct a table from which a may be found directly by inspection, P being given.

We have
$$a = \frac{1}{P+d} - 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

$$a + \Delta a = \frac{1}{P + \Delta P + d} - 1 \quad . \; . \; . \; . \; . \; . \quad (8)$$

and
$$\Delta a = \frac{1}{P + \Delta P + d} - \frac{1}{P+d} \quad . \; . \; . \; . \; . \quad (9)$$

from which it appears that the quantities, by the continuous addition of which to the initial value the successive values of a will be formed, are the differences of the series of reciprocals belonging to the successive values of $P + d$.

It is not necessary here to pursue the subject further. The student can work out an example for himself.

32. The last table which calls for attention in this chapter is one for finding by inspection the annual premium from the single premium.

By Chap. vii, equation 37,

$$P=\frac{dA}{1-A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$P+\Delta P=\frac{d(A+\Delta A)}{1-(A+\Delta A)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\Delta P=\frac{d(A+\Delta A)}{1-(A+\Delta A)}-\frac{dA}{1-A} \quad . \quad . \quad . \quad . \quad (12)$$

Therefore, to construct a table, giving P by inspection from the value of A, we must first of all calculate (by logarithms will be most easy) the values of $\frac{dA}{1-A}$ for all values of A; and, differencing these values, we shall have the differences of P. There is no convenient arithmetical process for computing this table; and, if computed, the table would be of very little use in practice.

33. Throughout this chapter we have omitted the suffixes to the symbols a, A, and P, and this has been done on purpose, because we have been dealing with the relations which exist between the three functions, quite independent of the rate of mortality. The investigations, and the resulting tables, apply equally to all cases in which the fundamental equations Nos. 1, 2, 7, and 10 hold; that is, where the annuity is 1 per annum, the assurance unity, the annual premium uniform throughout the whole duration of the status, and the assurance certainly payable one year after the last payment of the annuity.

Conversion tables can therefore be used for annuities and assurances on single lives, or on any number of joint lives, or on the last survivor or the last r survivors of any number of lives; also for endowment assurances, remembering that $a_{x:\overline{n-1}|}$ is the annuity corresponding to $A_{x\overline{n}|}$.

In Chap. xvii we shall also learn that conversion tables are available for annuities and assurances on successive lives.

34. As a matter of curiosity and instruction, but scarcely for practical use, it may be noticed that, by forming suitable annuity-values, Conversion Tables may be applied to the calculation of single and annual premiums for temporary assurances, although in the case of this class of

benefits the condition does not apply that the assurance is certainly payable one year after the last payment of the annuity.

For the single premium the special annuity-value may be derived from the general expression $A=1-d(1+a)$. Equation No. 42 of Chap. vii may be thrown into the form $A^{1}_{x\overline{t}|}=1-d\left(1+a_{x\overline{t}|}+\dfrac{v^t{}_tp_x}{i}\right)$, whence it follows that the special annuity-value with which to enter the Conversion Table in order to find the single premium for an assurance temporary for t years is $a_{x\overline{t}|}+\dfrac{v^t{}_tp_x}{i}$.

Similarly, for the annual premium, the general expression is $P=\dfrac{1}{1+a}-d$, and Equation No. 53 of Chap. vii may be thrown into the form $P^{1}_{x\overline{t}|}=\dfrac{1}{1+\dfrac{N_x-N_{x+t}}{D_x-D_{x+t}}}-d$. Whence the special annuity-value with which to enter the Conversion Table in order to obtain the annual premium for the temporary assurance is $\dfrac{N_x-N_{x+t}}{D_x-D_{x+t}}$.

35. What will be the result of entering the Single Premium Conversion Table with $a_{\overline{n}|}$, the value of an annuity-certain? The table will evidently give the value of 1 payable certainly one year after the last payment of the annuity; that is, v^{n+1}.

36. Again, what will be the result of entering with $a_{\overline{n}|}$ the Annual Premium Conversion Table? We have seen that entering with a we obtain P, that is, the sum payable at the beginning of each year throughout the whole duration of the status, to provide 1 one year after the last payment of an annuity on the same status. Therefore, if for a we employ $a_{\overline{n}|}$, we shall have for P the sinking fund payable at the beginning of each year for $n+1$ years, to provide unity at the end of $n+1$ years.

[37]. In Chap. x, Art. 16, the value of $\overline{A}$, an assurance payable at the moment of death, is shown to be $1-\delta\bar{a}$. If also the annual premium is to be paid by infinitesimal instalments throughout the year, it will be found by dividing $\overline{A}$ by the continuous annuity $\bar{a}$. That is, $\overline{P}=\dfrac{1}{\bar{a}}-\delta$.

These expressions show that Conversion Tables may be formed for continuous functions on the same principles as those for annual functions discussed in this chapter. By means of these, $\overline{A}$ and $\overline{P}$ may be obtained by entering the tables with the continuous annuity, $\bar{a}$. In 1893, Messrs. Rothery and Ryan, in their *Premium Conversion Tables*, included extensive tables of this kind, which are very useful where, as with Friendly Societies and Industrial Assurance Companies, the premiums are collected by weekly or other frequent instalments.

CHAPTER IX.

Annuities and Premiums payable Fractionally throughout the Year.

1. In Chap. vii we discussed annuities and premiums payable annually. The purpose of the investigation in the present chapter is to ascertain the changes that take place in the ordinary formulas if the annuities and premiums be made payable at more frequent intervals.

2. We shall still consider that the total of an annuity payable in any one year is unity; so that, for instance, if an annuity be payable by m instalments at equal intervals throughout the year, each instalment will be $\frac{1}{m}$, and all the instalments added together will be equal to unity.

3. Let $a_x^{(m)}$ represent the present value of an annuity of 1 per annum payable by m equal instalments throughout each year during the life of (x). To find $a_x^{(m)}$.

The value of an annuity of which the first payment is to be made at once is $1+a_x$; and the value of the same annuity when the first payment is to be made at the end of a year is a_x; therefore, by simple proportion, the value of the same annuity when the first payment is to be made at the end of the mth part of a year will be $a_x+\frac{m-1}{m}$. Let there be m such annuities, of which the first payments are to be made at the end of $\frac{1}{m}$, $\frac{2}{m}$, &c., of a year respectively. Then the sum of their values will be

$$\left(a_x+\frac{m-1}{m}\right)+\left(a_x+\frac{m-2}{m}\right)+\text{\&c.}+\left(a_x+\frac{m-m}{m}\right)$$

$$=ma_x+\frac{m(m-1)}{2m},$$

whence, dividing by m, we have approximately,

$$a_x^{(m)} = a_x + \frac{m-1}{2m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation 1 gives a first approximation to the value sought, and it is the value very commonly used in practice. It will be noticed that no account is taken of the age of the life, or of the rates of mortality and interest; also that the argument will equally apply to annuities on joint lives, and to annuities on the last survivor or survivors of any number of lives. The approximation is, however, a very rough one, and more exact formulas must be investigated.

4. It may be well in passing, to point out that in the extreme case of a perpetual annuity, that is, an annuity calculated on the basis of q_x being equal to 0 for all values of x, formula 1 gives the proper correction, provided we assume simple and not compound interest for fractional parts of a year. Thus, at simple interest, the gain through having $\frac{1}{m}$ paid at the end of the first mth part of the year instead of at the end of the year, is $\frac{1}{m} \cdot \frac{m-1}{m} \cdot i$, and of having another $\frac{1}{m}$ paid at the end of the second mth part of the year $\frac{1}{m} \cdot \frac{m-2}{m} \cdot i$, and so on. Therefore each year's rent of the annuity when it is payable $m$thly, is greater than the same when it is payable yearly, by the sum of the series $\frac{i}{m}\left(\frac{m-1}{m} + \frac{m-2}{m} + \&c. + \frac{m-m}{m}\right)$, that is by $\frac{i(m-1)}{2m}$; and the increase in the value of a perpetuity when it is made payable $m$thly is therefore itself the value of a yearly perpetuity of $\frac{i(m-1)}{2m}$. That is

$$a_\infty^{(m)} = a_\infty + a_\infty \frac{i(m-1)}{2m}$$

$$= a_\infty + \frac{m-1}{2m},$$

since $$a_\infty = \frac{1}{i}.$$

5. To find a second approximate formula for the correction in the value of a life annuity when it is made payable $m$thly we will make the usual assumption that the deaths are equally distributed throughout the year.

6. It will be convenient to consider in the first place the value of an annuity payable half-yearly.

The difference between the values of an annuity when payable yearly and half-yearly respectively, consists of two elements:—the first being the interest gained on the sums paid in the *middle* of each year which the life completes instead of at the *end* thereof; and the second being the value of the chance of receiving an extra payment in the middle of the year in which the life drops, in consequence of the life dropping in the second and not in the first half of the year. When the annuity is payable half-yearly, there is $\frac{1}{2}$ payable in the middle instead of at the end of each year which the life survives; and this payment at the end of the year will amount to $\frac{1}{2}(1+i)^{\frac{1}{2}}$; and the excess of the value of this over the value of the same payment made at the end of the year will be $\frac{(1+i)^{\frac{1}{2}}-1}{2}$. The present value of all such differences, supposed to be payable at the end of each year which the life survives, is therefore $a_x \times \frac{(1+i)^{\frac{1}{2}}-1}{2}$.

The second element in the difference is the chance of receiving an extra payment of $\frac{1}{2}$ in the middle of the year in which the life drops. If this extra $\frac{1}{2}$ were certainly payable at the end of the year in which the life drops, its present value would be $\frac{1}{2}\mathrm{A}_x$; but as, if paid at all, it will be paid in the middle of the year, it will by the end of the year have increased by the operation of interest to $\frac{1}{2}(1+i)^{\frac{1}{2}}$, and, still supposing it to be certainly receivable, its present value is therefore $\frac{1}{2}(1+i)^{\frac{1}{2}}\mathrm{A}_x$. Now, the deaths being assumed to be distributed uniformly, the chance of the life dropping in the second half of the year is equal to the chance of its dropping in the first half: hence the value of this portion of the correction is one-half of $\frac{1}{2}$ to be certainly received in the middle of the year in which the life drops, that is, $\frac{1}{4}(1+i)^{\frac{1}{2}}\mathrm{A}_x$. Hence

$$a_x^{(2)} = a_x + a_x \times \frac{(1+i)^{\frac{1}{2}}-1}{2} + \tfrac{1}{4}(1+i)^{\frac{1}{2}}\mathrm{A}_x$$

$$= a_x \times \frac{1+(1+i)^{\frac{1}{2}}}{2} + \tfrac{1}{4}(1+i)^{\frac{1}{2}}\mathrm{A}_x \quad . \ . \ . \ . \ . \quad (2)$$

7. It is easy to express $a_x^{(2)}$ in terms of a_x without involving A_x. The difference between $a_x^{(2)}$ and a_x is, $a_x \times \frac{(1+i)^{\frac{1}{2}}-1}{2} + \frac{1}{4}(1+i)^{\frac{1}{2}}\mathrm{A}_x$; and, since $\mathrm{A}_x = \frac{1-ia_x}{1+i}$, this difference becomes

$$a_x\times\frac{(1+i)^{\frac{1}{2}}-1}{2}+\frac{1-ia_x}{4(1+i)^{\frac{1}{2}}}$$

$$=\frac{a_x}{4}\left\{2(1+i)^{\frac{1}{2}}-2-\frac{i}{(1+i)^{\frac{1}{2}}}\right\}+\frac{1}{4(1+i)^{\frac{1}{2}}}$$

$$=\frac{a_x}{4}\left\{\frac{2+i}{(1+i)^{\frac{1}{2}}}-2\right\}+\frac{1}{4(1+i)^{\frac{1}{2}}}$$

$$=\frac{a_x}{4}\left\{(1+i)^{-\frac{1}{2}}+(1+i)^{\frac{1}{2}}-2\right\}+\tfrac{1}{4}(1+i)^{-\frac{1}{2}}$$

Adding a_x to this, we have

$$a_x^{(2)}=\frac{a_x}{4}\left\{2+(1+i)^{-\frac{1}{2}}+(1+i)^{\frac{1}{2}}\right\}+\tfrac{1}{4}(1+i)^{-\frac{1}{2}}.\quad . \quad . \quad (3)$$

8. If in equation 3 we expand $(1+i)^{\frac{1}{2}}$ and $(1+i)^{-\frac{1}{2}}$ in powers of i, the value of the correction just found can readily be measured. Thus,

$$a_x^{(2)}=\frac{a_x}{4}\left\{2+(1+i)^{-\frac{1}{2}}+(1+i)^{\frac{1}{2}}\right\}+\tfrac{1}{4}(1+i)^{-\frac{1}{2}}$$

$$=\frac{a_x}{4}\left\{2+1-\frac{i}{2}+\frac{3i^2}{8}-\frac{5i^3}{16}+\text{\&c.}\right.$$

$$\left.+1+\frac{i}{2}-\frac{i^2}{8}+\frac{i^3}{16}-\text{\&c.}\right\}$$

$$+\frac{1}{4}\left\{1-\frac{i}{2}+\frac{3i^2}{8}-\text{\&c.}\right\}$$

$$=a_x+\tfrac{1}{4}+a_x\left\{\frac{i^2}{16}-\frac{i^3}{16}+\text{\&c.}\right\}-\left\{\frac{i}{8}-\frac{3i^2}{32}+\text{\&c.}\right\}\quad . \quad . \quad (4)$$

9. Comparing formulas 1 and 4, we see that in formula 1 all the terms involving i and its powers have been ignored, and we are now enabled in half-yearly cases to estimate the closeness of the approximation given in formula 1. In practice a_x is always less than 25 at 3 per-cent interest, and less than 15 at 6 per-cent interest. Therefore when a_x has its greatest value at 3 per-cent interest, the error that arises in using formula 1 instead of formula 4, is ·00136—·00367, or —·00231; while as the value of a_x diminishes the error numerically increases, its extreme limit being —·00367. Similarly, at 6 per-cent interest, the error when a_x has its greatest value is ·00317—·00716 or —·00399; and as a_x diminishes the error numerically increases, its extreme limit being —·00716.

10. We shall get a good approximation if we write

$$a_x^{(2)} = a_x + \tfrac{1}{4} - \frac{i}{8} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

In formula 5 the error will be greatest when a_x has its greatest value, and the limit of error therefore is, at 3 per-cent interest ·00144, and at 6 per-cent interest ·00351. For practical purposes it may be assumed that formula 5 gives the value of $a_x^{(2)}$ correct to three places of decimals.

11. It is worthy of note that formula 3 may be derived directly from the mortality table, by calculating the value of each payment by means of the formula

$$a_x^{(2)} = \frac{1}{2l_x}\left\{\tfrac{1}{2}(l_x + l_{x+1})v^{\frac{1}{2}} + l_{x+1}v + \tfrac{1}{2}(l_{x+1} + l_{x+2})v^{\frac{3}{2}} + \&c.\right\} \quad . \quad (6)$$

12. We have found the value of a half-yearly annuity on the assumption of a uniform distribution of deaths, and in a similar manner the value may be found of an annuity payable m times a year, $a_x^{(m)}$. As before, the correction is twofold, first for interest and second for mortality.

As regards interest, the gain on the first payment in each year of the $m$thly annuity is $\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}} - 1\right\}$, and on the second, $\frac{1}{m}\left\{(1+i)^{\frac{m-2}{m}} - 1\right\}$, and so on; and therefore the total gain from interest in each year which the life survives, on account of the annuity being payable $m$thly, is the sum of the series of m terms

$$\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}} - 1\right\} + \frac{1}{m}\left\{(1+i)^{\frac{m-2}{m}} - 1\right\} + \&c. + \frac{1}{m}\left\{(1+i)^{\frac{m-m}{m}} - 1\right\}$$

The sum of this series is $\frac{1}{m}\left\{\frac{i}{(1+i)^{\frac{1}{m}} - 1} - m\right\}$, and therefore the value for the whole of life of the correction due to interest is $\frac{a_x}{m}\left\{\frac{i}{(1+i)^{\frac{1}{m}} - 1} - m\right\}$; and, adding this to a_x, we have as the value of the $m$thly annuity, $a_x \times \frac{i}{m\{(1+i)^{\frac{1}{m}} - 1\}}$, when for the moment we leave out of account the correction due to mortality.

For the value of this second correction we have the expression

$$\mathrm{A}_x\left\{\frac{m-1}{m} \cdot \frac{(1+i)^{\frac{m-1}{m}}}{m} + \frac{m-2}{m} \cdot \frac{(1+i)^{\frac{m-2}{m}}}{m} + \&c. + \frac{1}{m}\,\frac{(1+i)^{\frac{1}{m}}}{m}\right\}.$$

This we can easily see to be correct, because, by hypothesis, there is an equal chance of the life failing in each of the mth parts of the year of death. Taking the first payment of the annuity in the year of death; if it were certainly receivable its value would be $\text{A}_x\dfrac{(1+i)^{\frac{m-1}{m}}}{m}$; but as, by hypothesis, death certainly takes place within the year, the probability of its being received is $\dfrac{m-1}{m}$, and therefore its actual present value is $\dfrac{m-1}{m}\text{A}_x\dfrac{(1+i)^{\frac{m-1}{m}}}{m}$. Similarly for each of the other payments in the year of death. To find a concise form for the expression:—for the sake of brevity writing $(1+i)^{\frac{1}{m}}=c$, and reversing the series, it becomes

$$\frac{\text{A}_x}{m^2}\left\{c+2c^2+3c^3+\&\text{c.}+(m-1)c^{m-1}\right\}.$$

To find the value of the factor in the brackets, we multiply and divide by $(c-1)$, and the result is

$$\frac{mc^m-(c+c^2+\&\text{c.}+c^m)}{c-1}$$

$$=\frac{mc^m}{c-1}-c\cdot\frac{c^m-1}{(c-1)^2}.$$

Writing now for c its value, $(1+i)^{\frac{1}{m}}$, the correction due to mortality becomes

$$\text{A}_x\left[\frac{1+i}{m\{(1+i)^{\frac{1}{m}}-1\}}-\frac{i(1+i)^{\frac{1}{m}}}{m^2\{(1+i)^{\frac{1}{m}}-1\}^2}\right]$$

$$=\text{A}_x\frac{m(1+i)\{(1+i)^{\frac{1}{m}}-1\}-i(1+i)^{\frac{1}{m}}}{m^2\{(1+i)^{\frac{1}{m}}-1\}^2},$$

and the whole value of the $m$thly annuity is

$$a_x^{(m)}=a_x\times\frac{i}{m\{(1+i)^{\frac{1}{m}}-1\}}+\text{A}_x\times\frac{m(1+i)\{(1+i)^{\frac{1}{m}}-1\}-i(1+i)^{\frac{1}{m}}}{m^2\{(1+i)^{\frac{1}{m}}-1\}^2}\qquad(7)$$

But $m\{(1+i)^{\frac{1}{m}}-1\}$ is the nominal annual rate of interest convertible m times a year, corresponding to the effective rate i; and if we call this nominal rate $j_{(m)}$, and remember that $(1+i)^{\frac{1}{m}}=1+\frac{j_{(m)}}{m}$, equation 7 becomes

$$a_x^{(m)}=a_x\times\frac{i}{j_{(m)}}+\mathrm{A}_x\times\frac{(1+i)j_{(m)}-\left(1+\frac{j_{(m)}}{m}\right)i}{j^2_{(m)}}\quad .\ .\ .\ (8)$$

13. Equation 8 may be modified in a very instructive manner. Because $\mathrm{A}_x=\frac{1-ia_x}{1+i}$, we have

$$a_x^{(m)}=a_x\times\frac{i}{j_{(m)}}+\frac{1-ia_x}{1+i}\times\frac{1+i}{j_{(m)}}-\mathrm{A}_x\frac{\left(1+\frac{j_{(m)}}{m}\right)i}{j^2_{(m)}}$$

$$=\frac{1}{j_{(m)}}-\mathrm{A}_x\frac{i}{j_{(m)}}\left(\frac{1}{j_{(m)}}+\frac{1}{m}\right)\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ (9)$$

But $\frac{1}{j_{(m)}}$ is the value of a perpetuity payable m times a year, which we may write $a_\infty^{(m)}$; and $\left(\frac{1}{j_{(m)}}+\frac{1}{m}\right)$ is the value of a similar perpetuity-due. Also, on the supposition of a uniform distribution of deaths, it can easily be proved that $\mathrm{A}_x\times\frac{i}{j_{(m)}}$ is the value of a unit payable at the end of the mth part of the year in which (x) dies. The proof is as follows:—The beneficiaries gain in interest by the earlier payment of the sum assured. If (x) die in the first mth part of the year, the unit paid at the end of that part is equivalent to $(1+i)^{\frac{m-1}{m}}$ paid at the end of the year. If (x) die in the second mth part, the equivalent is $(1+i)^{\frac{m-2}{m}}$; and so on. Now as, by hypothesis, the probability of his death is equal for each of the parts, by the end of the year of the death of (x) the amount of the unit paid at the end of the mth part of the year in which he dies will, on the average, be

$$\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}}+(1+i)^{\frac{m-2}{m}}+\&c.+(1+i)^{\frac{m-m}{m}}\right\}$$

$$=\frac{1}{m}\left\{(1+i)^{\frac{0}{m}}+(1+i)^{\frac{1}{m}}+\&c.+(1+i)^{\frac{m-1}{m}}\right\}$$

$$=\frac{1}{m}\times\frac{(1+i)^{\frac{m}{m}}-1}{(1+i)^{\frac{1}{m}}-1}=\frac{1}{m}\cdot\frac{i}{\left(1+\frac{j_{(m)}}{m}\right)-1}=\frac{i}{j_{(m)}}.$$

Therefore, a unit payable at the end of the *m*th part of the year of death is equivalent to $\frac{i}{j_{(m)}}$ payable at the end of the year of death. Writing $A_x^{(m)}$ for the value of a unit payable at the end of the *m*th part of the year of death, we have

$$A_x^{(m)}=A_x\times\frac{i}{j_{(m)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and equation 9 becomes

$$a_x^{(m)}=a_\infty^{(m)}-\left(\frac{1}{m}+a_\infty^{(m)}\right)A_x^{(m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

14. If in equation 11 we make m equal to unity, we at once have $a_x=a_\infty-(1+a_\infty)A_x$, which is the ordinary formula for a_x in terms of A_x. In fact the interpretations of the two equations are the same. If from the value of a perpetuity payable at any assigned equal intervals, we deduct the value of a similar perpetuity, first payment at the end of the interval in which (x) dies, the remainder is evidently an annuity on the life of (x), payable at the same intervals as the perpetuities.

15. Having the value of $j_{(m)}$, we can by means of equation 9 very easily calculate that of $a_x^{(m)}$; but $j_{(m)}$ itself is rather troublesome to compute. If, however, there be a table of the values of $j_{(m)}$, that difficulty is overcome; and therefore such a table is here appended for the rates of interest most commonly used in practice, and for half-yearly, quarterly, and continuous annuities.* It should be observed that, when the annuity is continuous, $j_{(m)}$ is the function called the force of interest or discount, and denoted by δ.

Table of the Nominal Rates of Interest, $j_{(m)}$, corresponding to Effective Rates, i.

i	$j_{(2)}$	$j_{(4)}$	$j_{(\infty)}=\delta$
·025	·024846	·024769	·024693
·03	·029778	·029668	·029559
·035	·034699	·034550	·034401
·04	·039608	·039414	·039221
·045	·044505	·044260	·044017
·05	·049390	·049089	·048790
·06	·059126	·058695	·058269
·07	·068816	·068234	·067659
·08	·078461	·077706	·076961
·09	·088061	·087113	·086178
·10	·097618	·096455	·095310

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* A very extensive table of the values of $j_{(m)}$ will be found in *J.I.A.*, vol. xxiii, p. 184.

16. The expression for $a_x^{(m)}$ having been now reduced to such simple and self-evident principles, it proves itself; but because equations 2 and 7 are so different in appearance it may be useful to show algebraically that they are identical. In equation 7, making $m=2$, when we multiply and divide by $\{(1+i)^{\frac{1}{2}}+1\}$ the coefficient of a_x becomes $\frac{(1+i)^{\frac{1}{2}}+1}{2}$. Multiplying out the coefficient of A_x we have

$$\frac{2(1+i)\{(1+i)^{\frac{1}{2}}-1\}-i(1+i)^{\frac{1}{2}}}{4\{(1+i)-2(1+i)^{\frac{1}{2}}+1\}}$$

$$=\frac{(1+i)^{\frac{1}{2}}}{4}\times\frac{2(1+i)-2(1+i)^{\frac{1}{2}}-i}{(1+i)-2(1+i)^{\frac{1}{2}}+1}$$

$$=\frac{(1+i)^{\frac{1}{2}}}{4}$$

Therefore $a_x^{(2)}=a_x\times\frac{(1+i)^{\frac{1}{2}}+1}{2}+A_x\frac{(1+i)^{\frac{1}{2}}}{4}$ as in formula 2.

17. Equation 8 displays the difference between an annuity-certain and a life annuity. The value of an annuity-certain payable m times a year, interest convertible yearly, is $a_{\overline{n}|}\times\frac{i}{j_{(m)}}$, and this corresponds exactly with the first term of equation 8 for the value of a similar life annuity. We therefore see that the correction for interest is precisely the same for a life annuity as for an annuity-certain, but that in the case of a life annuity we add another term, so as to make a correction for mortality. In practice, in dealing with life annuities, we always assume interest to be convertible yearly, but when annuities-certain are in question, it is usual to assume that interest is convertible at the periods at which the annuities are payable.

18. The correction due to interest as found in equation 8 is perfectly accurate, and is not affected by any supposition as to distribution of deaths. Not so, however, the correction due to mortality. We have made the gratuitous assumption that the deaths are evenly distributed in each year of age; and we have thereby introduced an error which, though small, is quite appreciable. The error will sometimes be in one direction and sometimes in the other, according to the age, x, of the life on which the annuity depends; and it will only be an accident if at any particular age it disappears. By using the Calculus of Finite Differences we can avoid the assumption of a uniform distribution of mortality, and so obtain more accurate formulas than any of those already given; and in fact we can reduce the error to as small dimensions as we please by making use of a sufficient number of orders of differences.

[19] We have

$$\begin{aligned} a_x^{(m)} = \frac{1}{D_x}\cdot\frac{1}{m}\Big\{ & D_{x+\frac{1}{m}} + D_{x+\frac{2}{m}} + D_{x+\frac{3}{m}} + \&c. + D_{x+1} \\ & + D_{x+1+\frac{1}{m}} + D_{x+1+\frac{2}{m}} + D_{x+1+\frac{3}{m}} + \&c. + D_{x+2} \\ & + \qquad \&c. \qquad \&c. \qquad \&c. \qquad \Big\}. \end{aligned}$$

Now, applying Lubbock's formula of summation (Chap. xxiv, formula 15) and writing in it $u_0 = D_x$, and $t = \frac{1}{m}$, and $n = m$; and also deducting u_0 from each side of the equation, and for the sake of conciseness using the symbols Δ_x and $\Delta^2{}_x$ for ΔD_x and $\Delta^2 D_x$, we have

$$\begin{aligned} & \Big\{ D_{x+\frac{1}{m}} + D_{x+\frac{2}{m}} + \&c. + D_{x+1} + D_{x+1+\frac{1}{m}} + \&c. \Big\} \\ & = m(D_{x+1} + D_{x+2} + D_{x+3} + \&c.) + \frac{m-1}{2} D_x \\ & \qquad + \frac{m^2-1}{12m}\Delta_x - \frac{m^2-1}{24m}\Delta^2{}_x + \&c. \\ & = mD_x a_x + \frac{m-1}{2} D_x + \frac{m^2-1}{12m}(\Delta_x - \tfrac{1}{2}\Delta^2{}_x) - \&c. \end{aligned}$$

Therefore

$$a_x^{(m)} = a_x + \frac{m-1}{2m} + \frac{m^2-1}{12m^2}\cdot\frac{\Delta_x - \frac{1}{2}\Delta^2{}_x}{D_x} - \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$= a_x + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\cdot\frac{3D_x - 4D_{x+1} + D_{x+2}}{2D_x} + \&c. \quad . \quad (13)$$

[20] If it were desired, we could, by means of Lubbock's formula, add further terms to formulas 12 and 13, but it will be found that if we stop at second differences the results will be abundantly accurate for practical purposes.

[21] Instead of using Lubbock's formula to get $a_x^{(m)}$, we may have recourse to that of Woolhouse. In formula 25 of Chap. xxiv, writing $u_0 = D_x$, we have

$$\begin{aligned} & \frac{1}{m}(D_x + D_{x+\frac{1}{m}} + D_{x+\frac{2}{m}} + \&c.) \\ & \quad = D_x + D_{x+1} + D_{x+2} + \&c. - \frac{m-1}{2m} D_x + \frac{m^2-1}{12m^2}\cdot\frac{dD_x}{dx} - \&c. \end{aligned}$$

Whence, dividing by D_x and deducting $\frac{1}{m}$ from each side,

$$a_x^{(m)}=a_x+\frac{m-1}{2m}+\frac{m^2-1}{12m^2}\cdot\frac{1}{D_x}\cdot\frac{dD_x}{dx}-\&c. \quad . \quad . \quad . \quad (14)$$

[22] To obtain the value of the differential coefficient, we have

$$\frac{1}{D_x}\cdot\frac{dD_x}{dx}=\frac{d\log D_x}{dx}=\frac{d\log l_x}{dx}+\frac{d\log v^x}{dx}$$

$$=\frac{1}{l_x}\frac{dl_x}{dx}+\frac{1}{v^x}\frac{dv^x}{dx}=-(\mu_x+\delta)$$

Therefore, in equation 14, making the needful substitution, we have

$$a_x^{(m)}=a_x+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu_x+\delta) \quad . \quad . \quad . \quad . \quad (15)$$

[23] In formula 22 of Chap. ii it was shown that, approximately,

$$\mu_x=\frac{d_{x-1}+d_x}{2l_x}=\frac{l_{x-1}-l_{x+1}}{2l_x}.$$

In precisely the same manner from formula 25 of Chap. xxiii, it follows that, approximately,

$$\mu_x+\delta=\frac{D_{x-1}-D_{x+1}}{2D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Using this approximation, we have

$$a_x^{(m)}=a_x+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}\cdot\frac{D_{x-1}-D_{x+1}}{2D_x} \quad . \quad . \quad . \quad (17)$$

[24] By Woolhouse's formula we can insert further terms, and so increase the accuracy of the approximation. Thus, if for the third differential coefficient we take its approximate value as given in Chap. xxiii, formula 17, we shall have

$$a_x^{(m)}=a_x+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu_x+\delta)-\frac{m^4-1}{720m^4}\,\frac{\Delta^3D_{x-2}+\Delta^3D_{x-1}}{2D_x} \quad . \quad (18)$$

the last term of which enables us to measure the error that arises from retaining only three terms in equations 12 and 15. It will be found that the error seldom affects the sixth place of decimals in the annuity.

25. If in the foregoing formulas we make m infinitely great, that is, if we assume the annuity to be payable by infinitesimal instalments in every moment throughout the year, but so that the amount paid in each year completed is unity, we shall have an annuity payable momently, called a *continuous annuity*, and which is represented by $\bar{a}_x$.

From formula 1

$$\bar{a}_x = a_x + \tfrac{1}{2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (19)$$

From formula 9

$$\bar{a}_x = \frac{1}{\delta} - \mathrm{A}_x \times \frac{i}{\delta^2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (20)$$

From formula 12

$$\bar{a}_x = a_x + \tfrac{1}{2} + \frac{\Delta_x - \frac{1}{2}\Delta^2{}_x}{12\mathrm{D}_x} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (21)$$

From formula 15

$$\bar{a}_x = a_x + \tfrac{1}{2} - \tfrac{1}{12}(\mu_x + \delta) \quad . \ . \ . \ . \ . \ . \ . \ . \quad (22)$$

[26] The values of continuous annuities can best be found by the direct application of the differential and integral calculus. By Chap. xxiv, formula 24, $\int_0^\infty u_t dt = \Sigma^{(1)}u_t - \frac{1}{2}u_0 + \frac{1}{12} \cdot \frac{du_0}{dt} - \&c.$ For an annuity $u_t = v^t l_{x+t}$, and $\frac{d(v^t l_{x+t})}{dt} = -l_x(\mu_x + \delta)$. Also $\int_0^\infty u_t dt = \int_0^\infty v^t l_{x+t} dt = l_x \bar{a}_x$. Therefore $l_x \bar{a}_x = (l_x + v l_{x+1} + v^2 l_{x+2} + \&c.) - \frac{1}{2} l_x - \frac{l_x}{12}(\mu_x + \delta)$; and $\bar{a}_x = a_x + \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$ as already found in equation 22.

Or the process may be slightly modified.

$$\begin{aligned}
\bar{a}_x &= \int_0^\infty {}_tp_x v^t dt \\
&= \Sigma^{(1)}({}_tp_x v^t) - \tfrac{1}{2} + \tfrac{1}{12} \cdot \frac{d({}_tp_x v^t)}{dt} \\
&= \mathbf{a}_x - \tfrac{1}{2} - \tfrac{1}{12}(\mu_x + \delta) \\
&= a_x + \tfrac{1}{2} - \tfrac{1}{12}(\mu_x + \delta) \quad \text{as before.}
\end{aligned}$$

27. It may be useful for reference to tabulate here the expressions for half-yearly, quarterly, and continuous annuities, as given by the various formulas investigated. In each case, to find the continuous annuity, we have merely to make m infinite.

Formula	$a_x^{(2)}$	$a_x^{(4)}$	$\bar{a}_x$
1	$a_x+\frac14$	$a_x+\frac38$	$a_x+\frac12$
9	$\frac{1}{j_{(2)}}-A_x\frac{i}{j_{(2)}}\left(\frac{1}{j_{(2)}}+\frac12\right)$	$\frac{1}{j_{(4)}}-A_x\frac{i}{j_{(4)}}\left(\frac{1}{j_{(4)}}+\frac14\right)$	$\frac{1}{\delta}-A_x\frac{i}{\delta^2}$
12	$a_x+\frac14+\frac{\Delta_x-\frac12\Delta^2{}_x}{16D_x}$	$a_x+\frac38+\frac{5(\Delta_x-\frac12\Delta^2{}_x)}{64D_x}$	$a_x+\frac12+\frac{\Delta_x-\frac12\Delta^2{}_x}{12D_x}$
15	$a_x+\frac14-\frac{1}{16}(\mu_x+\delta)$	$a_x+\frac38-\frac{5}{64}(\mu_x+\delta)$	$a_x+\frac12-\frac{1}{12}(\mu_x+\delta)$
17	$a_x+\frac14-\frac{D_{x-1}-D_{x+1}}{32D_x}$	$a_x+\frac38-\frac{5(D_{x-1}-D_{x+1})}{128D_x}$	$a_x+\frac12-\frac{D_{x-1}-D_{x+1}}{24D_x}$

28. The numerical effect of the formulas may be estimated from examples calculated at 3 per-cent interest, the ages of the lives being 30 and 60 respectively.

Formula	$a_{30}^{(2)}$	$a_{30}^{(4)}$	$\bar{a}_{30}$
1	20·14502	20·27002	20·39502
9	20·14250	20·26685	20·39152
12	20·14269	20·26711	20·39192
15	20·14269	20·26711	20·39192
17	20·14269	20·26711	20·39192

Formula	$a_{60}^{(2)}$	$a_{60}^{(4)}$	$\bar{a}_{60}$
1	10·47349	10·59849	10·72349
9	10·47024	10·59447	10·71942
12	10·46982	10·59390	10·71859
15	10·46982	10·59390	10·71859
17	10·46982	10·59390	10·71859

29. Formulas 12, 15, and 17 give identical results to five places of decimals if the mortality table be well graduated. When the force of mortality is tabulated, No. 15 is the most convenient to use, but in other cases either No. 12 or No. 17 should be employed. Formula 9 is not only less accurate, but also more cumbrous.

30. For deferred and temporary annuities, we shall have in all cases

$$ {}_{t|}a_x^{(m)} = v^t {}_tp_x \times a_{x+t}^{(m)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

$$ {}_{|t}a_x^{(m)} = a_x^{(m)} - {}_{t|}a_x^{(m)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

31. The last two equations may be used with any of the formulas with equal facility, and it is worthy of note that the factor $v^t{}_tp_x$, $\left(=\frac{\mathrm{D}_{x+t}}{\mathrm{D}_x}\right)$ is not affected by the way in which the annuity is payable. Taking formula 1, which is the most generally used in practice, and employing commutation columns,

$$ {}_{t|}a_x^{(m)} = \frac{\mathrm{N}_{x+t} + \frac{m-1}{2m}\mathrm{D}_{x+t}}{\mathrm{D}_x} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (25) $$

$$ {}_{|t}a_x^{(m)} = \frac{\mathrm{N}_x - \mathrm{N}_{x+t} + \frac{m-1}{2m}(\mathrm{D}_x - \mathrm{D}_{x+t})}{\mathrm{D}_x} \quad . \; . \; . \quad (26) $$

32. It has already been remarked, Art. 3, that formula 1 applies to annuities on joint lives. Formulas 2 to 11 do not so apply, as they are avowedly based upon the hypothesis of a uniform distribution of deaths. The remaining formulas can be used for joint lives if for D_x, Δ_x, $\Delta^2{}_x$, and μ_x, we substitute the corresponding joint-life functions.

33. Those formulas that are applicable to joint-life annuities may also be used for annuities on the last survivor or survivors of any number of lives, because such annuities can always be resolved into others involving only single and joint lives.

[**34**] To find the value of an annuity the first payment of which is to be made $\frac{1}{t}$ of a year hence, ${}_{\frac{1}{t}|}\mathrm{a}_x$. We have by Chap. xxiii, formula 5,

$$ {}_{\frac{1}{t}|}\mathrm{a}_x \times \mathrm{D}_x = \mathrm{D}_{x+\frac{1}{t}} + \mathrm{D}_{x+1+\frac{1}{t}} + \mathrm{D}_{x+2+\frac{1}{t}} + \&c. $$

$$ = \mathrm{D}_x + \frac{1}{t}\Delta_x - \frac{t-1}{t^2\,\underline{|2}}\Delta^2{}_x + \frac{(t-1)(2t-1)}{t^3\,\underline{|3}}\Delta^3{}_x - \&c. $$

$$ + \mathrm{D}_{x+1} + \frac{1}{t}\Delta_{x+1} - \frac{t-1}{t^2\,\underline{|2}}\Delta^2{}_{x+1} + \frac{(t-1)(2t-1)}{6t^3}\Delta^3{}_{x+1} + \&c. $$

$$ + \quad \&c. \qquad\qquad \&c. \qquad\qquad \&c. $$

$$ = \mathrm{N}_{x-1} - \frac{1}{t}\mathrm{D}_x + \frac{t-1}{2t^2}\Delta_x - \frac{(t-1)(2t-1)}{t^3\,\underline{|3}}\Delta^2{}_x + \&c. $$

$$ = \mathrm{N}_x + \frac{t-1}{t}\mathrm{D}_x + \frac{t-1}{2t^2}\Delta_x - \frac{(t-1)(2t-1)}{6t^3}\Delta^2{}_x + \&c. $$

Therefore

$$_{\frac{1}{t}|}\text{a}_x = a_x + \frac{t-1}{t} + \frac{t-1}{2t^2}\frac{\Delta_x}{\text{D}_x} - \frac{(t-1)(2t-1)}{6t^3}\frac{\Delta^2{}_x}{\text{D}_x} \quad . \quad . \quad (27)$$

[35] The correctness of the last expression can be tested by using the formula to find $a_x^{(m)}$. If we write $t=\frac{r}{s}$, so that $\frac{1}{t}=\frac{s}{r}$, we have

$$_{\frac{s}{r}|}\text{a}_x = a_x + \left(1-\frac{s}{r}\right) + \frac{1}{2}\left(\frac{s}{r}-\frac{s^2}{r^2}\right)\frac{\Delta_x}{\text{D}_x} - \frac{1}{6}\left(\frac{2s}{r}-\frac{3s^2}{r^2}+\frac{s^3}{r^3}\right)\frac{\Delta^2{}_x}{\text{D}_x}$$

If in this last expression we give s every value from 1 to r, and take the sum, we shall have the value of an annuity of r per annum, payable r times a year. Thus

$$\Sigma\left(1-\frac{s}{r}\right) = \frac{1}{r}\Sigma(r-s) = \frac{1}{r}\left(r^2-\frac{r(r+1)}{2}\right) = \frac{r-1}{2}.$$

Also,

$$\Sigma\frac{1}{2}\left(\frac{s}{r}-\frac{s^2}{r^2}\right) = \frac{1}{2r^2}\Sigma(rs-s^2)$$

$$= \frac{1}{2r^2}\left\{\frac{r^2(r+1)}{2} - \frac{r(r+1)(2r+1)}{6}\right\} = \frac{r^2-1}{12r};$$

and finally,

$$\Sigma\frac{1}{6}\left(\frac{2s}{r}-\frac{3s^2}{r^2}+\frac{s^3}{r^3}\right) = \frac{1}{6r^3}\Sigma(2sr^2-3s^2r+s^3)$$

$$= \frac{1}{6r^3}\left\{\frac{2r^3(r+1)}{2} - \frac{3r^2(r+1)(2r+1)}{6} + \frac{r^4+2r^3+r^2}{4}\right\}$$

$$= \frac{r+1}{6} - \frac{2r^2+3r+1}{12r} + \frac{r^2+2r+1}{24r}$$

$$= \frac{r^2-1}{24r}.$$

Therefore the whole expression becomes

$$\Sigma_{\frac{s}{r}|}\text{a}_x = ra_x + \frac{r-1}{2} + \frac{r^2-1}{12r}\frac{\Delta_x}{\text{D}_x} - \frac{r^2-1}{24r}\frac{\Delta^2{}_x}{\text{D}_x}$$

Dividing now by r, and remembering that $\frac{1}{r}\Sigma_{\frac{s}{r}|}\text{a}_x = a_x^{(r)}$, we have

$$a_x^{(r)} = a_x + \frac{r-1}{2r} + \frac{r^2-1}{12r^2}\cdot\frac{\Delta_x - \frac{1}{2}\Delta^2{}_x}{\text{D}_x},$$

which is the same expression as in equation 12, when $r=m$.

[36] Returning now to equation 27, and writing $t=2$, we have, for the value of an annuity first payment half a year hence,

$$ {}_{\frac{1}{2}|}\mathrm{a}_x = a_x + \tfrac{1}{2} + \frac{\Delta_x - \frac{1}{2}\Delta^2{}_x}{8\mathrm{D}_x} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (28) $$

[37] The value usually assigned in practice to this annuity is $a_x+\frac{1}{2}$, but that is too great, the third term in formula 28 being essentially negative. In practice it is usual to employ the same value for $\bar{a}_x$ and ${}_{\frac{1}{2}|}\mathrm{a}_x$, namely, $a_x+\frac{1}{2}$; but these annuities are not really equal. The third term of $\bar{a}_x$ is $\frac{2}{24}\cdot\frac{\Delta_x-\frac{1}{2}\Delta^2{}_x}{\mathrm{D}_x}$, while the third term of ${}_{\frac{1}{2}|}\mathrm{a}_x$ is $\frac{3}{24}\frac{\Delta_x-\frac{1}{2}\Delta^2{}_x}{\mathrm{D}_x}$, the latter of which is numerically greater than the former by $\frac{1}{24}\cdot\frac{\Delta_x-\frac{1}{2}\Delta^2{}_x}{\mathrm{D}_x}$; and as this term is negative, it follows that $\bar{a}_x > {}_{\frac{1}{2}|}\mathrm{a}_x$.

38. If we take the usual approximation for $a_x^{(m)}$, namely, $a_x+\frac{m-1}{2m}$, and if further we assume that the value of an annuity first payment due at the expiration of one-half of the period intervening between the payments, is the arithmetical mean between $\mathrm{a}_x^{(m)}$ and $a_x^{(m)}$, we shall have in all cases ${}_{\frac{1}{2m}|}\mathrm{a}_x^{(m)}=\bar{a}_x$. This approximate formula will be found useful in some investigations.

39. To find the annual premium payable by m equal instalments throughout each year, for an assurance on the life of (x).

In analogy with the notation for the annuity, let $\mathrm{P}_x^{(m)}$ represent the total amount payable per annum, when the premium is payable by m equal instalments.

From the nature of the case, the total value of all the premiums must be equal to the value of the sum assured. But the premiums constitute an annuity-due of $\mathrm{P}_x^{(m)}$ per annum, payable m times a year, and their present value is $\mathrm{P}_x^{(m)}\left(\frac{1}{m}+a_x^{(m)}\right)$. Therefore

$$ \mathrm{P}_x^{(m)} = \frac{\mathrm{A}_x}{\frac{1}{m}+a_x^{(m)}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (29) $$

40. In equation 29 any one of the approximate formulas for $a_x^{(m)}$ may be used in order to get a corresponding approximate formula for $\mathrm{P}_x^{(m)}$. It will be found, however, that small differences in the value of $a_x^{(m)}$ do not appreciably affect $P_x^{(m)}$, and that for practical purposes it will be quite sufficient to employ formula 1, and write $a_x^{(m)} = a_x + \frac{m-1}{2m}$. We thus have

$$P_x^{(m)} = \frac{A_x}{\frac{m+1}{2m} + a_x} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

41. If we express both a_x and A_x in terms of the annual premium, P_x, we have

$$P_x^{(m)} = \frac{P_x}{P_x + d} \div \left(\frac{m+1}{2m} + \frac{1}{P_x + d} - 1\right)$$

$$= \frac{2mP_x}{2m - (m-1)(P_x + d)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

42. It should be observed that when $P_x^{(m)}$ is found by any of the foregoing formulas, the last payment of premium takes place at the beginning of that mth part of the year in which (x) dies. The difference between $P_x^{(m)}$ and P_x is due to two causes, namely, first, the loss of interest on that portion of the premium not paid at the beginning of the year; and secondly, the chance of losing part of the premium payable for the year of death. In practice there is a third cause of difference, namely, the extra trouble and expense incurred in collecting premiums at frequent intervals. It thus happens that the theoretical formulas for $P_x^{(m)}$ are very seldom used, and it is common to increase P_x by an arbitrary percentage, such as $2\frac{1}{2}$ per-cent in half-yearly cases. Caution must, however, be observed in so doing, because a percentage addition which would be more than sufficient at the younger ages might be inadequate at the older. Further, it is the custom with some companies to deduct from the sum assured the instalments of premium for the current year of assurance, unpaid at the death of the life assured.

CHAPTER X.

ASSURANCES PAYABLE AT ANY OTHER MOMENT THAN THE END OF THE YEAR OF DEATH.

1. According to the formula $\mathrm{A}_x = 1 - d(1 + a_x)$, (Chap. vii, formula 25), or any of those that may be algebraically deduced from it, the unit assured is supposed to be payable at the end of the year of death. But, on the assumption that the deaths are uniformly distributed over the year, as many persons die in the first half of the year as in the second; and, taken one with another, they may be considered all to die in the middle of the year; and, therefore, the usual formula for A_x may be considered to give the value of a unit payable six months after death.

2. In dealing with annuities the symbol $a_x^{(m)}$ was used to represent an annuity the payments of which are made at the end of each $\frac{1}{m}$ of a year, so that $a_x^{(1)}$ is an annuity payable at the end of each year which (x) survives. By analogy, as A_x or $\mathrm{A}_x^{(1)}$ is an assurance payable at the end of the year in which (x) dies, $\mathrm{A}_x^{(m)}$ will represent an assurance payable at the end of the interval $\frac{1}{m}$ in which (x) dies. Also, as $\overline{a}_x$ represents an annuity payable every moment that (x) survives, so $\overline{\mathrm{A}}_x$ will represent an assurance payable at the moment when (x) dies.

3. According to the same reasoning by which we reach the conclusion that A_x represents an assurance payable half a year after the death of (x), we are led to the result that $\mathrm{A}_x^{(m)}$ represents an assurance payable half an interval after the death of (x), that is, payable $\frac{1}{2m}$ of a year after the death of (x). Therefore, if we wish to represent an assurance payable the tth part of a year after the death of (x), we must write $A_x^{\left(\frac{t}{2}\right)}$. Thus an assurance payable three months (that is a quarter of a year) after the death of (x), is $A_x^{(2)}$.

4. The student should make himself thoroughly familiar with the notation, which is strictly analogous to that adopted for annuities, but which, when applied to assurances, is apt at first to be a little confusing. The notation is based upon the assumption of a uniform distribution of deaths, but it is retained even when that assumption is discarded.

5. From the value of the assurance payable at the end of the year of death we can find its value payable at any other period, by simply considering what will be its amount at interest by the end of the year. Thus, if we wish to find the value of an assurance payable three months after death, we have merely to increase A_x by three months' interest; that is $A_x^{(2)}=A_x(1+i)^{\frac{1}{4}}$; and, generally, if we wish to find the value of a sum payable $\frac{1}{t}$ of a year after death, we have merely to increase A_x by interest for $\left(\frac{1}{2}-\frac{1}{t}\right)$ of a year. That is

$$A_x^{\left(\frac{t}{2}\right)}=A_x(1+i)^{\frac{1}{2}-\frac{1}{t}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

6. If t be made infinitely great, we have the value of an assurance payable at the instant of death

$$\overline{A}_x=A_x(1+i)^{\frac{1}{2}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

7. In Chap. ix, Art. 13, we found the value of the assurance payable at the end of the mth part of the year of death; that is, on the assumption of a uniform distribution of deaths, payable $\frac{1}{2m}$ of a year after death. In the expression given in that article, writing $m=\frac{t}{2}$, to conform to the present notation, we have

$$A_x^{\left(\frac{t}{2}\right)}=A_x \times \frac{i}{j_{\left(\frac{t}{2}\right)}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

where $j_{\left(\frac{t}{2}\right)}$ is, as before, the nominal yearly rate of interest convertible $\frac{1}{2}t$ times a year corresponding to the effective rate i.

In equation 3, making t infinite,

$$\overline{A}_x=A_x \times \frac{i}{\delta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

8. Equations 3 and 4 are perfectly accurate on the assumption of a uniform distribution of deaths. By them we obtain the values of the same functions as by equations 1 and 2 respectively, which, however, are only approximations. The expressions are not identical algebraically, yet arithmetically they are nearly equivalent to each other, and will produce practically the same results. For example, by equation 2 the coefficient of A_x, when interest is 6 per cent., is 1·029563, and by equation 4, 1·029709. The largest possible value of A_x is v, which at 6 per cent. interest is ·943396. Thus the greatest possible difference between the results of the two formulas is, at 6 per cent. interest, ·000138. At lower rates of interest the difference will be less. Thus at 3 per cent. the greatest possible difference will be ·000037. It is an extreme case to make $\mathrm{A}_x = v$. Of course, for the values of A_x that occur in practice, the differences between the results of the two formulas will be much smaller than those above mentioned.

9. We have found two expressions for the value of $\overline{\mathrm{A}}_x$, but they are both based on the assumption that the deaths are equally distributed throughout each year of age, and they are therefore only approximations. It is also evident that they are not applicable to joint lives, because of the assumption on which they stand. It is true that in practice it is common to assume that $\overline{\mathrm{A}}_{xy} = \mathrm{A}_{xy} \times (1+i)^{\frac{1}{2}}$, and no material error is thereby introduced; but it is easy to deduce a general formula which avoids this trifling error, and which does not present any arithmetical difficulties, at any rate in those cases where the function μ_x is tabulated.

10. To find, without hypothesis as to the distribution of deaths, $\mathrm{A}_x^{(m)}$, the value of a unit payable at the end of that mth part of the year in which (x) shall die:

If the unit were payable for certain at the end of the first mth part of the first year, its value would be $v^{\frac{1}{m}}$; but, as payment is deferred during the life of (x), we must deduct the value of the interest on $v^{\frac{1}{m}}$. The interest on $v^{\frac{1}{m}}$ for the mth part of a year is $v^{\frac{1}{m}}\{(1+i)^{\frac{1}{m}}-1\} = 1 - v^{\frac{1}{m}}$; and therefore the value of the interest during the life of (x) is $m(1-v^{\frac{1}{m}})a_x^{(m)}$. We therefore have

$$\mathrm{A}_x^{(m)} = v^{\frac{1}{m}} - m(1-v^{\frac{1}{m}})a_x^{(m)}$$

$$= 1 - m(1-v^{\frac{1}{m}})\left(\frac{1}{m} + a_x^{(m)}\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

11. Here we perceive the complete analogy to the equation $A_x=1-d(1+a_x)$: for $1-v^{\frac{1}{m}}$ is the discount on 1 for the mth part of a year, and $m(1-v^{\frac{1}{m}})$ the discount on 1 for a year, the operation of discount being effected m times in the year. If in equation 5 we make $m=1$, we at once have $1-d(1+a_x)$.

12. Formula 5 in its present shape does not enable us to find the value of a continuous assurance, but the end may be attained by means of a slight modification. We have

$$m(1-v^{\frac{1}{m}})=m\left\{1-\frac{1}{1+\frac{\dot{j}_{(m)}}{m}}\right\}=\frac{\dot{j}_{(m)}}{1+\frac{\dot{j}_{(m)}}{m}},$$

and equation 5 becomes

$$A_x^{(m)}=1-\frac{\dot{j}_{(m)}}{1+\frac{\dot{j}_{(m)}}{m}}\left\{\frac{1}{m}+a_x^{(m)}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

whence, making m infinite,

$$\bar{A}_x=1-\delta\bar{a}_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

13. If in formula 5 we write for $a_x^{(m)}$ its value as given in formulas 12 or 15 of Chap. ix, we get an accurate expression for $A_x^{(m)}$; and similarly in formula 7, writing for $\bar{a}_x$ its value as given in formulas 21 or 22 of Chap. ix, we have $\bar{A}_x$.

Having now found $\bar{A}_x$ accurately, we can easily pass to the value of an assurance payable at any assigned period after death. Representing, as before, by $A_x^{(\frac{t}{2})}$ the value of an assurance payable the tth part of a year after the death of (x), we simply have

$$A_x^{(\frac{t}{2})}=\bar{A}_x(1+i)^{-\frac{1}{t}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus the value of an assurance payable three months after death is $\bar{A}_x(1+i)^{-\frac{1}{4}}$, or payable one month after death, $\bar{A}_x(1+i)^{-\frac{1}{12}}$.

14. The reasoning by which formula 7 was deduced applies equally to endowment assurances and assurances on joint-lives, or on the last survivor or survivors of any number of lives, so that

$$\bar{A}_{x\overline{n}|} \quad =1-\delta\bar{a}_{x\overline{n}|} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\bar{A}_{wxyz\ldots(r)}=1-\delta\bar{a}_{wxyz\ldots(r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\bar{A}_{\overline{wxyz\ldots}}^{\;r} \quad =1-\delta\bar{a}_{\overline{wxyz\ldots}}^{\;r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

15. For assurances deferred t years, or temporary for t years, it is self-evident that

$$_{t|}\overline{\mathrm{A}}_x = v^t {}_tp_x \overline{\mathrm{A}}_{x+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$_{|t}\overline{\mathrm{A}}_x = \overline{\mathrm{A}}_x - {}_{t|}\overline{\mathrm{A}}_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

[16] The value of an assurance payable at the moment of death may very easily be found by means of the differential and integral calculus.

When h is a fractional part of a year, and $\mathrm{A}_x^{\left(\frac{1}{h}\right)}$ is consequently the value of 1 payable at the end of that hth part of the year in which (x) dies, we have

$$\mathrm{A}_x^{\left(\frac{1}{h}\right)} = -\frac{1}{l_x}\left\{v^h(l_{x+h}-l_x)+v^{2h}(l_{x+2h}-l_{x+h})+\&\mathrm{c.}\right\}$$

$$= \frac{1}{l_x}\Sigma_{n=1}^{n=\infty} v^{nh}(-\Delta l_{x+\overline{n-1}h})$$

where by $\Delta l_{x+\overline{n-1}h}$ is meant the finite differences of l_x in respect of the interval h. If now h diminish without limit, the symbols Σ and Δ are replaced by $\int$ and d, and $\mathrm{A}_x^{\left(\frac{1}{h}\right)}$ becomes $\overline{\mathrm{A}}_x$. Writing now nh and $\overline{n-1}h$ both $=t$, (h being indefinitely small), the expression takes the form

$$\overline{\mathrm{A}}_x = \frac{1}{l_x}\int v^t(-dl_{x+t})$$

$$= -\frac{1}{l_x}\int v^t dl_{x+t};$$

or, integrating by parts,

$$\overline{\mathrm{A}}_x = \frac{1}{l_x}\left\{-l_{x+t}v^t + \log_e v \int l_{x+t}v^t dt\right\}.$$

Taking the integral between the limits $t=0$ and $t=\infty$, we have, seeing that after the limiting age of the table l_x permanently and absolutely vanishes,

$$\overline{\mathrm{A}}_x = \frac{1}{l_x}(l_x - l_x\delta\bar{a}_x)$$

$$= 1-\delta\bar{a}_x,$$

as before.

[17] Or we may proceed differently, and find other and useful formulas. By the last preceding article,

$$l_x\bar{A}_x=-\int v^t dl_{x+t}$$
$$=-\int v^t\left(\frac{d}{dt}l_{x+t}\right)dt.$$

But
$$l_x\bar{a}_x=\int v^t l_{x+t}dt.$$

Therefore, differentiating each side with respect to x,

$$\frac{d}{dx}\left(l_x\bar{a}_x\right)=\int v^t\left(\frac{d}{dx}l_{x+t}\right)dt.$$

But
$$\frac{d}{dx}l_{x+t}=\frac{d}{dt}l_{x+t},$$

and therefore
$$\frac{d}{dx}\left(l_x\bar{a}_x\right)=\int v^t\left(\frac{d}{dt}l_{x+t}\right)dt$$
$$=\int v^t dl_{x+t};$$

and therefore
$$l_x\bar{A}_x=-\frac{d}{dx}\left(l_x\bar{a}_x\right).$$

Also
$$-\frac{d}{dx}\left(l_x\bar{a}_x\right)=-\left(\frac{dl_x}{dx}\bar{a}_x+l_x\frac{d\bar{a}_x}{dx}\right)$$
$$=l_x\mu_x\bar{a}_x-l_x\frac{d\bar{a}_x}{dx};$$

and therefore
$$\bar{A}_x=\mu_x\bar{a}_x-\frac{d\bar{a}_x}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

By Chap. xxiii, formula 27,

$$-\frac{d\bar{a}_x}{dx}=\tfrac{1}{2}(\bar{a}_{x-1}-\bar{a}_{x+1}),$$

and therefore,
$$\bar{A}_x=\mu_x\bar{a}_x+\tfrac{1}{2}(\bar{a}_{x-1}-\bar{a}_{x+1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

or writing $\bar{a}_x=\frac{1}{2}+a_x$, an approximation quite sufficiently accurate in this case for practical purposes,

$$\bar{A}_x=\mu_x(\tfrac{1}{2}+a_x)+\tfrac{1}{2}(a_{x-1}-a_{x+1}) \quad . \quad . \quad . \quad . \quad (16)$$

Again, we have
$$\bar{A}_x=-\frac{1}{l_x}\int v^t dl_{x+t}$$
$$=\frac{1}{l_x}\int v^t l_{x+t}\mu_{x+t}dt$$
$$=\frac{1}{l_x}\int v^t\frac{l_{x-1+t}-l_{x+1+t}}{2l_{x+t}}\cdot l_{x+t}dt$$
$$=\frac{1}{2}\int\left\{v^t\frac{l_{x-1+t}}{l_x}-v^t\frac{l_{x+1+t}}{l_x}\right\}dt$$
$$=\frac{1}{2}\int\left\{\frac{1}{p_{x-1}}v^t\frac{l_{x-1+t}}{l_{x-1}}-p_xv^t\frac{l_{x+1+t}}{l_{x+1}}\right\}dt$$
$$=\frac{1}{2}\left(\frac{\bar{a}_{x-1}}{p_{x-1}}-p_x\bar{a}_{x+1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

[18] The arguments by which formulas 15 and 17 were deduced apply equally to joint lives, so that

$$\bar{A}_{xy}=(\mu_x+\mu_y)\bar{a}_{xy}+\tfrac{1}{2}(\bar{a}_{x-1:y-1}-\bar{a}_{x+1:y+1}) \quad . \quad . \quad . \quad (18)$$

$$=\frac{1}{2}\left(\frac{\bar{a}_{x-1:y-1}}{p_{x-1:y-1}}-p_{xy}\bar{a}_{x+1:y+1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

[19] The following are the values by the various formulas at 3 per-cent interest of assurances on lives aged 30 and 60, payable at the moment of death.

Formula	$\bar{A}_{30}$	$\bar{A}_{60}$
2	·3972351	·6831242
4	·3972498	·6831495
7	·3972393	·6831713
15	·3973500	·6830800
16	·3973600	·6830600
17	·3973000	·6829400

It should be noticed that by formulas Nos. 15, 16, and 17 the annuity to five figures will not give more than five figures in the assurance.

[20] It may here be noted that the force of mortality may be expressed in terms of the expectation of life. If in formula 15 it be assumed that money yields no interest, then $\bar{A}_x=1$: also the annuities become the corresponding complete expectations of life. Whence

$$1=\mu_x\mathring{e}_x+\tfrac{1}{2}(\mathring{e}_{x-1}-\mathring{e}_{x+1})$$

and

$$\mu_x=\frac{1}{\mathring{e}_x}\left\{1-\tfrac{1}{2}(\mathring{e}_{x-1}-\mathring{e}_{x+1})\right\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

[21] In the case of the endowment assurance, the *endowment* portion is not affected in value when the time is changed at which the *assurance* portion is to become payable in the event of the failure of the life. We have already seen, formula 9, that it is correct to write $\bar{A}_{x\overline{n}|}=1-\delta\bar{a}_{x\overline{n}|}$. If, however, we wish to apply formulas 2 and 4 to this class of benefits we must separate the assurance into its component elements and write, by means of commutation columns,

$$\left.\begin{aligned}\bar{\mathbf{A}}_{x\overline{n}|}&=\frac{(M_x-M_{x+n})(1+i)^{\frac{1}{2}}+D_{x+n}}{D_x}\\ \bar{A}_{x\overline{n}|}&=\frac{(M_x-M_{x+n})\dfrac{i}{\delta}+D_{x+n}}{D_x}\end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

This point is of some importance in connection with the valuations of life offices, as will be seen in Chap. xviii.

CHAPTER XI.

Complete Annuities.

1. The annuities hitherto considered have been *curtate*; that is, they have not included a proportionate part of the yearly payment for the period between the last payment of the annuity and the moment of death. But such annuities are not the annuities of practice. These run up to the day of death, and are said to be *complete*, or *apportionable.* By Act of Parliament all annuities are apportionable unless a special stipulation to the contrary appears in the deed creating the annuity. In 1738, by 11 Geo. II. c. 19, annuities arising out of rents and profits from real estate were made apportionable; and in 1834, by 4 & 5 Wm. IV. c. 22, and again in 1870, by 33 & 34 Vict. c. 35, the rule was extended to all other annuities, except such annual sums as are made payable under policies of assurance. Therefore the premiums under life policies are not apportionable.

2. Seeing that both by law and by custom the great majority of annuities are apportionable, it becomes of importance to investigate the difference in value between a curtate and a complete annuity.

3. Taking first a yearly annuity, and assuming a uniform distribution of deaths, we see that, on the average, half a yearly payment will be due for the year of death, and will be payable at the moment of death. Therefore we shall have, by Chap. x, formula 2,

$$\mathring{a}_x = a_x + \tfrac{1}{2}\mathrm{A}_x(1+i)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

A little consideration will show that this formula gives too large a correction. On the supposition of a uniform distribution of deaths, we are right as regards the portion, ($\frac{1}{2}$), payable, on the average, for the year of death; but we assign to that portion too large a present value. If the life fail in the second half of the year, a larger sum will be payable than if it fail in the first half, and the payment will be longer delayed. Therefore the larger payments are in reality more affected by discount than the smaller ones, and are proportionately more reduced in present value. But the formula assumes that the payment, whether large or small, is made in the middle of the year of death, and therefore that all payments, of whatsoever magnitude, are equally affected by discount.

4. If the annuity be payable by m instalments throughout the year, on the average a sum of $\frac{1}{2m}$ will be payable at the moment of death, and we shall have

$$\mathring{a}_x^{(m)}=a_x^{(m)}+\frac{1}{2m}\mathrm{A}_x(1+i)^{\frac{1}{2}}\quad.\ .\ .\ .\ .\ .\ .\quad(2)$$

Equation 2 contains the same error as equation 1, and we must seek more correct expressions.

[5] Examining first yearly annuities:—Let the year of death be divided into r equal parts, r being very large, and presently to become infinite. If death take place in the sth in order of these parts, the sum of $\frac{s}{r}$ will become payable.* Also on the supposition of uniform distribution of deaths, the chance that $\frac{s}{r}$ will become payable is $\frac{1}{r}$. Therefore the value at the beginning of the year of death of the sum of $\frac{s}{r}$ is $\frac{s}{r}\times\frac{1}{r}\times v^{\frac{s}{r}}$. Giving now to s every integral value from 1 to r, and taking the sum, we have the value at the beginning of the year of death of the correction to make the annuity complete, that is, $\frac{1}{r^2}\left\{v^{\frac{1}{r}}+2v^{\frac{2}{r}}+3v^{\frac{3}{r}}+\&c.+rv^{\frac{r}{r}}\right\}$. Multiplying and dividing by $1-v^{\frac{1}{r}}$, the correction becomes

$$\frac{1}{r^2}\,\frac{v^{\frac{1}{r}}+v^{\frac{2}{r}}+v^{\frac{3}{r}}+\&c.+v^{\frac{r}{r}}-rv^{\frac{r+1}{r}}}{1-v^{\frac{1}{r}}}$$

$$=\frac{v^{\frac{1}{r}}(1-v)}{r^2(1-v^{\frac{1}{r}})^2}-\frac{v^{\frac{1}{r}}v}{r(1-v^{\frac{1}{r}})}$$

$$=\frac{(1+i)^{\frac{1}{r}}vi}{r^2\{(1+i)^{\frac{1}{r}}-1\}^2}-\frac{v}{r\{(1+i)^{\frac{1}{r}}-1\}}.$$

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* This is strictly accurate only when r becomes infinite.

If now r become infinite, $(1+i)^{\frac{1}{r}}$ becomes unity, and $r\{(1+i)^{\frac{1}{r}}-1\}$ becomes δ, and we have for the correction

$$v\left\{\frac{i}{\delta^2}-\frac{1}{\delta}\right\}, \text{ or } v\frac{i-\delta}{\delta^2}.$$

This is the value of the correction at the *beginning* of the year of death, and therefore the value at the end of that year will be $\frac{i-\delta}{\delta^2}$. Taking now the present value for every possible year of death, the correction becomes $\mathrm{A}_x\frac{i-\delta}{\delta^2}$, and we have

$$\mathring{a}_x=a_x+\mathrm{A}_x\frac{i-\delta}{\delta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But, by Chap. x, formula 4, $\mathrm{A}_x\frac{i}{\delta}=\overline{\mathrm{A}}_x$: therefore

$$\mathring{a}_x=a_x+\overline{\mathrm{A}}_x\frac{i-\delta}{i\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is rigidly accurate on the hypothesis of a uniform distribution of deaths.

[6] Formula 3 may be brought into a convenient form for numerical calculation by expanding the coefficient of A_x in powers of i by the relation $\delta=i-\frac{i^2}{2}+\frac{i^3}{3}-$&c. This gives,

$$\mathring{a}_x=a_x+\mathrm{A}_x\left(\frac{1}{2}+\frac{i}{6}-\frac{i^2}{24}+\&c.\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

[7] If we similarly expand the coefficient of A_x in formula 1, it becomes $\frac{1}{2}+\frac{i}{4}-\frac{i^2}{16}+$&c., and comparing this with formula 5, the difference between the two expansions is approximately $\frac{i-\frac{1}{4}i^2}{12}$; and therefore the error already referred to in formula 1 is approximately equal to $\mathrm{A}_x\frac{i-\frac{1}{4}i^2}{12}$.

[8] Passing now to the general case of an annuity payable m times a year:—If death take place in the $(n+1)$th year, at the end of the first tth part of a year in that mth part of the year following s of the m parts, that is, at the moment $n+\frac{s}{m}+\frac{1}{t}$, there will become payable the

sum of $\frac{1}{t}$; and the value at the beginning of the year of that sum will be $\frac{1}{t^2}\times v^{\frac{s}{m}}\times v^{\frac{1}{t}}$, because the chance of receiving it is $\frac{1}{t}$. Similarly, if death take place at the end of the second tth part of a year in that mth part of the $(n+1)$th year now under consideration, the sum of $\frac{2}{t}$ will become payable, the value of which at the beginning of the year will be $\frac{2}{t^2}\times v^{\frac{s}{m}}\times v^{\frac{2}{t}}$: and so on for all the $\frac{t}{m}$ tth parts of a year contained in that mth part now under consideration. We therefore have the total correction for the $(s+1)$th mth part of the year,

$$\frac{1}{t}v^{\frac{s}{m}}\left\{\frac{1}{t}v^{\frac{1}{t}}+\frac{2}{t}v^{\frac{2}{t}}+\text{\&c.}+\frac{1}{m}v^{\frac{1}{m}}\right\},$$

Multiplying and dividing by $1-v^{\frac{1}{t}}$, this becomes,

$$v^{\frac{s}{m}}\left\{\frac{(1+i)^{\frac{1}{t}}(1-v^{\frac{1}{m}})}{t^2\{(1+i)^{\frac{1}{t}}-1\}^2}-\frac{1}{m}\frac{v^{\frac{1}{m}}}{t\{(1+i)^{\frac{1}{t}}-1\}}\right\},$$

or, when t is infinite,

$$v^{\frac{s}{m}}\left\{\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{1}{m}\frac{v^{\frac{1}{m}}}{\delta}\right\}.$$

We must now sum for all integral values of s from 0 to $m-1$ inclusive, so as to get the total value of the correction at the beginning of the year should death take place in the $(n+1)$th year. Thus,

$$\frac{1-v}{1-v^{\frac{1}{m}}}\left\{\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{1}{m}\frac{v^{\frac{1}{m}}}{\delta}\right\}$$

$$=\frac{1-v}{\delta^2}-\frac{1}{m}\frac{v^{\frac{1}{m}}(1-v)}{(1-v^{\frac{1}{m}})\,\delta}$$

$$=\frac{vi}{\delta^2}-\frac{vi}{m\{(1+i)^{\frac{1}{m}}-1\}\delta}$$

$$=v\left\{\frac{i}{\delta^2}-\frac{i}{j_{(m)}\delta}\right\}.$$

This being the value of the correction at the beginning of the year of death, its value at the end of that year will be $\left\{\frac{i}{\delta^2}-\frac{i}{j_{(m)}\delta}\right\}$; and, taking the present value for every possible year of death, and adding the same to $a_x^{(m)}$, we have

$$\mathring{a}_x^{(m)}=a_x^{(m)}+\mathrm{A}_x\left\{\frac{i}{\delta^2}-\frac{i}{j_{(m)}\delta}\right\}$$

$$=a_x^{(m)}+\mathrm{A}_x\frac{i(j_{(m)}-\delta)}{j_{(m)}\delta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$=a_x^{(m)}+\bar{\mathrm{A}}_x\frac{j_{(m)}-\delta}{j_{(m)}\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

[9] We can test this result by assigning different values to m. If m be unity, then $j_{(m)}=i$, and the correction is $\bar{\mathrm{A}}_x\frac{i-\delta}{i\delta}$, which we already have in equation 4. Again, if m be infinite, $j_{(m)}=\delta$, and the correction vanishes. This is as it should be, because the annuity is then continuous, and is of necessity complete.

[10] We may now investigate formulas for the values of complete annuities without making any supposition as to the distribution of deaths. It will be an interesting study in the first place to find expressions by means of the Calculus of Finite Differences only. Afterwards, by means of the Differential and Integral Calculus, more convenient working formulas will be deduced.

[11] Let $\bar{\mathrm{M}}$ (omitting suffixes) be the value in the commutation column corresponding to an assurance payable at the instant of death, so that $\bar{\mathrm{M}}=\mathrm{D}\times\bar{\mathrm{A}}=\mathrm{D}(1-\delta\bar{a})$.

Considering first a yearly annuity:—If death occur in the first tth part of any year, (t being very large and presently to be made infinite), the sum of $\frac{1}{t}$ will be payable at the moment of death. Omitting the denominator D_x, the value of this is $\frac{1}{t}\left(\bar{\mathrm{M}}-\bar{\mathrm{M}}_{\frac{1}{t}}\right)$. Similarly, if death occur in the second tth part of the year, the sum of $\frac{2}{t}$ will be payable, the value of which is $\frac{2}{t}\left\{\bar{\mathrm{M}}_{\frac{1}{t}}-\bar{\mathrm{M}}_{\frac{2}{t}}\right\}$: and so on. The total value of the correction for the year under consideration is therefore

$$\frac{1}{t}\left\{\overline{M}-\overline{M}_{\frac{1}{t}}\right\}+\frac{2}{t}\left\{\overline{M}_{\frac{1}{t}}-\overline{M}_{\frac{2}{t}}\right\}+\&c.+\frac{t}{t}\left\{\overline{M}_{\frac{t-1}{t}}-\overline{M}_{\frac{t}{t}}\right\}$$

$$=\frac{1}{t}\left\{\overline{M}+\overline{M}_{\frac{1}{t}}+\overline{M}_{\frac{2}{t}}+\&c.+\overline{M}_{\frac{t-1}{t}}\right\}-M_{+1}.$$

If now we express the various values of $\overline{M}$ in the brackets in terms of the initial value and its differences, we shall have for the general term, $\overline{M}_{\frac{r}{t}}$, stopping at second differences, as those of higher orders are insignificant,

$$\overline{M}_{\frac{r}{t}}=\overline{M}+\frac{r}{t}\Delta\overline{M}-\frac{r}{t}\cdot\frac{t-r}{2t}\Delta^2\overline{M}.$$

Giving now to r every value from 0 to $t-1$, and taking the sum, by means of Chap. xxiv, formula 12, we shall have the value of the correction for the year under consideration,

$$\frac{1}{t}\left\{t\overline{M}+\Sigma_0^{t-1}\frac{r}{t}\Delta\overline{M}-\Sigma_0^{t-1}\frac{r}{t}\cdot\frac{t-r}{t}\cdot\frac{\Delta^2\overline{M}}{2}\right\}-\overline{M}_{+1}$$

$$=\frac{1}{t}\left\{t\overline{M}+\frac{t-1}{2}\Delta\overline{M}-\left[\frac{t-1}{4}-\frac{(t-1)\,t(2t-1)}{12t^2}\right]\Delta^2\overline{M}\right\}-\overline{M}_{+1}$$

$$=\overline{M}+\frac{t-1}{2t}\Delta\overline{M}-\frac{t^2-1}{12t^2}\Delta^2\overline{M}-\overline{M}_{+1}$$

and when t is infinite this becomes

$$\overline{M}+\tfrac{1}{2}\Delta\overline{M}-\tfrac{1}{12}\Delta^2\overline{M}-\overline{M}_{+1}.$$

Adding the suffixes and inserting the denominator D_x, we have the value of the correction for the $(n+1)$th year,

$$\frac{\overline{M}_{x+n}+\frac{1}{2}\Delta\overline{M}_{x+n}-\frac{1}{12}\Delta^2\overline{M}_{x+n}-\overline{M}_{x+n+1}}{D_x}.$$

To get the total correction for the whole of life we must now give to n every integral value from 0 upwards, and take the sum. We thus obtain the expression

$$\frac{\overline{M}_x-\frac{1}{2}\overline{M}_x+\frac{1}{12}\Delta\overline{M}_x}{D_x}$$

$$=\frac{\overline{M}_x}{2D_x}-\frac{\overline{M}_x-\overline{M}_{x+1}}{12D_x}$$

$$=\tfrac{1}{2}\overline{A}_x-\tfrac{1}{12}\overline{A}^{1}_{x\rceil},$$

where $\overline{A}^1_{x\overline{1}|}$ represents the assurance payable at the moment of the death of (x) if that event happen in the first year. Adding this correction to the value of the annuity, we have

$$\mathring{a}_x = a_x + \tfrac{1}{2}\overline{A}_x - \tfrac{1}{12}\overline{A}^1_{x\overline{1}|} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

[12] Passing now to the general case of an annuity payable m times a year, the investigation may be made brief, as it is identical in principle with what has gone before.

Dividing each mth of the year into t parts, we have the value of the correction for that mth part immediately following s parts of any particular year,

$$\frac{1}{mt}\left\{\overline{M}_{\frac{s}{m}} - \overline{M}_{\frac{s}{m}+\frac{1}{mt}}\right\} + \frac{2}{mt}\left\{\overline{M}_{\frac{s}{m}+\frac{1}{mt}} - \overline{M}_{\frac{s}{m}+\frac{2}{mt}}\right\} + \&\text{c.}$$

$$+ \frac{t}{mt}\left\{\overline{M}_{\frac{s}{m}+\frac{t-1}{mt}} - \overline{M}_{\frac{s}{m}+\frac{t}{mt}}\right\}$$

$$= \frac{1}{mt}\left\{\overline{M}_{\frac{s}{m}} + \overline{M}_{\frac{s}{m}+\frac{1}{mt}} + \overline{M}_{\frac{s}{m}+\frac{2}{mt}} + \&\text{c.} + \overline{M}_{\frac{s}{m}+\frac{t-1}{mt}}\right\} - \frac{1}{m}\overline{M}_{\frac{s+1}{m}}$$

$$= \frac{1}{mt}\Sigma_0^{t-1}\overline{M}_{\frac{s}{m}+\frac{r}{mt}} - \frac{1}{m}\overline{M}_{\frac{s+1}{m}}$$

$$= \frac{1}{mt}\Sigma_0^{t-1}\left\{\overline{M} + \frac{st+r}{mt}\Delta\overline{M} - \frac{st+r}{mt}\cdot\frac{mt-st-r}{2mt}\Delta^2\overline{M}\right\} - \frac{1}{m}\overline{M}_{\frac{s+1}{m}}$$

$$= \frac{1}{mt}\left\{t\overline{M} + \frac{2st^2+t(t-1)}{2mt}\Delta\overline{M}\right\}$$

$$- \frac{\Delta^2\overline{M}}{2m^3t^3}\left\{st^3(m-s) - \frac{t^2(2s-m)(t-1)}{2} - \frac{(t-1)t(2t-1)}{6}\right\}$$

$$- \frac{1}{m}\overline{M}_{\frac{s+1}{m}}.$$

Making t infinite and writing $\overline{M}_{\frac{s+1}{m}}$ in terms of $\overline{M}$ and its differences, this becomes

$$\frac{1}{m}\overline{M} + \frac{2s+1}{2m^2}\Delta\overline{M} - \frac{6ms-6s^2-6s+3m-2}{12m^3}\Delta^2\overline{M}$$

$$- \frac{1}{m}\left\{\overline{M} + \frac{s+1}{m}\Delta\overline{M} - \frac{s+1}{m}\cdot\frac{m-s-1}{2m}\Delta^2\overline{M}\right\}$$

$$= -\frac{\Delta\overline{M}}{2m^2} - \frac{6s-3m+4}{12m^3}\Delta^2\overline{M}.$$

In order to get the correction for the whole of the year under consideration, we must now assign to s every value from 0 to $m-1$ and take the sum. This gives

$$-\frac{\Delta\overline{\text{M}}}{2m}-\frac{3(m-1)\,m-3m^2+4m}{12m^3}\Delta^2\overline{\text{M}}$$

$$=-\frac{\Delta\overline{\text{M}}}{2m}-\frac{\Delta^2\overline{\text{M}}}{12m^2}.$$

Taking now the value for the whole of life and dividing by D_x, and adding the result to $a_x^{(m)}$, we finally have

$$\mathring{a}_x^{(m)}=a_x^{(m)}+\frac{1}{2m}\overline{\text{A}}_x-\frac{1}{12m^2}\overline{\text{A}}^1_{x\overline{1}|}\quad.\quad.\quad.\quad.\quad.\quad.\quad(9)$$

[13] The correctness of formula 9 is proved, because, if $m=1$, it at once becomes formula 8.

[14] The value of $\overline{\text{A}}^1_{x\overline{1}|}$ is troublesome to calculate, but without much loss of accuracy $v^{\frac{1}{2}}q_x$ may be substituted for it, which will give

$$\left.\begin{aligned}\mathring{a}_x&=a_x+\tfrac{1}{2}\overline{\text{A}}_x-\frac{v^{\frac{1}{2}}q_x}{12}\\ \mathring{a}_x^{(m)}&=a_x^{(m)}+\frac{1}{2m}\overline{\text{A}}_x-\frac{v^{\frac{1}{2}}q_x}{12m^2}\end{aligned}\right\}\quad.\quad.\quad.\quad.\quad.\quad(10)$$

[15] Making use now of the differential and integral calculus, and assuming to begin with a uniform distribution of deaths; to find the value of a complete annuity payable annually.

If death occur in the $(n+1)$th year, in the instant dt following the time t, then there becomes payable the sum of t. The deaths being equally distributed, the chance of this sum becoming payable is $({}_np_x-{}_{n+1}p_x)\,dt$, and the present value of the chance of receiving it is $tv^{n+t}({}_np_x-{}_{n+1}p_x)\,dt$. The value of all possible payments in the $(n+1)$th year is the integral with respect to t of the last expression, between the limits 0 and 1, namely $\int_0^1 tv^{n+t}({}_np_x-{}_{n+1}p_x)\,dt$, or, removing the constants from under the sign of integration, $v^n({}_np_x-{}_{n+1}p_x)\int_0^1 tv^t dt$.

The value of all possible payments throughout life is found by summing this last expression with respect to n, giving to n the values

0, 1, 2, &c., to the extremity of life. Now $\Sigma v^{n+1}({}_np_x - {}_{n+1}p_x) = \mathrm{A}_x$. Hence the value of the correction is $\dfrac{\mathrm{A}_x}{v}\displaystyle\int_0^1 tv^t dt$, or, $\mathrm{A}_x \displaystyle\int_0^1 tv^{t-1}dt$.

The value of the definite integral involved may be found as follows:—

$$tv^{t-1} = t(1+i)^{-(t-1)}$$

$$= t - t(t-1)i + t(t-1)t\frac{i^2}{2} - t(t-1)t(t+1)\,\frac{i^3}{6} + \&c.$$

$$= t - (t^2 - t)i + (t^3 - t^2)\,\frac{i^2}{2} - (t^4 - t^2)\,\frac{i^3}{6} + \&c.$$

Whence

$$\int tv^{t-1}dt = \mathrm{C} + \frac{t^2}{2} - \left(\frac{t^3}{3} - \frac{t^2}{2}\right)i + \left(\frac{t^4}{4} - \frac{t^3}{3}\right)\frac{i^2}{2}$$

$$- \left(\frac{t^5}{5} - \frac{t^3}{3}\right)\frac{i^3}{6} + \&c.$$

and

$$\int_0^1 tv^{t-1}dt = \frac{1}{2} + \frac{i}{6} - \frac{i^2}{24} + \frac{i^3}{45} - \&c.$$

Therefore the correction is $\mathrm{A}_x\left(\dfrac{1}{2} + \dfrac{i}{6} - \dfrac{i^2}{24} + \dfrac{i^3}{45} - \&c.\right)$, as found in formula 5.

[16] Or the correction may be found otherwise. Integrating by parts, and remembering that $v^t dt = \dfrac{dv^t}{\log_e v}$,

$$\int tv^t dt = \frac{tv^t}{\log_e v} - \int \frac{v^t}{\log_e v}\,dt$$

$$= \frac{tv^t}{\log_e v} - \frac{v^t}{(\log_e v)^2} + c$$

$$= c - \frac{tv^t}{\delta} - \frac{v^t}{\delta^2}$$

since $\log_e v = -\delta$.

Therefore $\displaystyle\int_0^1 tv^t dt = \frac{1}{\delta^2} - \frac{v}{\delta} - \frac{v}{\delta^2}$

and $\displaystyle\int_0^1 tv^{t-1}dt = \frac{1}{v\delta^2} - \frac{1}{\delta} - \frac{1}{\delta^2} = \frac{i-\delta}{\delta^2}.$

The correction therefore is $\mathrm{A}_x\dfrac{i-\delta}{\delta^2}$, as already found in formula 3.

[17] Let the annuity now be supposed to be payable m times a year, and let each year be divided into m equal parts. Then if death occur in

the $(s+1)$th part of the $(n+1)$th year at the instant dt following the time $n+\frac{s}{m}+t$ there is payable the sum of t. The probability of death occurring in that instant on the supposition of a uniform distribution of deaths is, as before, $({}_np_x-{}_{n+1}p_x)dt$; and the value of the chance of receiving t by such death is, $tv^{n+\frac{s}{m}+t}({}_np_x-{}_{n+1}p_x)dt$. The value of all possible payments in the $(s+1)$th portion of the $(n+1)$th year, is the integral of the last expression with respect to t, between the limits 0 and $\frac{1}{m}$, namely, $v^{n+\frac{s}{m}}({}_np_x-{}_{n+1}p_x)\int_0^{\frac{1}{m}}tv^tdt$. But it has already been shown that $\int tv^tdt=c-\frac{tv^t}{\delta}-\frac{v^t}{\delta^2}$. Therefore $\int_0^{\frac{1}{m}}tv^tdt=\frac{1}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}-\frac{v^{\frac{1}{m}}}{\delta^2}=\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}$. Therefore the value of all possible payments accruing through death taking place in the $(s+1)$th part of the $(n+1)$th year, is $v^{n+\frac{s}{m}}({}_np_x-{}_{n+1}p_x)\left\{\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}\right\}$.

The value of all possible payments in the $(n+1)$th year will now be found by summing with respect to s, giving it the values 0, 1, 2, &c., to $(m-1)$, and is

$$v^n\left(1+v^{\frac{1}{m}}+v^{\frac{2}{m}}+\&c.+v^{\frac{m-1}{m}}\right)({}_np_x-{}_{n+1}p_x)\left\{\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}\right\}$$

$$=v^n\frac{1-v}{1-v^{\frac{1}{m}}}({}_np_x-{}_{n+1}p_x)\left\{\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}\right\}.$$

Lastly, summing with respect to n, giving to it the values 0, 1, 2, &c., to the extremity of life, the total value of the correction is

$$\frac{\mathrm{A}_x}{v}\frac{1-v}{1-v^{\frac{1}{m}}}\left\{\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}\right\}=\mathrm{A}_x\left\{\frac{i}{\delta^2}-\frac{i}{m\{(1+i)^{\frac{1}{m}}-1\}\delta}\right\}$$

$$=\mathrm{A}_x\frac{i(j_{(m)}-\delta)}{j_{(m)}\delta^2},$$

as already given in equation 6.

[18] Abandoning now the hypothesis of a uniform distribution of deaths, and taking at once the general case of the annuity payable m times a year, we have

$$\overline{\mathrm{M}}=\mathrm{D}(1-\delta\overline{a})$$

$$=\mathrm{D}-\delta\int\mathrm{D}dt$$

by Chap. ix, Art. 26. The correction is found by means of the integral $-\int td\overline{\mathrm{M}}$, or $-\overline{\mathrm{M}}t+\int\overline{\mathrm{M}}dt$, which must be taken between limits for each successive interval $\frac{1}{m}$.

For the 1st interval it is $-\frac{1}{m}\overline{\mathrm{M}}_{\frac{1}{m}}+\int_0^{\frac{1}{m}}\overline{\mathrm{M}}dt$;

" 2nd " " $-\frac{1}{m}\overline{\mathrm{M}}_{\frac{2}{m}}+\int_{\frac{1}{m}}^{\frac{2}{m}}\overline{\mathrm{M}}dt$,

&c. &c. &c.

Hence, by summing these, we get

$$-\int_0^\infty td\overline{\mathrm{M}}=-\frac{1}{m}\left(\overline{\mathrm{M}}_{\frac{1}{m}}+\overline{\mathrm{M}}_{\frac{2}{m}}+\&c.\right)+\int_0^\infty\overline{\mathrm{M}}dt$$

$$=\frac{1}{m}\overline{\mathrm{M}}-\Sigma^{(m)}\overline{\mathrm{M}}+\int_0^\infty\overline{\mathrm{M}}dt$$

in the notation of Chap. xxiv, Art. 32. But by Chap. xxiv, formula 22,

$$\Sigma^{(m)}\overline{\mathrm{M}}=\int\overline{\mathrm{M}}dt+\frac{1}{2m}\overline{\mathrm{M}}-\frac{1}{12m^2}\frac{d\overline{\mathrm{M}}}{dt}+\&c.,$$

the subsequent terms being insignificant. Therefore,

$$-\int_0^\infty td\overline{\mathrm{M}}=\frac{1}{2m}\overline{\mathrm{M}}+\frac{1}{12m^2}\cdot\frac{d\overline{\mathrm{M}}}{dt}-\&c.$$

But since $$\overline{\mathrm{M}}=\mathrm{D}-\delta\int\mathrm{D}dt,$$

therefore $$\frac{d\overline{\mathrm{M}}}{dt}=\frac{d\mathrm{D}}{dt}+\delta\mathrm{D}=-(\mu+\delta)\mathrm{D}+\delta\mathrm{D}=-\mu\mathrm{D}.$$

Therefore $$-\int_0^\infty td\overline{\mathrm{M}}=\frac{1}{2m}\overline{\mathrm{M}}-\frac{\mu\mathrm{D}}{12m^2},$$

and, dividing by D, we have the correction, $\frac{1}{2m}\overline{\mathrm{A}}_x-\frac{\mu_x}{12m^2}$; whence

$$\mathring{a}_x^{(m)}=a_x^{(m)}+\frac{1}{2m}\overline{\mathrm{A}}_x-\frac{\mu_x}{12m^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Or we may write the correction $\frac{1}{2m}(1-\delta\bar{a}_x)-\frac{\mu_x}{12m^2}$, whence

$$\mathring{a}_x^{(m)}=a_x^{(m)}+\frac{1}{2m}(1-\delta\bar{a}_x)-\frac{\mu_x}{12m^2} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

[19] The value of the complete annuity may be expressed conveniently in terms of the continuous annuity, thus,

$$\begin{aligned}
a_x^{(m)} &= a_x + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x+\delta) \\
&= \bar{a}_x - \tfrac{1}{2} + \frac{\mu_x+\delta}{12} + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x+\delta) \\
&= \bar{a}_x - \frac{1}{2m} + \frac{\mu_x+\delta}{12m^2}
\end{aligned}$$

Adding to this value of $a_x^{(m)}$ the correction $\frac{1}{2m}(1-\delta\bar{a}_x)-\frac{\mu_x}{12m^2}$, we have

$$\begin{aligned}
\mathring{a}_x^{(m)} &= \bar{a}_x - \frac{1}{2m} + \frac{\mu_x+\delta}{12m^2} + \frac{1}{2m}(1-\delta\bar{a}_x) - \frac{\mu_x}{12m^2} \\
&= \bar{a}_x\left(1-\frac{\delta}{2m}\right) + \frac{\delta}{12m^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)
\end{aligned}$$

[20] Comparing formulas 9 and 11, the difference between them is, that the term $\frac{1}{12m^2}\bar{A}^{1}_{x\overline{1}|}$ in equation 9 is replaced by $\frac{\mu_x}{12m^2}$ in equation 11. The numerical results of the two formulas will not differ materially at those ages which most frequently occur in practice, but equation 11 is the more correct of the two. The first term omitted in deducing formula 9 is $\frac{1}{24m^2}\left\{\bar{A}^{1}_{x\overline{1}|}-\frac{D_{x+1}}{D_x}\bar{A}_{\overset{1}{x+1}:\overline{1}|}\right\}$, and this is larger than the first term omitted in deducing formula 11, which is $\frac{1}{720m^4}\cdot\frac{1}{D_x}\cdot\frac{d^3\bar{M}_x}{dx^3}$.

[21] All the foregoing formulas which do not assume a uniform distribution of deaths, apply to joint lives as well as to single lives. We therefore have,

$$\left.\begin{aligned}
\mathring{a}_{xy}^{(m)} &= a_{xy}^{(m)} + \frac{1}{2m}\bar{A}_{xy} - \frac{1}{12m^2}v^{\frac{1}{2}}q_{xy} \\
\mathring{a}_{xy}^{(m)} &= a_{xy}^{(m)} + \frac{1}{2m}\bar{A}_{xy} - \frac{\mu_x+\mu_y}{12m^2} \\
\mathring{a}_{xy}^{(m)} &= \bar{a}_{xy}\left(1-\frac{\delta}{2m}\right) + \frac{\delta}{12m^2}
\end{aligned}\right\} \quad . \; . \; . \; . \quad (14)$$

[22] Also we have for deferred and temporary annuities,

$$_{t|}\mathring{a}_x^{(m)} = v^t\,{}_tp_x\,\mathring{a}_{x+t}^{(m)} \quad . \; . \; . \; . \; . \; . \quad (15)$$

$$_{|t}\mathring{a}_x^{(m)} = \mathring{a}_x^{(m)} - {}_{t|}\mathring{a}_x^{(m)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

[23] The relative effects of the various formulas for the value of a complete annuity will best be measured by numerical examples, and it will be convenient to take yearly annuities at 3 per-cent interest, on lives aged respectively 30 and 60. In making the calculations it is assumed that $\bar{A}_x = 1 - \delta\bar{a}_x$, where $\bar{a}_x = a_x + \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$. As the differences between the results of the formulas will always be greater when the annuity is payable yearly, it is not necessary to give examples for annuities payable fractionally throughout the year.

Formula.	$\mathring{a}_{30}$	$\mathring{a}_{60}$
1	20·09364	10·56505
3	20·09267	10·56338
5	20·09267	10·56338
10	20·09301	10·56262
11	20·09300	10·56265
13	20·09300	10·56264

CHAPTER XII.

JOINT-LIFE ANNUITIES.

1. In previous chapters the theory of joint-life annuities has been discussed, and it does not present any difficulty to the mathematician. The object of the present chapter is to show how the values of annuities on joint lives may be arrived at with sufficient accuracy for practical purposes, without going through the immense labour of calculating them at full length directly from the mortality table.

2. It has already been seen that De Moivre's formula (Chap. iii, Arts. 9 to 11, Chap. vi, Arts. 1 to 4, Chap. vii, Arts. 104 to 109) enables us to calculate the values of annuities, because by it the value of l_x is expressed in such a way as to admit of the summation of the series consisting of the values of the successive payments of the annuity, without calculating separately the value of each such payment. The formulas, however, become very complicated when more than two lives are involved; and, besides, De Moivre's hypothesis does not represent the law of mortality with sufficient faithfulness to be at the present day admissible in actuarial calculations.

3. Thomas Simpson, in the supplement to his work "Doctrine of Annuities and Reversions", which was first published in 1752 in his "Select Exercises for Young Proficients in the Mathematicks", gave a rule, which goes by the name of *Simpson's Rule*, for finding the value of an annuity on three joint lives by means of tables of annuities on single lives and on two joint lives. In his own words the rule is as follows:—
"Let A be the youngest and C the oldest of the three proposed lives:
" take the value of the two joint lives B and C, and find the age of a
" single life D of the same value: then find the value of the joint lives
" A and D; which will be the answer."

The rule may be expressed in modern notation as follows:—Required

a_{xyz} where $x<y<z$. Take w such that $a_w=a_{yz}$, and find a_{xw}; which will be the answer. That is, when $a_w=a_{yz}$,

$$a_{xyz}=a_{xw} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

4. It is desirable to give an example of the working of Simpson's rule, but the tables in this volume do not serve the present purpose, because complete tables of annuities on two lives are not here given. We therefore make use of the "Institute of Actuaries' Life Tables", published in 1872.*

5. Let it be required to find the value of an annuity, at 4 per-cent interest, on three joint lives, aged respectively 30, 36 and 41. The value of an annuity on the two oldest lives, $a_{36:41}$, (=12·626), lies between a_{49}, (=12·817) and a_{50}, (=12·536), and is the same as that of an annuity on a single life aged 49·680, when the interpolation is done by first differences. It remains now to find the value of an annuity on two joint lives aged respectively 30 and 49·680. Interpolating by first differences between $a_{30:49}$, (=11·550), and $a_{30:50}$, (=11·332), we have $a_{30:49·680}=11·402$, which is the required approximation to $a_{30:36:41}$.

6. If the true value of the required annuity be found by exact calculation from the Institute of Actuaries' Mortality Table, it is ascertained to be 11·345; and thus, in the case of this example, Simpson's rule gives a value too great by ·057. It will be found that, except when all the lives are advanced in years, Simpson's rule gives values too large. An age w can always be found such that $a_w=a_{yz}$ exactly, no matter what may be the ages y and z; but by making the assumption that because $a_w=a_{yz}$, therefore $a_{xw}=a_{xyz}$, we also assume that ${}_np_w={}_np_{yz}$ for all values of n; and this latter assumption is not legitimate. In fact, on trial, it will be found that, except at old ages, ${}_np_w>{}_np_{yz}$ for the smaller values of n, but that, as n increases, a point is reached at which the two probabilities have the same value, and as n still increases the relative magnitude is reversed, and ${}_np_w<{}_np_{yz}$. Since then for the smaller values of n, ${}_np_w>{}_np_{yz}$, and that for these values the operation of discount has the least effect, it follows that Simpson's rule gives too large values for the annuity.

7. On account of this error, always in one direction, produced by Simpson's rule, Price, in his "Observations on Reversionary Payments", recommended the uniform deduction of ·05 from the result given by the rule; and he also suggested that the age w should be taken to the nearest integer and not to a fraction. That is, when w is the nearest integer in the equation $a_{yz}=a_w$,

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* In this chapter much use has been made of the paper on Joint-Life Annuities, by J. J. M'Lauchlan, published in the Transactions of the Actuarial Society of Edinburgh. The numerical example is borrowed from that paper.

$$a_{xyz}=a_{xw}-\cdot05 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Adopting these corrections in the foregoing example, we have $a_{30:36:41}=a_{30:50}-\cdot05=11\cdot282$, which is too small by ·063. The corrections therefore have not improved the result.

8. Milne, in his "Annuities and Assurances", gave a modification of Simpson's rule, which greatly increases its accuracy. He said (substituting our notation for his) :—"Let (x) be the youngest and (z) the "oldest of the three proposed lives (x), (y), and (z). Find the value of "the two joint lives (y) and (z), and let (w) be the equivalent single "life. Then—

"If (z), the oldest life proposed, be under 45 years of age, let the "age of the substituted life be the whole number next greater than that "which expresses the age of (w). But—

"If the age of (z) be not under 45 years, let the age of the sub-"stituted life be the next greater than that of (w), which does not require "more than one decimal figure to express it." That is, when $z<45$ and w is the next higher integer in the equation $a_{yz}=a_w$; or when $z =$ or > 45 and w is the next greater number which can be represented by one decimal place in the equation $a_{yz}=a_w$,

$$a_{xyz}=a_{xw} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

9. Applying Milne's modification we have $w=50$, because the interpolated value is 49·68, and the age of the oldest of the three lives is 41, which is less than 45. By assuming then $a_{30:36:41}=a_{30:50}$, we have 11·332, which is only ·013 less than the true value.

Had the oldest of the proposed lives been not less than 45, we should have taken $w=49\cdot7$ in accordance with the second of the instructions given by Milne.

10. Meikle has proposed another modification of Simpson's rule, which is obtained as follows :—

In applying Simpson's rule, we shall generally find that $a_{yz}<a_{w-1}>a_w$ where w is an integer. Let it be assumed that the following proportions hold :—

$$a_w \;:\; a_{yz} :: a_{xw} \;:\; a_{xyz}$$

$$a_{w-1} : a_{yz} :: a_{x:w-1} : a_{xyz}.$$

Compounding the proportions, we have

$$(a_w)(a_{w-1}):(a_{yz})^2::(a_{xw})(a_{x:w-1}):(a_{xyz})^2,$$

whence
$$a_{xyz}=a_{yz}\sqrt{\left(\frac{a_{xw}\times a_{x:w-1}}{a_w\times a_{w-1}}\right)}\quad.\ .\ .\ .\ .\ .\ .\quad(4)$$

Applying Meikle's modification to find $a_{30:36:41}$, we have $w=50$; and, making the calculations, the values of the various annuities involved having already been given above, $a_{30:36:41}=11{\cdot}396$ which is ·051 in excess.

11. The proportions above given may be treated somewhat differently. Multiplying together the extremes and the means respectively, we have

$$a_w\quad\times a_{xyz}=a_{xw}\quad\times a_{yz}$$
$$a_{w-1}\times a_{xyz}=a_{x:w-1}\times a_{yz}$$

whence
$$(a_w+a_{w-1})a_{xyz}=(a_{xw}+a_{x:w-1})a_{yz}$$

and
$$a_{xyz}=a_{yz}\times\frac{a_{xw}+a_{x:w-1}}{a_w+a_{w-1}}\quad.\ .\ .\ .\ .\ .\ .\quad(5)$$

This last expression gives $a_{30:36:41}=11{\cdot}395$, which is ·050 in excess.

12. Simpson, at pages 26 and 58 of his "Supplement to the Doctrine of Annuities and Reversions", gives the application of his rule to the finding of the value of an annuity on the last survivor of three lives, $a_{\overline{xyz}}$, where $x<y<z$.

His rule for finding $a_{\overline{xyz}}$ (*see* his problem 10) is,—Find (w) so that $a_w=a_{yz}$. Then $a_{\overline{xyz}}=a_{\overline{xy}}+a_{\overline{xz}}-a_{\overline{xw}}$.

The *rationale* of this rule may be seen as follows:—

$$a_{\overline{xyz}}=a_x+a_y+a_z-a_{xy}-a_{xz}-a_{yz}+a_{xyz}$$
$$=(a_x+a_y-a_{xy})+(a_x+a_z-a_{xz})-(a_x+a_{yz}-a_{xyz})$$
$$=(a_x+a_y-a_{xy})+(a_x+a_z-a_{xz})-(a_x+a_w-a_{xw}).$$

That is,
$$a_{\overline{xyz}}=a_{\overline{xy}}+a_{\overline{xz}}-a_{\overline{xw}}\quad.\ .\ .\ .\ .\ .\ .\ .\quad(6)$$

13. Where, as is the case with the Institute of Actuaries' Life Tables, the values of annuities on the survivor of two lives are tabulated, we can find $a_{\overline{xyz}}$ by formula 6 with very little labour. Taking an annuity at the same ages as before, namely, $a_{\overline{30:36:41}}$, we have, by Art. 5, $w=49{\cdot}680$. By the Institute Tables at 4 per-cent $a_{\overline{30:49}}=18{\cdot}398$; and $a_{\overline{30:50}}=18{\cdot}335$: whence $a_{\overline{30:49{\cdot}680}}=18{\cdot}355$; also $a_{\overline{30:36}}=19{\cdot}341$, and $a_{\overline{30:41}}=18{\cdot}958$. Therefore, by formula 6, $a_{\overline{30:36:41}}=19{\cdot}944$. The true value of the annuity is 19·887, so that the value given by formula 6 is ·057 in excess.

14. If, instead of using the fractional age 49·680 for w, we adopt Milne's amendment, and write $w=50$, the value comes out 19·964, or ·077 in excess, so that, although Milne's amendment improves the rule for a joint-life annuity, it does not do so for a last-survivor annuity. The reason for this is apparent when we analyze the annuity $a_{\overline{xw}}$. In terms of single and joint-life annuities, we have $a_{\overline{xw}}=a_x+a_w-a_{xw}$, where the substituted life (w) appears twice. In using the fractional age, $w=49·680$, the value of a_w is exact, while a_{xw} is too large, and as a_{xw} is subtractive, the value of $a_{\overline{xw}}$ is too small, and, consequently, the value of $a_{\overline{xyz}}$ is too large by exactly the error in the value of a_{xw}. When, however, we use Milne's value, 50, for w, we reduce the values of both a_w and a_{xw}; but the reduction is greater in a_w, and consequently, a_w-a_{xw} is diminished, and $a_{\overline{xw}}$ is also diminished. Therefore, as in formula 6 $a_{\overline{xw}}$ is subtractive, $a_{\overline{xyz}}$ is increased, and since it was already too large, the error is augmented. It is better therefore, in seeking by means of formula 6 the value of a last-survivor annuity, to use Simpson's rule in its original shape, and not as amended by Milne.

15. Another method of finding $a_{\overline{xyz}}$ from tables of the values of annuities on the survivor of two lives has been suggested, namely, to find a life u such that $a_{\overline{yz}}=a_u$ and then to write

$$a_{\overline{xyz}}=a_{\overline{xu}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Taking the same example as before, we have $a_u=a_{\overline{36:41}}=18·272$; and since $a_{22}=18·384$, and $a_{23}=18·251$, it follows that $18·272=a_{22·842}$, and $u=22·842$. Taking now an annuity on the survivor of 30 and 22·842, we have 20·401 which is ·514 too large. Formula 7 is therefore inadmissible. It always gives values considerably too large.

16. From the foregoing investigation we learn, that if we seek the value of $a_{\overline{xyz}}$, having only tables of single and joint-life annuities, we must take the most accurate value attainable for a_{xyz}, and that therefore it is better to use Milne's formula No. 3 rather than Simpson's No. 1, but that when we have at our disposal tables of annuities on the survivor of two lives, it is better to take the fractional value of w as given by Simpson's formula No. 1, rather than to adopt Milne's modification as given in formula 3.

[**17**] De Morgan, in the Philosophical Magazine, November 1839, in a paper reprinted in *J.I.A.*, x, 27, showed that if Gompertz's law of mortality were accurately true for the whole of life, then Simpson's rule would also be rigidly correct; and subsequently he proved, *J.I.A*, viii,

181, that if Simpson's rule were accurate, then Gompertz's law would prevail. His proof of the converse proposition was, however, complex, and not entirely satisfactory; and in the following paragraphs we have used mainly the method of R. Henderson (*J.I.A.* xxxii, 293), extended by A. Levine (*J.I.A.* xxxiii, 538), which has also the great advantage of including Makeham's law.*

[18] The assumption made in Simpson's rule may be stated in the form:—Given two lives aged x and y respectively, w may always be found so that the equation, $a_w = a_{xy}$, may be an identity; or, in other words, so that ${}_tp_w = {}_tp_{xy}$ for all values of t.

[19] We have, $\log {}_tp_{xy} = \log {}_tp_x + \log {}_tp_y$; and, assuming Gompertz's law, Chap vi, Art. 10,

$$\begin{aligned}\log {}_tp_{xy} &= \log k + c^{x+t}\log g - \log k - c^x \log g \\ &\quad + \log k + c^{y+t}\log g - \log k - c^y \log g \\ &= c^t(c^x + c^y)\log g - (c^x + c^y)\log g \\ &= c^{w+t}\log g - c^w \log g \\ &= \log {}_tp_w,\end{aligned}$$

when we take $c^w = c^x + c^y$.

Thus, under Gompertz's law, we can always find a single life (w), which may be substituted for the joint lives (x) and (y); and the probability of living t years will not be changed. To find the value of an annuity on two joint lives (x) and (y), we should, therefore, proceed by finding from a table of the powers of Gompertz's constant, c, the value of c^w, from the relation $c^x + c^y = c^w$. Then, finding from the table the age w corresponding to that value, we should take a_w from the table of single life annuities, and this would be equal to a_{xy}.

[20] It should be noted, moreover, that there need not be only one substituted life. If there be n lives, (x), (y), (z), &c., we may substitute m lives, (w), provided we retain the relation $mc^w = (c^x + c^y + c^z + \&c.)$; and m may be less than, equal to, or greater than, n. Also, the substituted lives need not all be of one age. We may replace n lives, (x), (y), (z), &c., by m lives, (u), (v), (w), &c., provided that $(c^u + c^v + c^w + \&c.) = (c^x + c^y + c^z + \&c.)$.

[21] For the calculation of the values of joint-life annuities by Gompertz's law it would not be necessary to have a table of the powers of c, if the much more useful function μ_x were tabulated. By Chap. vi, formula 1, on the supposition of Gompertz's law, $\mu_x = \mathrm{B}c^x$, where B is a constant, independent of x. Therefore, if $c^x + c^y = c^w$, we shall also have $\mu_x + \mu_y = \mu_w$; and to find a_{xy} we should simply require to take from the table of μ the age w at which the relation $\mu_w = \mu_x + \mu_y$ would hold, and

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* In the first edition of this work De Morgan's proof of the converse proposition

find a_w from the table of single life annuities. Or we might employ the function $\log p_x$ instead of μ_x, because, according to Gompertz's law, $\log p_x = c^x(c-1)\log g$; so that if $c^x + c^y = c^w$, then also will $\log p_x + \log p_y = \log p_w$.

[22] To the property of Gompertz's law by which a single life may be substituted for two or more joint lives, the name *uniform seniority* has been given, because the *seniority* of the substituted life, (w), is the same for all ages of the original lives, (x) and (y), where $y-x$ is constant. This may be proved as follows:

Assuming $y-x=r$, and $c^{-r}+1=b$, we have

$$c^w = c^x + c^y = c^y(c^{-r}+1) = c^y \times b.$$

Therefore $\quad w\log c = y\log c + \log b \quad$ and $\quad w = y + \dfrac{\log b}{\log c}$.

Thus $\dfrac{\log b}{\log c}$, which is a constant quantity for the difference $y-x=r$, represents the seniority of the substituted life, (w), over the elder of the two lives, (x) and (y).

[23] A mortality table constructed according to Makeham's modification of Gompertz's law, such as the mortality table at the end of this volume, affords great facility for calculating the values of joint-life annuities. It has been shown in Chap. vi, Art. 28, that if the age w be selected so that $c^x + c^y = 2c^w$, then ${}_np_{xy} = {}_np_{ww}$, from which it follows that

$$a_{xy} = a_{ww} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

That is, we may substitute for any joint lives, (x), (y), (z), &c., the same number of lives of equal ages; and if tables of annuities on joint-lives of equal ages be given, the values of joint-life annuities for lives of any ages may be derived.

Instead of using the powers of the constant c to find the equivalent age w, it is equally correct and much more convenient to use, as in Art. 21, the force of mortality. By Chap. vi, formula 5, μ_x is of the form $A + Bc^x$, so that if there be m joint lives, (x), (y), (z), &c., and if $mc^w = c^x + c^y + c^z + \&c.$, then also $m\mu_w = \mu_x + \mu_y + \mu_z + \&c.$

[24] The force of mortality, with its differences to facilitate calculation, is tabulated in Table I, and in Tables XLI and XLII there is shown for two and three lives respectively the uniform seniority for every difference of age of from one to fifty years, calculated on the principles explained in the last preceding article, a note being added to explain the use of these Tables; and other Tables, Nos. VI, VIII, &c., are added, giving the values of annuities at various rates of interest on single lives, and on two, three, and four joint lives of equal ages. By means of these, the values of annuities on any joint lives up to four in number may be found with

perfect accuracy, provided none of the lives be less than 28 years of age (*see* page 208); and, by the method of differences, a close approximation may be made to the values of annuities on five or more lives. The values so found will correspond to the mortality table as constructed by Makeham's formula, but as that table adheres very closely to the original facts, (*see* Chap. vi, Art. 52), the values of joint-life annuities so found may be used without much loss of accuracy along with values of annuities taken from the volume of "Institute of Actuaries' Life Tables", these being based on the same original facts, but graduated by a different formula.

[25] To take the same example as before:—Let it be required to find, at 4 per-cent interest, the value of an annuity on three joint lives aged 30, 36 and 41, respectively.

$$\begin{aligned}
\mu_{30} &= {\cdot}00768 \\
\mu_{36} &= {\cdot}00876 \\
\mu_{41} &= \underline{{\cdot}01025} \\
3)&\underline{{\cdot}02669} \\
\mu_w = \mu_{36{\cdot}56} &= \underline{\underline{{\cdot}00890}}
\end{aligned}
\qquad\qquad
\begin{aligned}
a_{36:36:36} &= 11{\cdot}497 \\
a_{37:37:37} &= \underline{11{\cdot}280} \\
\Delta &= \underline{\quad{\cdot}217} \\
\Delta \times {\cdot}56 &= \underline{\underline{\quad{\cdot}122}} = \text{Deduction}
\end{aligned}$$

$$a_{30:36:41} = a_{36:36:36} - {\cdot}122 = \underline{11{\cdot}375}$$

[26] The above result is correct according to the mortality table at the end of this volume, and it is only ·030 in excess of the value by the "Institute of Actuaries' Life Table"; so that, regarded as an approximation to the value of the annuity by the Institute Table, the error is only one-half of that produced by the use of Simpson's rule.

[27] To take another example:—Let it be required to find the value of an annuity on four joint lives aged 30, 36, 41 and 50, respectively, interest at 4 per-cent.

$$\begin{aligned}
\mu_{30} &= {\cdot}00768 \\
\mu_{36} &= {\cdot}00876 \\
\mu_{41} &= {\cdot}01025 \\
\mu_{50} &= \underline{{\cdot}01542} \\
4)&\underline{{\cdot}04211} \\
\mu_w = \mu_{41{\cdot}72} &= \underline{\underline{{\cdot}01053}}
\end{aligned}
\qquad\qquad
\begin{aligned}
a_{41:41:41:41} &= 9{\cdot}101 \\
a_{42:42:42:42} &= \underline{8{\cdot}868} \\
\Delta &= \underline{{\cdot}233} \\
\Delta \times {\cdot}72 &= \underline{\underline{{\cdot}168}} = \text{Deduction}
\end{aligned}$$

$$a_{30:36:41:50} = a_{41:41:41:41} - {\cdot}168 = \underline{\underline{8{\cdot}933}}$$

[28] Passing now to the converse problem, let it be assumed that for an annuity on n joint lives of different ages we may substitute an annuity on m joint lives of equal ages. It is required to find what law of mortality is involved.

The hypothesis implies that ${}_tp_{www\,..\,(m)} = {}_tp_{xyz\,\dots\,(n)}$ for all values of t, where x, y, z, &c., are independent of each other, but where w is a function of them all. The equation may be written

$$m\log {}_tp_w=\log {}_tp_x+\log {}_tp_y+\log {}_tp_z+\&c.$$

Differentiating with respect to t, and changing the sign of both sides of the equation, we have

$$m\mu_{w+t}=\mu_{x+t}+\mu_{y+t}+\mu_{z+t}+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

for all values of t. Differentiating again with respect to t, we have

$$m\frac{d\mu_{w+t}}{dt}=\frac{d\mu_{x+t}}{dt}+\frac{d\mu_{y+t}}{dt}+\frac{d\mu_{z+t}}{dt}+\&c.$$

or, what is the same thing,

$$m\frac{d\mu_{w+t}}{dw}=\frac{d\mu_{x+t}}{dx}+\frac{d\mu_{y+t}}{dy}+\frac{d\mu_{z+t}}{dz}+\&c. \quad . \quad . \quad . \quad \text{(ii)}$$

Putting now $t=0$ in equations (i) and (ii), we get

$$m\mu_w=\mu_x+\mu_y+\mu_z+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

$$m\frac{d\mu_w}{dw}=\frac{d\mu_x}{dx}+\frac{d\mu_y}{dy}+\frac{d\mu_z}{dz}+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Now, remembering that (x), (y), (z), &c., are absolutely independent of each other, while w is a function of them all, we have, differentiating (iii) and (iv) with respect to x,

$$m\frac{d\mu_w}{dw}\cdot\frac{dw}{dx}=\frac{d\mu_x}{dx}$$

$$m\frac{d^2\mu_w}{dw^2}\cdot\frac{dw}{dx}=\frac{d^2\mu_x}{dx^2}.$$

Whence

$$\frac{\dfrac{d^2\mu_x}{dx^2}}{\dfrac{d\mu_x}{dx}}=\frac{\dfrac{d^2\mu_w}{dw^2}}{\dfrac{d\mu_w}{dw}},$$

and it is evident that a similar equation holds for each of the other independent variables y, z, &c. Whence

$$\frac{\dfrac{d^2\mu_x}{dx^2}}{\dfrac{d\mu_x}{dx}}=\frac{\dfrac{d^2\mu_y}{dy^2}}{\dfrac{d\mu_y}{dy}}=\frac{\dfrac{d^2\mu_z}{dz^2}}{\dfrac{d\mu_z}{dz}}=\&c.=\text{(say) }\log c\text{, a constant.}$$

We thus have, for all values of x, the equation

$$\frac{\dfrac{d^2\mu_x}{dx^2}}{\dfrac{d\mu_x}{dx}}=\log c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

Whence, integrating, we have

$$\log \frac{d\mu_x}{dx} = \log c^x + \log b \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vi})$$

where $\log b$ is the constant introduced by integration; and from (vi) we have

$$\frac{d\mu_x}{dx} = bc^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vii})$$

Integrating again, and writing A for the constant so introduced, we have

$$\mu_x = \text{A} + \frac{b}{\log c} c^x,$$

or, writing $\frac{b}{\log c} = \text{B}$, we have finally, for all values of x,

$$\mu_x = \text{A} + \text{B}c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{viii})$$

Hence from equations (i), (iii), and (viii),

$$m\text{A} + m\text{B}c^w = n\text{A} + \text{B}(c^x + c^y + c^z + \text{\&c.}) \quad . \quad . \quad . \quad (\text{ix})$$

$$m\text{A} + m\text{B}c^{w+t} = n\text{A} + \text{B}(c^{x+t} + c^{y+t} + c^{z+t} + \text{\&c.}) \quad . \quad . \quad (\text{x})$$

and from (ix) and (x) it follows that

$$m\text{B}(c^t - 1)c^w = \text{B}(c^t - 1)(c^x + c^y + c^z + \text{\&c.}) \ldots$$

whence

$$mc^w = (c^x + c^y + c^z + \text{\&c.}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xi})$$

and from equations (ix) and (xi) we see that, of necessity, either A must be equal to zero, or m must be equal to n.

When $\text{A} = 0$, we have by equation (vii), $\mu_x = \text{B}c^x$, which form for the force of mortality is proved in Chap. vi, Art. 10, to imply Gompertz's formula for the law of mortality.

When $m = n$, A need not be equal to zero; and in equation (viii) we have the form for the force of mortality which in Chap. vi, Art. 15, is proved to imply Makeham's formula for the law of mortality.

[29] Hence we reach the following conclusions:—

(α). If for n joint lives of different ages we may substitute one, or any number, m, joint lives of equal ages, m not being equal to n, then Gompertz's formula for the law of mortality must hold.

(β). If for n joint lives of different ages we may substitute the same number of joint lives of equal ages, then Makeham's formula for the law of mortality must hold.

[30] We saw in Art. 20 of this chapter that under Gompertz's law we may for n joint lives of different ages substitute the same number of joint lives of equal ages. This, however, does not contradict conclusion β above. It is only because Gompertz's formula is merely that particular case of the more general formula of Makeham where the constant, A, in

the force of mortality is equal to zero, or, what is the same thing, where the constant, s, in the expression for l_x is equal to unity.

[**31**] From the foregoing it appears that Simpson's rule assumes precisely the same law of mortality as does Gompertz's law; and from this it follows that, in order to secure the greatest amount of accuracy attainable when applying Simpson's rule, we should take the oldest two lives, (y) and (z), rather than any other pair, for the purpose of substituting for them an equivalent life. It is in this substitution that, when using Simpson's rule, the assumption of Gompertz's law lies; and as Gompertz's law more correctly represents human mortality at the older ages than at the younger, the older the lives dealt with, the less will be the error.

[**32**] It is only when a mortality table follows Makeham's law that the method of substituting lives of equal age, w, for given joint lives, can be resorted to. From age 28 upwards the mortality table at the end of this volume does follow Makeham's law, and, therefore, if none of the joint lives are younger than 28, the results of the substitution are rigidly accurate. In Chap. vi, Arts. 46 to 51 and Arts. 54 and 55, it is fully explained how the portion of the mortality table below age 28 was constructed, and from the table in Art. 51 of that chapter it will be seen that down to age 20 the mortality table so nearly follows Makeham's law, that no error of material importance will be introduced into the values of joint-life annuities by resorting to the method of substitution, if none of the lives are under 20 years of age. For still younger lives, however, there might be a considerable error, and we must investigate a method by means of which that error may be reduced or eliminated.

[**33**] Retaining the nomenclature and notation of Chap. vi, let the "normal table" be the table down to age 0 constructed by Makeham's constants, and the "final table" the table as given at the end of the volume.

Also let $_tp_x$ = probability of living t years by the final table,
l_x = living by final table,
λ_x = living by normal table,
$r_x = l_x \div \lambda_x$;

and therefore

$$\log l_x = \log \lambda_x + \log r_x.$$

For two lives (x) and (y), assuming $x<y$,

$$\begin{aligned}\log {}_tp_{xy} &= \log l_{x+t} + \log l_{y+t} - \log l_x - \log l_y \\ &= \log\lambda_{x+t} + \log\lambda_{y+t} - \log\lambda_x - \log\lambda_y \\ &\quad + \log r_{x+t} + \log r_{y+t} - \log r_x - \log r_y\end{aligned}$$

But in the normal table Makeham's law holds, and therefore if w be so taken that $c^x+c^y=2c^w$, we have

$$\log\lambda_{x+t}+\log\lambda_{y+t}-\log\lambda_x-\log\lambda_y=2\log\lambda_{w+t}-2\log\lambda_w.$$

Therefore
$$\begin{aligned}\log {}_tp_{xy}&=2\log\lambda_{w+t}-2\log\lambda_w\\&\quad+\log r_{x+t}+\log r_{y+t}-\log r_x-\log r_y\\&=2\log\lambda_{w+t}-2\log\lambda_w+\log\frac{r_{x+t}\cdot r_{y+t}}{r_x\cdot r_y}.\end{aligned}$$

Returning now to the final table, this becomes

$$\begin{aligned}\log {}_tp_{xy}&=2\log l_{w+t}-2\log l_w-2\log r_{w+t}+2\log r_w+\log\frac{r_{x+t}\cdot r_{y+t}}{r_x\cdot r_y}\\&=\log {}_tp_{ww}+\log\frac{r_{x+t}\cdot r_{y+t}\cdot(r_w)^2}{r_x\cdot r_y(r_{w+t})^2}\quad\ldots\ldots\ldots\ldots\quad(9)\end{aligned}$$

[34] If the second term of the right-hand member of the last equation vanish—that is, if the fraction of which the logarithm is taken be equal to unity,—then ${}_tp_{xy}={}_tp_{ww}$; and this actually does happen if both the lives be not less than 28 years of age; and it is very nearly the case if both the lives be over 20 years of age. For younger ages it will be noticed that the fraction varies for every value of t. The fraction may be written $\frac{(r_w)^2}{r_x\cdot r_y}\cdot\frac{r_{x+t}\cdot r_{y+t}}{(r_{w+t})^2}$; and it will be observed that the first factor $\frac{(r_w)^2}{r_x\cdot r_y}$ is constant, while the second factor $\frac{r_{x+t}\cdot r_{y+t}}{(r_{w+t})^2}$ varies with t, approaching towards unity as t increases, and becoming permanently unity when $x+t$ becomes greater than 27. Seeing that $x<w<y$, it is also apparent that for all values of t this second factor of the fraction cannot differ much from unity. If perfect accuracy were required, then the value of each payment of the annuity until $x+t>27$ would have to be calculated separately, but the labour would be prohibitive. The considerations already advanced show, however, that the variable term may be neglected, and that we may write for all values of t

$$\log {}_tp_{xy}=\log {}_tp_{ww}+\log\frac{(r_w)^2}{r_x\cdot r_y},$$

whence
$$a_{xy}=a_{ww}\times\frac{(r_w)^2}{r_x\cdot r_y}\quad\ldots\ldots\ldots\quad(10)$$

[35] *If, however, either of the lives is below* 15, *the formula must be used with caution unless the lives be of nearly equal age.* It will be better in such cases to employ one of the formulas of approximate summation given in Chap. xxiv, and illustrated in Arts. 54 to 62 of this chapter.

[36] Precisely the same principles apply for more lives than two. For m joint lives we must multiply the value of the annuity on m joint lives all aged w, by $\dfrac{(r_w)^m}{r_x . r_y . r_z . \&c.}$.

[37] In Art. 33, when speaking of lives below 28, it was purposely said that the equivalent age, w, is to be found from the relation $c^x+c^y=2c^w$, and it was not suggested to employ μ instead of c. Throughout the whole mortality table c remains the same, as explained in Chap. vi, and therefore the equivalent age found by means of c must always be correct; but because of the change in the other constants, μ is not available for this purpose below age 28. The property of uniform seniority, however, comes to our aid. If x be the younger life and $y-x=b$, we have

$$\begin{aligned} c^w &= \tfrac{1}{2}(c^x+c^y) \\ &= c^x \tfrac{1}{2}(1+c^b) \\ &= \mathrm{C}c^x, \end{aligned}$$

where C is a constant depending, not on x and y, but on their difference b. Therefore

$$w=x+\frac{\log \mathrm{C}}{\log c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

[38] It thus appears that, to find the age w equivalent to the joint lives x and y, we must add to the younger age a constant depending on the difference in age of the proposed two lives. Should therefore x, or y, or both, be less than 28, we can substitute for them any other two ages we please having the same difference, and by that means find the addition to be made to x, the younger of the proposed lives.

[39] As an example, let it be required to find the age equivalent to 12 and 20. These lives being both below 28, we must move along the table. Using 32 and 40

$$\begin{aligned} \mu_{32} &= \cdot 00798 \\ \mu_{40} &= \underline{\cdot 00990} \\ 2)&\underline{\cdot 01788} \\ \mu_w &= \underline{\underline{\cdot 00894}} \end{aligned}$$

By interpolating in the table, this value is found to correspond to age 36·72, showing a seniority of 4·72 years over the age of the younger life, and we conclude that the age equivalent to 12 and 20 is 16·72. The seniority of 4·72 years might have been obtained at once by reference to Table XLI.

[40] Precisely similar reasoning applies to cases involving three or more lives.

[41] In order to apply formula 10 to the calculation of joint-life annuities, it is necessary to know the values of r_x, and therefore these are given in the following table. It will be remembered that r_x is the ratio by which λ_x of the *normal* table must be multiplied in order to get l_x of the *final* table.

x	$\log r_x$	r_x	x	$\log r_x$	r_x
0	0·06475	1·16078	15	$\bar{1}$·99378	0·98578
1	·01551	1·03636	16	·99506	·98869
2	·00273	1·00631	17	·99622	·99133
3	$\bar{1}$·99591	·99063	18	·99722	·99362
4	·99189	·98150	19	·99805	·99552
5	·98913	·97528	20	·99870	·99701
6	·98741	·97143	21	·99917	·99809
7	·98656	·96953	22	·99951	·99887
8	·98640	·96917	23	·99972	·99936
9	·98678	·97002	24	·99986	·99968
10	·98756	·97176	25	·99993	·99984
11	·98858	·97405	26	·99997	·99993
12	·98978	·97674	27	·99999	·99998
13	·99109	·97969	28	0·00000	1·00000
14	·99743	·98272			

[42] As an example, let it be required to find, at 4 per-cent interest, the value of an annuity on two lives aged respectively 12 and 20. Here by Art. 39 the equivalent age, w, is 16·72 and $a_{12:20} = a_{ww} \times \dfrac{(r_w)^2}{r_{12} \cdot r_{20}}$.

$a_{16:16} =$	16·829	$r_{16} =$	98869	$\log a_{ww} =$	1·22253
$a_{17:17} =$	16·641	$r_{17} =$	99133	$2 \log r_w =$	$\bar{1}$·99178
	·188		264		1·21431
·72 reversed	27	·72 reversed	27	$\log r_{12} + \log r_{20} =$	$\bar{1}$·98848
	133		185	$\log a_{12:20} =$	1·22583
	4		5		
Deduction	·136	addition	190		
$a_{ww} =$	16·693	$r_w =$	·99059	$a_{12:20} =$	16·820

As another example, let it be required to find the value of an annuity on the three lives 10, 15 and 25: interest 4 per-cent. In order to get the equivalent age we may make use of the force of mortality at 30, 35 and 45; thus

$$\begin{array}{r} \mu_{30}=\cdot00768 \\ \mu_{35}=\cdot00854 \\ \mu_{45}=\underline{\cdot01204} \\ 3)\underline{\cdot02826} \\ \underline{\underline{\cdot00942}}=\mu_{38\cdot48} \end{array}$$

whence we gather that $w=18\cdot48$

$a_{18:18:18}=$	$14\cdot683$	$r_{18}=$	$\cdot99362$	$\log a_{www}=$	$1\cdot16409$
$a_{19:19:19}=$	$14\cdot490$	$r_{19}=$	$\cdot99552$	$3\log r_w=$	$\bar{1}\cdot99286$
	$\cdot193$		$\cdot00190$		$1\cdot15695$
$\cdot48$ reversed $=$	84	$\cdot48$ reversed $=$	84	$\log r_{10}r_{15}r_{25}$	$\bar{1}\cdot98127$
	077		00076	$\log a_{10:15:25}=$	$1\cdot17568$
	15		15		
Deduction $=$	$\cdot092$	addition $=$	$\cdot00091$		
$a_{www}=$	$14\cdot591$	$r_w=$	$\cdot99453$	$a_{10:15:25}=$	$14\cdot986$

[43] In the preceding articles the values of annuities on joint lives have been found by substituting for the given lives an equal number of lives of uniform age. Another method has been suggested of utilizing Makeham's formula, for the purpose of easily calculating the values of joint-life annuities. By Chap. vi, Art. 28, ${}_tp_x=s^t\times g^{(c^t-1)c^x}=s^t.\mathrm{G}^{c^x}$ when we write $g^{(c^t-1)}=\mathrm{G}$. Similarly, ${}_tp_{xy}=s^{2t}.\mathrm{G}^{(c^x+c^y)}=s^{2t}.\mathrm{G}^{c^w}$ when we take w such that $c^x+c^y=c^w$.

In like manner, if there are m lives,

$${}_tp_{xyz\ldots(m)}=s^{mt}.\mathrm{G}^{c^w}$$

when $$(c^x+c^y+c^z+\&c.)=c^w,$$

or $${}_tp_{xyz\ldots(m)}=s^{(m-1)t}.s^t.\mathrm{G}^{c^w}$$

$$=s^{(m-1)t}.{}_tp_w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Combining now interest with the rate of mortality,

$$a_{xyz\ldots(m)}=\Sigma v^t.{}_tp_{xyz\ldots(m)}=\Sigma v^t.s^{(m-1)t}.{}_tp_w$$

$$=\Sigma v^t_{(m)}.{}_tp_w=a'_w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

when we so change the rate of interest that $v_{(m)}=vs^{(m-1)}$. Thus, to find the value of an annuity on the *m* joint lives, we must substitute a single life aged *w*, and find the value of an annuity on that substituted life at a special rate of interest, which depends on the *number* of joint lives in question and not on the *ages* of the lives.

[44] In the case of the mortality table at the end of this volume, $\log s = -\cdot0026893$. Using *i* to denote the rate of interest at which the value of an annuity is required, and $i_{(2)}$, $i_{(3)}$, and $i_{(4)}$, the corresponding rates for two, three, and four joint lives, we therefore have, calculating $i_{(m)}$ from the formula $v_{(m)}=vs^{(m-1)}$,

i	$i_{(2)}$	$i_{(3)}$	$i_{(4)}$
·03	·0364	·0428	·0493
·035	·0414	·0479	·0544
·04	·0465	·0530	·0595
·05	·0565	·0631	·0697
·06	·0666	·0732	·0799

[45] It thus appears that, instead of giving tables of the values of annuities on joint lives of equal ages, we might have given tables of the values of annuities on single lives at the rates of interest shown in the above table, and the values of joint-life annuities would have been equally readily obtainable. Had this latter method been adopted however, we should have required a table of c^x for all integral values of *x*, in order to find the equivalent age *w*, because when $c^w=c^x+c^y+c^z+$&c. we do not have the convenient relation $\mu_w=\mu_x+\mu_y+\mu_z+$&c. To avoid this objection, the method has been followed in this volume of tabulating the values of annuities on joint lives of equal ages.

[46] The increase in *i* is approximately constant for all rates of interest used in practice, and in the case of the mortality table in this volume it is ·0065 for each additional life involved. That is, if the value of an annuity on two lives be required at rate *i*, we may substitute the value of an annuity on an equivalent single life calculated at rate $i+\cdot0065$; or on three lives, the value of an annuity on an equivalent single life calculated at rate $i+2\times\cdot0065$; or, generally, on *m* lives, the value of an annuity on an equivalent single life calculated at rate $i+(m-1)\times\cdot0065$, when *m* is not large.

[47] These considerations lead to a modification of Simpson's rule, whereby, when the requisite annuity-values are tabulated, it is much improved.

[48] We have seen, Art. 29, that in substituting a life, (w), for the joint lives, (y) and (z), Gompertz's law of mortality is assumed in Simpson's rule. But Makeham's formula represents the law of mortality much more accurately than Gompertz's formula does; and we have seen, Art. 43, that, by altering the rate of interest at the same time that we substitute the single life, (w), for the joint lives, (y) and (z), we replace Gompertz's formula by Makeham's. Therefore, if i be the given rate of interest, and $i_{(2)}$ the altered rate; (in the case of the table in this volume $i_{(2)}=i+\cdot0065$); and if a_x, a_{xy}, &c., denote the values of annuities at the rate i, and a'_x, a'_{xy}, &c., the values at the rate $i_{(2)}$; to find a_{xyz}, we must take w such that $a'_w=a_{yz}$, and we shall have very nearly

$$a'_{xw}=a_{xyz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

[49] The values of annuities at rate $i_{(2)}$ are not tabulated, but yet formula 14 may frequently be effectively used by resorting to a simple process of interpolation.

[50] We may take an example, again from the "Institute of Actuaries' Life Tables", and in so doing we may safely assume that $i_{(2)}=i+\cdot0065$. We cannot find by formula 14 the value of an annuity at 4 per-cent interest, because in the Institute volume joint-life annuities are tabulated only at 3, 3½, and 4 per-cent, and $i_{(2)}$ is greater than i. Let it, however, be required to find by formula 14 the value of an annuity at 3 per-cent on three joint lives aged 30, 40, and 50. Assuming first that $i_{(2)}=i+\cdot005=\cdot035$, we have $a_{40:50}=11\cdot8177=a'_{54\cdot200}$, and $a_{30:40:50}=a'_{30:54\cdot200}=10\cdot7884$. Taking now $i_{(2)}=i+\cdot01=\cdot04$, we have $a_{40:50}=11\cdot8177=a'_{52\cdot457}$ and $a_{30:40:50}=a'_{30:52\cdot457}=10\cdot7628$.

Interpolating now between the values at 3½ per-cent and 4 per-cent, we have

at 3½ per-cent	10·7884
at 4 per-cent	10·7628
Difference	·0256

and

$$a_{30:40:50}=10\cdot7884-\tfrac{15}{50}\times\cdot0256$$
$$=10\cdot7807$$

The true value of the annuity is 10·7840, so that by formula 14 the error is only ·0033.

[51] With other tables of mortality the value of $i_{(2)}$ will be different, but an approximate value may always be found. Thus, in Chap. vi, Art. 24, it was stated that for the Carlisle table $\log s = -\cdot0035114$; and therefore $i_{(2)} = i + \cdot0084$, and $i_{(m)} = i + (m-1) \times \cdot0084$.

[52] Two methods have been proposed for calculating the values of annuities on single and joint lives without the use of previously-prepared annuity tables. The one is explained by Makeham in a paper *J.I.A.*, xvii, 305; and the other by Emory M'Clintock, *J.I.A.*, xviii, 242. They both refer exclusively to mortality tables based on Makeham's hypothesis, and they both require special mathematical tables for their practical application. Seeing that the great majority of mortality tables do not follow Makeham's law, and that therefore only approximate values of the constants can be obtained, neither of the methods in question will give other than approximate results. Moreover, the arithmetical labour involved is very considerable, and is of an intricate character, so that for practical purposes the methods are of no importance, and better results will be obtained with less trouble by using the approximate formulas of Chap. xxiv, which we now proceed to illustrate. The great advantages of these approximate formulas are, that they give results which may safely be depended upon; that they apply to all mortality tables, whether these follow Makeham's law or not; and that it is almost as easy to calculate the values of annuities on several joint lives as on one life, each additional life merely necessitating another column in the calculations.

[53] In Chap. xxiv, examples are given of finding the value of an annuity on a single life. Here it will be convenient similarly to show how the values of annuities on joint lives may be calculated.

[54] Let it be required to find $a_{12:20}$ at 4 per-cent interest, by means of Lubbock's formula, No. 18 of Chap. xxiv. Here $u_t = \frac{v^t . l_{12+t} . l_{20+t}}{l_{12} . l_{20}}$. Taking $n=10$, we have

(1) t	(2) $\log v^t$	(3) $\log l_{12+t}$	(4) $\log l_{20+t}$	(5) (2)+(3)+(4)	(6) (5) + colog $l_{12:20}$ = $\log u_t$
0	0·00000	4·99661	4·98255	9·97916	0·00000
10	$\bar{1}$·82967	4·97741	4·95272	9·75980	$\bar{1}$·78064
20	$\bar{1}$·65933	4·94593	4·91528	9·52054	$\bar{1}$·54138
30	$\bar{1}$·48900	4·90638	4·86210	9·25748	$\bar{1}$·27832
40	$\bar{1}$·31867	4·84793	4·76969	8·93629	$\bar{2}$·95713
50	$\bar{1}$·14833	4·74239	4·57952	8·47024	$\bar{2}$·49108
60	$\bar{2}$·97800	4·51950	4·14572	7·64322	$\bar{3}$·66406
70	$\bar{2}$·80767	4·00415	3·10483	5·91665	$\bar{5}$·93749
80	$\bar{2}$·63733	2·75967	0·60206	1·99906	$\bar{8}$·01990
	$\bar{7}$·86800	40·69997	36·91447	71·48244	$\overline{19}$·67000

t	u_t		Δ		Δ^2		Δ^3		Δ^4
0	1·00000	−	·39655	+	·14094	−	·04336	+	·00460
10	·60345		·25561		·09758		·03876		
20	·34784		·15803		·05882				
30	·18981		·09921						
40	·09060								
50	·03098								
60	·00461								
70	·00009								
80	·00000								

$$22{\cdot}6738 = n(u_0+u_n+\&c.)$$
$$5{\cdot}8975$$
$$16{\cdot}7763 = a_{12:20}$$

$$\frac{n+1}{2}u_0 = 5{\cdot}50000$$
$$c_1\Delta = {\cdot}32715$$
$$c_2\Delta^2 = {\cdot}05814$$
$$c_3\Delta^3 = {\cdot}01132$$
$$c_4\Delta^4 = {\cdot}00085$$
$$5{\cdot}89746$$

[55] In Art. 42 we found by another process an approximate value of this annuity to be 16·820, so that the two results differ by ·044, or about 0·25 per-cent. In order to test them let it be required to find the value of the annuity by formula 39*a*, Chap. xxiv. The integral is $\bar{a}_{xy} = \frac{1}{l_x l_y}\int v^t . l_{x+t} . l_{y+t} . dt$, and here $7n = 102 - 20 = 82$; so that $n = 12$ nearly.

(1) t	(2) $\log v^t$	(3) $\log l_{12+t}$	(4) $\log l_{20+t}$	(5) log coeff.	(6) (2)+(3)+(4)+(5)
0	0·00000	4·99661	4·98255	$\bar{1}\cdot44716$	9·42632
12	$\bar{1}\cdot79560$	4·97170	4·94593	0·20952	9·92275
36	$\bar{1}\cdot38680$	4·87481	4·81393	0·34242	9·41796
60	$\bar{2}\cdot97800$	4·51950	4·14572	0·20952	7·85274
72	$\bar{2}\cdot77360$	3·83531	2·75967	$\bar{1}\cdot74819$	5·11677
	$\bar{4}\cdot93400$	23·19793	21·64780	$\bar{1}\cdot95681$	41·73654

	(7) (6) + colog $l_{12:20}$	(8) Antilog (7)
0	$\bar{1}\cdot44716$	·28000
12	$\bar{1}\cdot94359$	·87819
36	$\bar{1}\cdot43880$	·27466
60	$\bar{3}\cdot87358$	·00747
72	$\bar{5}\cdot13761$	·00001
	$\bar{9}\cdot84074$	$1\cdot44033\times12=17\cdot28396=\bar{a}_{12:20}$

The work here involved is seen to be remarkably limited, especially when it is remembered that the value of the joint-life annuity is found directly from the mortality table, without the intervention of any annuity tables whatever. There was no need to calculate the term for $t=0$, as in the case of immediate annuities that term is always ·28, being the value of the coefficient only.

[56] From the value of the continuous annuity thus found, deducting $\frac{1}{2}-\frac{1}{12}(\mu_{12}+\mu_{20}+\delta)=\cdot49597$, we have $a_{12:20}=16\cdot78799$, which differs by only ·0117 from the value found by Lubbock's formula. It may therefore be assumed that the approximate formulas of Chap. xxiv give better results for young lives, unless the ages be nearly equal, than the method explained in Arts. 33 to 41 of this chapter.

[57] As another example, let it be required to find $a_{10:15:25}$ at 4 per-cent interest, using formula 33 of Chap. xxiv, and making $n=5$. This formula will, as in the last example, give the value of the continuous annuity, from which the value of the ordinary annuity must afterwards be calculated. The integral is, $\bar{a}_{xyz}=\dfrac{1}{l_xl_yl_z}\displaystyle\int v^t.l_{x+t}.l_{y+t}.l_{z+t}.dt.$

(1)	(2)	(3)	(4)	(5)	(6)
t	$\log v^t$	$\log l_{10+t}$	$\log l_{15+t}$	$\log l_{25+t}$	(2)+(3)+(4)+(5)
−5	0·08517	5·01543	5·00000	4·98255	15·08315
5	$\bar{1}$·91483	4·99212	4·98255	4·95272	14·84222
15	$\bar{1}$·74450	4·96869	4·95272	4·91528	14·58119
25	$\bar{1}$·57417	4·93519	4·91528	4·86210	14·28674
35	$\bar{1}$·40383	4·89164	4·86210	4·76969	13·92726
45	$\bar{1}$·23350	4·82325	4·76969	4·57952	13·40596
55	$\bar{1}$·06317	4·69293	4·57952	4·14572	12·48134
65	$\bar{2}$·89283	4·40827	4·14572	3·10483	10·55165
75	$\bar{2}$·72250	3·73902	3·10483	0·60206	6·16841
	$\bar{6}$·63450	42·46654	41·31241	36·91447	115·32792

	(7)	(8)
	(6)+Colog $l_{10:15:25}$	antilog (7)
−5	0·12234	1·32538
5	$\bar{1}$·88141	·76104
15	$\bar{1}$·62038	·41723
25	$\bar{1}$·32593	·21180
35	$\bar{2}$·96645	·09257
45	$\bar{2}$·44515	·02787
55	$\bar{3}$·52053	·00332
65	$\bar{5}$·59084	·00004
75	$\bar{9}$·20760	·00000
	$\overline{20}$·68063	1·51387

1·32538
·76104
24) ·56434
·02351
1·51387
15·3738 = $\bar{a}_{10:15:25}$
14·8783 = $a_{10:15:25}$

[58] In Art. 42 an approximate value of this annuity was found by means of formula 10. The result came out 14·986, or ·108 in excess of the value just found. In order to see which approximation is the more accurate, let the value of the same annuity be found by Woolhouse's formula No. 27, stopping at the first differential coefficient, which in this case is $-(\mu_{10:15:25}+\delta)$, and making $n=7$.

(1) t	(2) $\log v^t$	(3) $\log l_{10+t}$	(4) $\log l_{15+t}$	(5) $\log l_{25+t}$	(6) (2)+(3)+(4)+(5)
7	$\bar{1}$·88077	4·98882	4·97741	4·94593	14·79293
14	$\bar{1}$·76153	4·97170	4·95603	4·91951	14·60877
21	$\bar{1}$·64230	4·94936	4·93143	4·88629	14·40938
28	$\bar{1}$·52307	4·92360	4·90166	4·84020	14·18853
35	$\bar{1}$·40383	4·89164	4·86210	4·76969	13·92726
42	$\bar{1}$·28460	4·84793	4·80398	4·65294	13·58945
49	$\bar{1}$·16537	4·78193	4·71069	4·44854	13·10653
56	$\bar{1}$·04613	4·67372	4·55075	4·07806	12·34866
63	$\bar{2}$·92690	4·48551	4·26453	3·39287	11·06981
70	$\bar{2}$·80767	4·14572	3·73902	2·11059	8·80300
	$\bar{7}$·44217	47·65993	46·69760	43·04462	130·84432

	(7) (6)+Colog $l_{10:15:25}$	(8) u_t
7	$\bar{1}$·83212	·67939
14	$\bar{1}$·64796	·44459
21	$\bar{1}$·44857	·28091
28	$\bar{1}$·22772	·16894
35	$\bar{2}$·96645	·09257
42	$\bar{2}$·62864	·04252
49	$\bar{2}$·14572	·01399
56	$\bar{3}$·38785	·00244
63	$\bar{4}$·10900	·00013
70	$\bar{7}$·84219	·00000
	$\overline{19}$·23622	1·72548 × 7 = 12·07836

$\mu_{10} =$ ·00428
$\mu_{15} =$ ·00353
$\mu_{25} =$ ·00701
$\delta =$ ·03922
·05404
4
·21616

12·07836
3·
15·07836
·21616
14·86220 $= a_{10:15:25}$

[59] Here the value comes out 14·8622, or only ·0161 less than by formula 33. If Lubbock's formula be employed, with $n=10$, the result is 14·8671, or only ·0049 greater than that by Woolhouse's formula. The formulas of approximate summation therefore mutually verify each other, and confirm the remark of Art. 35, that for ages below 15 formula 10 must be used with caution.

[60] It is perhaps worthy of note, that if formula 39*a* be used, the value comes out at 14·8859, and that in the process it is necessary to calculate only four terms excluding the initial term which, as pointed out in Art. 55, is always ·28 for immediate annuities. For practical

purposes, formula 39*a* gives results sufficiently accurate, and at a minimum of labour.

[61] The annual premium for a joint-life assurance may be found by means of the formulas of approximate summation with little trouble, and in such a way that the work checks itself.

We have, Chap. x, Art. 16,

$$\overline{A}_x = -\frac{1}{l_x}\int v^t . dl_{x+t}$$

$$= \frac{1}{l_x}\int v^t . l_{x+t} . \mu_{x+t} . dt$$

Similarly,

$$\overline{A}_{xyz} = \frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t} . l_{z+t}(\mu_{x+t} + \mu_{y+t} + \mu_{z+t})dt \quad . \quad (15)$$

[62] Formula 15 for the assurance, is in every respect identical with the corresponding formula for the annuity, except that for the assurance one more factor, namely $(\mu_{x+t}+\mu_{y+t}+\mu_{z+t})$, is brought in, necessitating one more column in the calculations. Therefore if the value of an assurance be sought, the best plan will be, first to calculate the annuity, and then add the additional column and complete the calculation of the assurance. Afterwards, by means of the relation $\overline{A}=1-\delta\bar{a}$, the two values thus found may be used to verify one another. Having, then, the values of the assurance and the continuous annuity, the annual premium may easily be found.

[63] In all the examples of this chapter the work has been performed as if no monetary tables had been already calculated from the mortality table. If the values of D_x have already been formed, then we have

$$\bar{a}_{xyz} = \frac{1}{D_x . l_y . l_z} . \int D_{x+t} . l_{y+t} . l_{z+t} . dt \quad . \quad . \quad . \quad . \quad (16)$$

and if the values of D_{xy} are available, then

$$\bar{a}_{xyz} = \frac{1}{D_{xy} . l_z} . \int D_{x+t:y+t} . l_{z+t} . dt \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In each of these cases the amount of arithmetical work is abridged; but unless the logarithms of D_x and D_{xy} have been tabulated, the economy in labour is more apparent than real.

CHAPTER XIII.

CONTINGENT, OR SURVIVORSHIP, ASSURANCES.

1. An assurance which is payable only if the lives on which it depends fail in a specified order, is called a *Contingent*, or a *Survivorship, Assurance.*

2. In Chap. iv, the probabilities of survivorship among lives were discussed with some fulness, and the object of this chapter is, to combine the rate of interest with the results there obtained.

3. To find A^1_{xy}, the value of 1 payable at the end of the year of the death of (x), if he die before (y).

The value of 1 payable at the end of the nth year, if (x) die in that year, (y) surviving him, is, by Chap. iv, Art. 3,

$$v^n \tfrac{1}{2}({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y + {}_np_y)$$

$$= \tfrac{1}{2}v^n\left({}_{n-1}p_{xy} - {}_np_{xy} + \frac{{}_np_{x-1:y}}{p_{x-1}} - \frac{{}_np_{x:y-1}}{p_{y-1}}\right),$$

and, to find A^1_{xy}, this expression must be summed for every integral value of n, from 1 to the extremity of life. But

$$\Sigma v^n({}_{n-1}p_{xy} - {}_np_{xy}) = A_{xy},$$

and
$$\Sigma v^n\left(\frac{{}_np_{x-1:y}}{p_{x-1}} - \frac{{}_np_{x:y-1}}{p_{y-1}}\right) = \frac{a_{x-1:y}}{p_{x-1}} - \frac{a_{x:y-1}}{p_{y-1}};$$

and therefore

$$A^1_{xy} = \frac{1}{2}\left\{A_{xy} + \frac{a_{x-1:y}}{p_{x-1}} - \frac{a_{x:y-1}}{p_{y-1}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

4. This formula agrees with No. 5 of Chap. iv, if we write A_{xy} for 1, and replace the expectations of life by the corresponding annuities. In fact, almost all the formulas of Chap. iv may thus be modified, so as to pass from the survivorship probabilities to the contingent assurances.

5. Formula 1 is based on the hypothesis of a uniform distribution of deaths, but that hypothesis may be discarded. From Chap. iv, formula 8,

$$\mathrm{A}_{xy}^{1}=\frac{1}{2}\left\{\mathrm{A}_{xy}+\frac{a_{x-1:y}}{p_{x-1}}-\frac{a_{x:y-1}}{p_{y-1}}\right\}$$

$$+\frac{1}{12}\left\{\frac{a_{x-1:y}-p_y a_{x-1:y+1}}{p_{x-1}}-\frac{a_{x:y-1}-p_x a_{x+1:y-1}}{p_{y-1}}\right.$$

$$\left.+p_y a_{x:y+1}-p_x a_{x+1:y}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

6. Formula 2 gives, theoretically, a very close approximation to the value of A_{xy}^{1}, but as the values of annuities on joint lives are not usually tabulated to more than three, or at most four, places of decimals, formula 1 gives in practice results as correct as the annuity tables admit of.

7. By taking different forms of the probability ${}_{n-1|}q_{xy}^{1}$, the expression for A_{xy}^{1} may be varied. Thus

$$\mathrm{A}_{xy}^{1}=\Sigma v^n\cdot\frac{l_{x+n-1}-l_{x+n}}{l_x}\cdot\frac{l_{y+n-\frac{1}{2}}}{l_y}$$

$$=\Sigma v^n\cdot\frac{l_{x-1}}{l_x}\cdot\frac{l_{y-\frac{1}{2}}}{l_y}\cdot\frac{l_{x-1+n}l_{y-\frac{1}{2}+n}}{l_{x-1}l_{y-\frac{1}{2}}}-\Sigma v^n\cdot\frac{l_{y-\frac{1}{2}}}{l_y}\cdot\frac{l_{x+n}l_{y-\frac{1}{2}+n}}{l_x l_{y-\frac{1}{2}}}$$

$$=\frac{l_{x-1}}{l_x}\cdot\frac{l_{y-\frac{1}{2}}}{l_y}\cdot a_{x-1:y-\frac{1}{2}}-\frac{l_{y-\frac{1}{2}}}{l_y}a_{x:y-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, since by Chap. ix, Art 37, ${}_{\frac{1}{2}|}\mathrm{a}_x=\bar{a}_x$ approximately, therefore, from the expression from which equation 3 was derived,

$$\mathrm{A}_{xy}^{1}=\frac{v^{\frac{1}{2}}}{l_x}\{\Sigma l_{x-\frac{1}{2}}\cdot v^{n-\frac{1}{2}}\cdot{}_{n-\frac{1}{2}}p_{x-\frac{1}{2}:y}-\Sigma l_{x+\frac{1}{2}}\cdot v^{n-\frac{1}{2}}\cdot{}_{n-\frac{1}{2}}p_{x+\frac{1}{2}:y}\}$$

$$=\frac{v^{\frac{1}{2}}}{l_x}\{l_{x-\frac{1}{2}}\bar{a}_{x-\frac{1}{2}:y}-l_{x+\frac{1}{2}}\bar{a}_{x+\frac{1}{2}:y}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

8. By omitting the factor $v^{\frac{1}{2}}$, we have an approximation to the value of $\bar{\mathrm{A}}_{xy}^{1}$, an assurance payable at the moment of the death of (x) if (y) be then living, namely,

$$\bar{\mathrm{A}}_{xy}^{1}=\frac{1}{l_x}\{l_{x-\frac{1}{2}}\bar{a}_{x-\frac{1}{2}:y}-l_{x+\frac{1}{2}}\bar{a}_{x+\frac{1}{2}:y}\} \quad . \quad . \quad . \quad . \quad (5)$$

9. In formula 1, transposing x and y, we have

$$\mathrm{A}_{x\overset{1}{y}}=\frac{1}{2}\left\{\mathrm{A}_{xy}+\frac{a_{x:y-1}}{p_{y-1}}-\frac{a_{x-1:y}}{p_{x-1}}\right\},$$

and adding this to the original expression, we have

$$A_{\overset{1}{x}y}+A_{x\overset{1}{y}}=A_{xy} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

This is as it should be, because it is evident that an assurance payable if (x) die before (y), together with an assurance payable if (y) die before (x), is equivalent to an assurance payable on the failure of the joint lives.

10. To find $A_{\overset{2}{x}y}$, an assurance payable at the end of the year of the death of (x), if he die after (y). The required assurance is equivalent to an assurance payable absolutely on the death of (x), less an assurance payable if he die before (y). That is,

$$A_{\overset{2}{x}y}=A_x-A_{\overset{1}{x}y} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

11. In Chap. iv, Art 9, it was shown that $Q_{\overset{2}{x}y}=Q_{x\overset{1}{y}}$; but it will be noticed now that the equality does not exist with the corresponding assurances; that $A_{\overset{2}{x}y}$ is *not* equal to $A_{x\overset{1}{y}}$. In determining the probabilities, we look only at the *order* of the deaths, and not at the *times* of the deaths; but when the payment of money is concerned, the case is different. Then the *time* at which payment is to be made is as important an element in finding the value, as is the probability on which the payment depends. Assuming (x) to die after (y), then $A_{x\overset{1}{y}}$ is payable on the first death, and $A_{\overset{2}{x}y}$ is deferred until the second death. Therefore, although $Q_{\overset{2}{x}y}=Q_{x\overset{1}{y}}$, yet $A_{\overset{2}{x}y}<A_{x\overset{1}{y}}$.

12. There may be deferred, or temporary, contingent assurances. Thus,

$$\left.\begin{aligned}
{}_{n|}A_{\overset{1}{x}y}&=v^n{}_np_{xy}A_{\overset{1}{\overline{x+n}}:\overline{y+n}}\\
{}_{n|}A_{\overset{2}{x}y}&={}_{n|}A_x-{}_{n|}A_{\overset{1}{x}y}\\
{}_{n|}A_{x\overset{1}{y}}&={}_{n|}A_{xy}-{}_{n|}A_{\overset{1}{x}y}\\
{}_{n|}A_{x\overset{2}{y}}&={}_{n|}A_y+{}_{n|}A_{\overset{1}{x}y}-{}_{n|}A_{xy}
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (8)$$

and

$$\left.\begin{aligned}
{}_{|n}A_{\overset{1}{x}y}&=A_{\overset{1}{x}y}-{}_{n|}A_{\overset{1}{x}y}\\
{}_{|n}A_{\overset{2}{x}y}&={}_{|n}A_x-{}_{|n}A_{\overset{1}{x}y}\\
{}_{|n}A_{x\overset{1}{y}}&={}_{|n}A_{xy}-{}_{|n}A_{\overset{1}{x}y}\\
{}_{|n}A_{x\overset{2}{y}}&={}_{|n}A_y+{}_{|n}A_{\overset{1}{x}y}-{}_{|n}A_{xy}
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (9)$$

13. All contingent assurances involving two lives have thus been reduced to assurances on single and joint lives, with contingent assurances depending on (x) dying before (y). The object is to limit the extent of the necessary tables. In consequence of the relations displayed in formulas 6 to 9, it is sufficient if A^1_{xy} be tabulated for only $x<y$ or $x>y$. In practice the table of A^1_{xy} for $x<y$ will be the more convenient, as assurances on the younger life against the older are more frequently met with than those on the older life against the younger.

14. Formula 1 may be translated into commutation symbols. If the commutation table be in Davies's form, that is if $D_{xy}=v^x l_x l_y$ when $x>y$, (*see* Chap. vii, Art. 53), then

$$\frac{a_{x-1:y}}{p_{x-1}}=\frac{N_{x-1:y}}{p_{x-1}D_{x-1:y}}=\frac{l_{x-1}}{l_x}\cdot\frac{N_{x-1:y}}{v^{x-1}l_{x-1}l_y}=\frac{vN_{x-1:y}}{D_{xy}}$$

$$\frac{a_{x:y-1}}{p_{y-1}}=\frac{N_{x:y-1}}{p_{y-1}D_{x:y-1}}=\frac{l_{y-1}}{l_y}\cdot\frac{N_{x:y-1}}{v^x l_x l_{y-1}}=\frac{N_{x:y-1}}{D_{xy}}$$

$$A_{xy}=\frac{M_{xy}}{D_{xy}}=\frac{vN_{x-1:y-1}-N_{xy}}{D_{xy}}.$$

Therefore, after collecting like terms, when $x>y$,

$$A^1_{xy}=\frac{v(N_{x-1:y-1}+N_{x-1:y})-(N_{x:y-1}+N_{xy})}{2D_{xy}} \quad . \quad . \quad (10)$$

Similarly, when $x<y$,

$$A^1_{xy}=\frac{v(N_{x-1:y-1}-N_{x:y-1})+(N_{x-1:y}-N_{xy})}{2D_{xy}} \quad . \quad . \quad (11)$$

15. If the commutation tables be in De Morgan's form, that is if $D_{xy}=v^{\frac{x+y}{2}}l_x l_y$, (*see* Chap. vii, Art. 53), then,

$$\frac{a_{x-1:y}}{p_{x-1}}=\frac{v^{\frac{1}{2}}N_{x-1:y}}{D_{xy}}, \text{ and } \frac{a_{x:y-1}}{p_{y-1}}=\frac{v^{\frac{1}{2}}N_{x:y-1}}{D_{xy}};$$

and when $x>y$, or $x<y$,

$$A^1_{xy}=\frac{vN_{x-1:y-1}-N_{xy}+v^{\frac{1}{2}}(N_{x-1:y}-N_{x:y-1})}{2D_{xy}} \quad . \quad . \quad (12)$$

16. With either form of table, when $x=y$,

$$A^1_{xy}=\tfrac{1}{2}A_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

17. Special commutation tables may be constructed for contingent assurances. By Chap. iv, formula 2, $_{n-1|}q^1_{xy}=\dfrac{d_{x+n-1}\cdot l_{y+n-\frac{1}{2}}}{l_x l_y}$; and therefore the value of 1 to be received at the end of the nth year if (x)

die in that year, leaving (y) surviving him, is $\dfrac{v^n . d_{x+n-1} . l_{y+n-\frac{1}{2}}}{l_x l_y}$. If the table is to be in Davies's form, and if $x>y$, the expression may be written $\dfrac{v^{x+n} . d_{x+n-1} . l_{y+n-\frac{1}{2}}}{\mathrm{D}_{xy}}$, or if $x<y$, $\dfrac{v^{y+n} . d_{x+n-1} . l_{y+n-\frac{1}{2}}}{\mathrm{D}_{xy}}$; while if the table is to be in De Morgan's form, the expression becomes, both for $x>y$ and $x<y$, $\dfrac{v^{\frac{x+y}{2}+n} . d_{x+n-1} . l_{y+n-\frac{1}{2}}}{\mathrm{D}_{xy}}$. Writing $\mathrm{C}_{\overset{1}{\overline{x+n-1}}:\overline{y+n-1}}$ for the numerator of these expressions, we have $\dfrac{\mathrm{C}_{\overset{1}{\overline{x+n-1}}:\overline{y+n-1}}}{\mathrm{D}_{xy}}$ as the value of 1 receivable at the end of the nth year if (x) die in that year, leaving (y) surviving him. In order to find A^1_{xy}, the expression must be summed for all integral values of n from 1 to the extremity of life: that is

$$\mathrm{A}^1_{xy} = \frac{\mathrm{C}^1_{xy} + \mathrm{C}_{\overset{1}{\overline{x+1}}:\overline{y+1}} + \mathrm{C}_{\overset{1}{\overline{x+2}}:\overline{y+2}} + \&\text{c.}}{\mathrm{D}_{xy}}$$

$$= \frac{\mathrm{M}^1_{xy}}{\mathrm{D}_{xy}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

when M^1_{xy} is written for the sum of the values of C^1_{xy}.

18. By means of these commutation columns the values of deferred and temporary contingent assurances may be found very conveniently. Thus

$$\left.\begin{aligned} {}_{n|}\mathrm{A}^1_{xy} &= \frac{\mathrm{M}_{\overset{1}{\overline{x+n}}:\overline{y+n}}}{\mathrm{D}_{xy}} \\ {}_{|n}\mathrm{A}^1_{xy} &= \frac{\mathrm{M}^1_{xy} - \mathrm{M}_{\overset{1}{\overline{x+n}}:\overline{y+n}}}{\mathrm{D}_{xy}} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (15)$$

19. David Chisholm has given extensive commutation tables for survivorship assurances, in his book on Life Contingencies, published in 1858. They are based on the Carlisle Mortality Table, at many rates of interest, and are complete for $x>y$ and $x<y$. They are in Davies's form.

20. It is usual when contingent assurances are granted at annual premiums, for the premiums to be payable during the joint lives of (x) and (y), so that

$$\mathrm{P}^1_{xy} = \frac{\mathrm{A}^1_{xy}}{1+a_{xy}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

$$= \frac{\mathrm{M}^1_{xy}}{\mathrm{N}_{x-1:y-1}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

21. To find $\mathrm{A}^{1}_{x:\overline{y(\overline{t|})}}$, the value of 1 payable at the end of the year of the death of (x) if he die before (y) or within t years after the death of (y). Following the reasoning by which formula 14 of Chap. iv was deduced:—The assurance sought consists of two portions: 1st, a temporary assurance for t years on the life of (x); and 2nd, an assurance payable if (x) die after t years, (y) surviving him, or having previously died, but at such a time that the interval between the deaths is not greater than t years. The value of the first portion of the assurance sought is therefore ${}_{|t}\mathrm{A}_x$, and that of the second, $\Sigma v^{t+n}.{}_tp_x.\frac{1}{2}({}_{n-1}p_{x+t}-{}_np_{x+t})({}_{n-1}p_y+{}_np_y)$, which is equal to $v^t.{}_tp_x.\mathrm{A}_{\overset{1}{\overline{x+t:y}}}$. Therefore

$$\mathrm{A}^{1}_{x:\overline{y(\overline{t|})}}={}_{|t}\mathrm{A}_x+v^t.{}_tp_x.\mathrm{A}_{\overset{1}{\overline{x+t:y}}}$$

$$=\mathrm{A}_x-{}_{t|}\mathrm{A}_x+\frac{\mathrm{D}_{x+t}}{\mathrm{D}_x}\mathrm{A}_{\overset{1}{\overline{x+t:y}}}$$

$$=\mathrm{A}_x-\frac{\mathrm{D}_{x+t}}{\mathrm{D}_x}\left(\mathrm{A}_{x+t}-\mathrm{A}_{\overset{1}{\overline{x+t:y}}}\right) \quad . \quad . \quad . \quad . \quad (18)$$

22. Assurances of this description are not unfrequently met with in connection with the loan transactions of Life Offices. For instance, a son, B, who will become entitled on the death of his father, A, to a life interest in property producing a large income, wishes to raise a sum of money to be repaid after he succeeds to the property; and the lender desires to secure himself by means of a life policy. If the amount of loan be small compared with the yearly income from the property, then it will be sufficient if a policy be effected on the life of B against the life of A, and for, say, three years after the death of A; because after A's death, B will have the means to repay the loan in a year or two out of income.

23. The annual premium for $\mathrm{A}^{1}_{x:\overline{y(\overline{t|})}}$ may be payable during the joint lives only, in which case

$$\mathrm{PA}^{1}_{x:\overline{y(\overline{t|})}}=\frac{\mathrm{A}^{1}_{x:\overline{y(\overline{t|})}}}{1+a_{xy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Or the annual premium may be payable until the risk determines, that is, during the joint lives and for t years longer if (x) live so long; in which case, by Chap. vii, formula 69,

$$\mathrm{PA}^{1}_{x:\overline{y(\overline{t|})}}=\frac{\mathrm{A}^{1}_{x:\overline{y(\overline{t|})}}}{1+a_{x:\overline{y(\overline{t|})}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

24. Passing now to problems involving three lives:—Let it be required to find A^1_{xyz}, the value of 1 payable on the death of (x) if he die first of the three lives (x), (y), and (z).

In Chap. iv, formula 18, Q^1_{xyz}, the corresponding probability, is given. In that probability the term consisting of unity results from $\Sigma({}_{n-1}p_{xyz}-{}_{n}p_{xyz})$, and when the rate of interest is involved this becomes A_{xyz}. Also the expectations of life are changed into the corresponding annuities. Therefore

$$A^1_{xyz}=\frac{1}{3}\left\{A_{xyz}-\frac{a_{x:y-1:z-1}}{p_{y-1:z-1}}+\frac{a_{x-1:y:z}}{p_{x-1}}\right\}$$

$$+\frac{1}{6}\left\{\frac{a_{x-1:y-1:z}}{p_{x-1:y-1}}-\frac{a_{x:y:z-1}}{p_{z-1}}+\frac{a_{x-1:y:z-1}}{p_{x-1:z-1}}-\frac{a_{x:y-1:z}}{p_{y-1}}\right\} \quad . \quad (21)$$

25. An assurance payable on the death of (x) if he die after (y) but before (z), $A^2_{x\underset{1}{y}z}$, is equivalent to an assurance payable if (x) die before (z), less an assurance payable if he die before both (y) and (z). That is

$$A^2_{x\underset{1}{y}z}=A^1_{xz}-A^1_{xyz} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

26. An assurance payable at the death of (x) if he die second of the three lives, A^2_{xyz}, is equal to an assurance payable at the death of (x), if he die after (y) and before (z), together with an assurance payable at the death of (x) if he die after (z) and before (y). That is

$$A^2_{xyz}=A^2_{x\underset{1}{y}z}+A^2_{xy\underset{1}{z}}$$

$$=A^1_{xy}+A^1_{xz}-2A^1_{xyz} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

Also
$$A^3_{xyz}=A_x-A^1_{xyz}-A^2_{xyz}$$

$$=A_x-A^1_{xy}-A^1_{xz}+A^1_{xyz} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

27. An assurance payable at the death of (x) if he die before the survivor of (y) and (z), $A^1_{x:\overline{yz}}$, is equivalent to an assurance payable if (x) die either first or second of the three lives. That is

$$A^1_{x:\overline{yz}}=A^1_{xyz}+A^2_{xyz}$$

$$=A^1_{xy}+A^1_{xz}-A^1_{xyz} \quad . \; . \; . \; . \; . \; . \quad (25)$$

28. An assurance payable on the failure of the joint lives (x) and (y) if that event take place before the death of (z), $A_{\overset{1}{\overline{xy}}:z}$, is equivalent to an assurance payable on the death of (x) if he die first of the three lives, together with an assurance payable on the death of (y) if he die first; that is

$$\mathrm{A}^{\;1}_{\overline{xy}:z}=\mathrm{A}_{\overset{1}{x}yz}+\mathrm{A}_{x\overset{1}{y}z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

29. Lastly, an assurance payable on the death of the survivor of (x) and (y), if such survivor die before (z), $\mathrm{A}_{\overline{xy}:\underset{3}{z}}$, is equivalent to an assurance payable if (x) die second, (y) having died first, together with an assurance payable if (y) die second, (x) having died first. That is

$$\begin{aligned}\mathrm{A}_{\overline{xy}:\underset{3}{z}}&=\mathrm{A}_{\overset{2}{x}\underset{1}{y}z}+\mathrm{A}_{\underset{1}{x}\overset{2}{y}z}\\&=\mathrm{A}_{\overset{1}{x}z}+\mathrm{A}_{\overset{1}{y}z}-\mathrm{A}_{\overset{1}{x}yz}-\mathrm{A}_{x\overset{1}{y}z} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)\end{aligned}$$

30. The foregoing formulas show that all the contingent assurances in which only one order of survivorship is involved, may be expressed in terms of assurances which determine on the first death. By this means the numerical calculation of the benefits is much facilitated.

31. In order to find by the formulas already given the numerical values of contingent assurances depending on more than two lives, a great amount of calculation would be required, even if the values of the necessary joint-life annuities were tabulated. But the values of annuities on more than two joint lives are never completely tabulated, because of the extent of tables that would be required. In practice it is therefore common to adopt one or other form of Simpson's rule, explained in Chap. xii, Arts. 3 to 11, and substitute a single life, assumed to be equivalent to two of the lives involved. Thus, to find $\mathrm{A}_{\overset{1}{x}yz}$, we might find w so that $a_{yz}=a_{w}$, and then write $\mathrm{A}_{\overset{1}{x}yz}=\mathrm{A}_{\overset{1}{x}w}$. Similarly, for $\mathrm{A}_{\overset{1}{x}:\overline{yz}}$ we might write $\mathrm{A}_{\overset{1}{x}w}$, where $a_{\overline{yz}}=a_{w}$; and for $\mathrm{A}_{\overline{xy}:\underset{3}{z}}$ we might write $\mathrm{A}_{\overset{1}{w}z}$, where $a_{\overline{xy}}=a_{w}$. These approximations are, however, very rough, and much more accurate as well as more simple formulas may be obtained by means of the differential and integral calculus. The analysis is very similar to that employed in Chap. x, Arts. 16 to 18.

[32] To find $\overline{\mathrm{A}}_{\overset{1}{x}y}$, the value of 1 payable at the moment of the death of (x), if (y) be then alive.

When h is the fractional part of a year, the value of 1 payable at the end of that hth part of the year in which (x) dies, if (y) be alive at the end of that part, is

$$\frac{1}{l_x l_y}\left\{v^h(l_x - l_{x+h})l_{y+h} + v^{2h}(l_{x+h} - l_{x+2h})l_{y+2h} + \&c.\right\}$$

$$= \frac{1}{l_x l_y}\Sigma_{n=1}^{n=\infty} v^{nh}(-\Delta l_{x+\overline{n-1}.h})l_{y+nh}$$

where $\Delta l_{x+\overline{n-1}.h}$ means the finite difference of l with respect to the interval h. If now h diminish without limit, the symbols Σ and Δ must be replaced by $\int$ and d. Writing $nh = \overline{n-1}.h = t$ (h being indefinitely small, we hence have

$$\overline{\mathrm{A}}^1_{xy} = -\frac{1}{l_x l_y}\int v^t . l_{y+t} . dl_{x+t}$$

$$= -\frac{1}{l_x l_y}\int\left(\frac{d}{dt}l_{x+t}\right)l_{y+t}.v^t.dt \quad . \quad . \quad . \quad . \quad . \quad (28)$$

That is $$l_x l_y \overline{\mathrm{A}}^1_{xy} = -\int\left(\frac{d}{dt}l_{x+t}\right)l_{y+t}.v^t.dt.$$

But $$l_x l_y \bar{a}_{xy} = \int l_{x+t}.l_{y+t}.v^t.dt,$$

and $$\frac{d}{dx}(l_x l_y \bar{a}_{xy}) = \int\left(\frac{d}{dx}l_{x+t}\right)l_{y+t}.v^t.dt$$

$$= \int\left(\frac{d}{dt}l_{x+t}\right)l_{y+t}.v^t.dt$$

$$= -l_x l_y \overline{\mathrm{A}}^1_{xy}$$

Also $$\frac{d}{dx}(l_x l_y \bar{a}_{xy}) = l_y\left(\frac{d}{dx}l_x\right)\bar{a}_{xy} + l_x l_y\left(\frac{d}{dx}\bar{a}_{xy}\right)$$

$$= -l_x l_y\left\{\mu_x \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dx}\right\}$$

Therefore $$l_x l_y \mathrm{A}^1_{xy} = l_x l_y\left\{\mu_x \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dx}\right\}$$

and since by Chap. xxiii, formula 28, $\frac{d\bar{a}_{xy}}{dx} = -\frac{1}{2}(\bar{a}_{x-1:y} - \bar{a}_{x+1:y})$ approximately, therefore finally

$$\overline{\mathrm{A}}^1_{xy} = \mu_x \bar{a}_{xy} + \tfrac{1}{2}(\bar{a}_{x-1:y} - \bar{a}_{x+1:y}) \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Or we may proceed more briefly as follows:—

$$\overline{\mathrm{A}}^1_{xy} = -\frac{1}{l_x l_y}\int v^t l_{y+t} dl_{x+t}$$

$$= \int v^t . \frac{l_{y+t}}{l_y} . \frac{l_{x-1+t} - l_{x+1+t}}{2l_x}dt$$

$$= \int\left\{\frac{v^t l_{y+t} l_{x-1+t}}{2l_{x-1}l_y} \cdot \frac{l_{x-1}}{l_x} - \frac{v^t l_{y+t} l_{x+1+t}}{2l_{x+1}l_y} \cdot \frac{l_{x+1}}{l_x}\right\}dt$$

$$= \tfrac{1}{2}\left(\frac{\bar{a}_{x-1:y}}{p_{x-1}} - p_x \bar{a}_{x+1:y}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)a$$

[33] The calculation of the value of the Continuous Contingent Assurance from an ordinary table of annuities will be expedited, and the value obtained will be sufficiently accurate for many practical purposes, if $\bar{a}_{xy}$ be replaced by $a_{xy}+\frac{1}{2}$. This substitution being a trifle in excess, and the coefficient μ_x being small, the result will be but little affected by the change, and will incline to the safe side. Thus we shall have the convenient formula

$$\overline{\mathrm{A}}{}^{1}_{xy}=\mu_x(a_{xy}+\tfrac{1}{2})+\tfrac{1}{2}(a_{x-1:y}-a_{x+1:y}) \quad . \quad . \quad . \quad . \quad (30)$$

[34] The process by which formula 29 is derived is not vitiated if for lives we substitute statuses, and make corresponding alterations in the annuities and force of mortality involved in the formula. Thus, taking the joint lives (y) and (z), for the single life (y), we have

$$\overline{\mathrm{A}}{}^{1}_{xyz}=\mu_x\bar{a}_{xyz}+\tfrac{1}{2}(\bar{a}_{x-1:yz}-\bar{a}_{x+1:yz}) \quad . \quad . \quad . \quad . \quad (31)$$

Or taking the survivor of the lives (y) and (z), for the single life (y), we have

$$\overline{\mathrm{A}}{}^{1}_{x:\overline{yz}}=\mu_x\bar{a}_{x:\overline{yz}}+\tfrac{1}{2}(\bar{a}_{x-1:\overline{yz}}-\bar{a}_{x+1:\overline{yz}}) \quad . \quad . \quad . \quad . \quad (32)$$

Or taking the joint lives (x) and (y), for the single life (x), we have

$$\overline{\mathrm{A}}_{\overset{1}{\overline{xy}}:z}=(\mu_x+\mu_y)\bar{a}_{xyz}+\tfrac{1}{2}(\bar{a}_{x-1:y-1:z}-\bar{a}_{x+1:y+1:z}) \quad . \quad . \quad (33)$$

[35] An equivalent result to that given in formula 33 may be arrived at in another way, by the relation $\overline{\mathrm{A}}_{\overset{1}{\overline{xy}}:z}=\overline{\mathrm{A}}{}^{1}_{xyz}+\overline{\mathrm{A}}_{x\overset{1}{y}z}$. Thus

$$\begin{aligned}\overline{\mathrm{A}}_{\overset{1}{\overline{xy}}:z}&=\overline{\mathrm{A}}{}^{1}_{xyz}+\overline{\mathrm{A}}_{x\overset{1}{y}z}\\&=\mu_x\bar{a}_{xyz}+\tfrac{1}{2}(\bar{a}_{x-1:yz}-\bar{a}_{x+1:yz})\\&\quad+\mu_y\bar{a}_{xyz}+\tfrac{1}{2}(\bar{a}_{x:y-1:z}-\bar{a}_{x:y+1:z})\\&=(\mu_x+\mu_y)\bar{a}_{xyz}+\tfrac{1}{2}(\bar{a}_{x-1:yz}+\bar{a}_{x:y-1:z}-\bar{a}_{x+1:yz}-\bar{a}_{x:y+1:z}).\end{aligned}$$

The two expressions for $\overline{\mathrm{A}}_{\overset{1}{\overline{xy}}:z}$ have practically the same numerical value. They differ only in that they involve different approximations to the differential coefficient of the annuity $\bar{a}_{xyz}$.

[36] There is one status to which formula 29 does not apply, namely, the survivor of (x) and (y), against (z). For l_{x+t} under the sign of integration we should have to substitute $l_{\overline{x+t:y+t}}$, but that symbol has not such a meaning that it could be dealt with like l_{x+t}, or $l_{x+t:y+t}$. To find $\overline{\mathrm{A}}_{\overline{xy}:\underset{3}{z}}$ we must proceed differently. Thus, by formula 27,

$$\begin{aligned}\bar{A}_{\overline{xy}:\underset{3}{z}} &= \bar{A}^1_{xz}+\bar{A}^1_{yz}-\bar{A}^1_{xyz}-\bar{A}_{x\overset{1}{y}z}\\
&=\mu_x\bar{a}_{xz}+\tfrac{1}{2}(\bar{a}_{x-1:z}-a_{x+1:z})\\
&\quad+\mu_y\bar{a}_{yz}+\tfrac{1}{2}(\bar{a}_{y-1:z}-\bar{a}_{y+1:z})\\
&\quad-\mu_x\bar{a}_{xyz}-\tfrac{1}{2}(\bar{a}_{x-1:yz}-\bar{a}_{x+1:yz})\\
&\quad-\mu_y\bar{a}_{xyz}-\tfrac{1}{2}(\bar{a}_{x:y-1:z}-\bar{a}_{x:y+1:z})\\
&=\mu_x(\bar{a}_{xz}-\bar{a}_{xyz})+\mu_y(\bar{a}_{yz}-\bar{a}_{xyz})\\
&\quad+\tfrac{1}{2}(\bar{a}_{x-1:z}-\bar{a}_{x+1:z}+\bar{a}_{y-1:z}-\bar{a}_{y+1:z})\\
&\quad-\tfrac{1}{2}(\bar{a}_{x-1:yz}-\bar{a}_{x+1:yz}+\bar{a}_{x:y-1:z}-\bar{a}_{x:y+1:z}) \quad . \quad . \quad (34)\end{aligned}$$

By Arts. 34 and 35, we may substitute $\frac{1}{2}(a_{x-1:y-1:z}-a_{x+1:y+1:z})$ for the last term in equation 34; and we may also throughout the formula substitute ordinary yearly annuities for the continuous annuities. With these alterations, formula 34 becomes

$$\begin{aligned}\bar{A}_{\overline{xy}:\underset{3}{z}} &= \mu_x(a_{xz}-a_{xyz})+\mu_y(a_{yz}-a_{xyz})\\
&\quad+\tfrac{1}{2}(a_{x-1:z}-a_{x+1:z}+a_{y-1:z}-a_{y+1:z})\\
&\quad-\tfrac{1}{2}(a_{x-1:y-1:z}-a_{x+1:y+1:z}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34a)\end{aligned}$$

[37] If the rate of interest be taken as zero, then the contingent assurances, $\bar{A}^1_{xy}$ and $\bar{A}^1_{xyz}$, in formulas 29 and 31, become the survivorship probabilities, Q^1_{xy} and Q^1_{xyz}, respectively; and the continuous annuities become the corresponding complete expectations of life. Whence

$$Q^1_{xy}=\mu_x\mathring{e}_{xy}+\tfrac{1}{2}(\mathring{e}_{x-1:y}-\mathring{e}_{x+1:y}) \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$Q^1_{xyz}=\mu_x\mathring{e}_{xyz}+\tfrac{1}{2}(\mathring{e}_{x-1:yz}-\mathring{e}_{x+1:yz}) \quad . \quad . \quad . \quad . \quad (36)$$

It is easier to calculate the numerical values of these probabilities by the expressions here given, than by formulas 5 and 18 of Chap. iv.

[38] By assuming Makeham's formula for the law of mortality, an expression may be found for the survivorship probabilities, which, under some circumstances, may be useful. We have, when, as in Chap. xii, Art. 43, $G=g^{c^t-1}$,

$$\log_e {}_tp_x=t\log_e s+c^x\log_e G$$

$$\log_e {}_tp_{xy}=2t\log_e s+(c^x+c^y)\log_e G.$$

Whence, transposing and dividing,

$$\frac{\log_e {}_tp_x-t\log_e s}{\log_e {}_tp_{xy}-2t\log_e s}=\frac{c^x}{c^x+c^y}=\frac{1}{1+c^{y-x}}$$

and
$$\log_e {}_tp_x = \frac{1}{1+c^{y-x}} \cdot \log_e {}_tp_{xy} - t \cdot \frac{1-c^{y-x}}{1+c^{y-x}} \cdot \log_e s.$$

Differentiating with respect to t, this becomes,

$$\frac{d_t p_x}{{}_tp_x} = \frac{1}{1+c^{y-x}} \cdot \frac{d_t p_{xy}}{{}_tp_{xy}} - \frac{1-c^{y-x}}{1+c^{y-x}} \cdot \log_e s \, . \, dt\,;$$

and, multiplying by ${}_tp_{xy}$,

$${}_tp_y \, . \, d_t p_x = \frac{1}{1+c^{y-x}} \cdot d_t p_{xy} - \frac{1-c^{y-x}}{1+c^{y-x}} \cdot \log_e s \, . \, {}_tp_{xy} \, . \, dt.$$

Changing signs, and integrating,

$$-\int {}_tp_y \, . \, d_t p_x = -\frac{1}{1+c^{y-x}} \int d_t p_{xy} + \frac{1-c^{y-x}}{1+c^{y-x}} \cdot \log_e s \int {}_tp_{xy} \, . \, dt\,;$$

and taking the integral between the limits 0 and ∞,

$$Q^1_{xy} = \frac{1}{1+c^{y-x}} + \frac{1-c^{y-x}}{1+c^{y-x}} \cdot \log_e s \, . \, \mathring{e}_{xy} \quad . \quad . \quad . \quad . \quad (37)$$

If x be less than y, then c^{y-x} is greater than unity, and the second term in the expression on the right of the equation is positive, log s being negative; while if x be greater than y the term is negative.

[39] If before integrating as above we multiply by v^t, we obtain the value of the contingent assurance as follows:—

$$-\int v^t \, . \, {}_tp_y \, . \, d_t p_x = -\frac{1}{1+c^{y-x}} \int v^t \, . \, d_t p_{xy} + \frac{1-c^{y-x}}{1+c^{y-x}} \cdot \log_e s \, . \int v^t \, . \, {}_tp_{xy} \, . \, dt$$

and between limits

$$\bar{A}^1_{xy} = \frac{1}{1+c^{y-x}} \bar{A}_{xy} + \frac{1-c^{y-x}}{1+c^{y-x}} \cdot \log_e s \, . \, \bar{a}_{xy} \quad . \quad . \quad . \quad (38)$$

[40] If Gompertz's formula be assumed instead of Makeham's, we have simply to write $s=1$ in the foregoing expressions, when consequently $\log_e s = 0$; and we have

$$Q^1_{xy} = \frac{1}{1+c^{y-x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\bar{A}^1_{xy} = \frac{\bar{A}_{xy}}{1+c^{y-x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

It must, however, be noted that the value of c in Gompertz's formula, differs from the value in Makeham's, so that the constant, c, of the mortality table in this volume cannot be used in applying formulas 39 and 40.

[41] By an analysis similar to the above, taking statuses of joint lives instead of single lives, a general expression may be obtained for an assurance on the m joint lives (x_1), (x_2) . . . (x_m), against the n joint lives (y_1), (y_2) . . . (y_n).

For conciseness, writing u for the status of the m lives (x_1), (x_2) . . . (x_m), and w for the status of the n lives (y_1), (y_2) . . . (y_n); and also writing $c^{x_1}+c^{x_2}+\ldots+c^{x_m}=\mathrm{K}_x$, and $c^{y_1}+c^{y_2}+\ldots+c^{y_n}=\mathrm{K}_y$, we have

$$\log_e {}_tp_u = mt\log_e s + \mathrm{K}_x \log_e \mathrm{G}$$

$$\log_e {}_tp_{uw} = (m+n)t\log_e s + (\mathrm{K}_x+\mathrm{K}_y)\log_e \mathrm{G}.$$

Whence
$$\frac{\log_e {}_tp_u - mt\log_e s}{\log_e {}_tp_{uw} - (m+n)t\log_e s} = \frac{\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}$$

and
$$\log_e {}_tp_u = \frac{\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot\log_e {}_tp_{uw} + \frac{m\mathrm{K}_y - n\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot t\cdot\log\ s.$$

Differentiating with respect to t, this becomes

$$\frac{d_t p_u}{{}_tp_u} = \frac{\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot\frac{d_t p_{uw}}{{}_tp_{uw}} + \frac{m\mathrm{K}_y - n\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot\log_e s.dt;$$

and, multiplying by ${}_tp_{uw}$,

$${}_tp_w\cdot d_t p_u = \frac{\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot d_t p_{uw} + \frac{m\mathrm{K}_y - n\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot\log_e s.{}_tp_{uw}.dt.$$

Changing signs, multiplying by v^t, and integrating,

$$-\int v^t.{}_tp_w.d_t p_u = -\frac{\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\int v^t.d_t p_{uw} - \frac{m\mathrm{K}_y - n\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot\log_e s.\int v^t.{}_tp_{uw}.dt$$

and taking the integral between the limits 0 and ∞,

$$\bar{\mathrm{A}}^1_{uw} = \frac{\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot\bar{\mathrm{A}}_{uw} - \frac{m\mathrm{K}_y - n\mathrm{K}_x}{\mathrm{K}_x+\mathrm{K}_y}\cdot\log_e s.\bar{a}_{uw} \quad . \quad . \quad (41)$$

[42] With a mortality table constructed by Makeham's formula, as is the case with the one at the end of this volume, formulas 38 and 41 will probably be found the most convenient for calculating contingent assurances, at any rate when more than two lives are involved. Only

one annuity-value is required, as the joint-life assurance can be found from the annuity by means of the general relation $\overline{A}=1-\delta\bar{a}$.

[43] In order that an idea may be formed of the effects of the various formulas, the following tabular statement is appended. The numerical values have been calculated at 4 per-cent interest, and the ages have been taken $x=30$, $y=45$, $z=60$.

Number of Formula.	Function.	Formula.	Value, 4 °/o, x=30, y=45, z=60.
1	A^1_{xy}	$\frac{1}{2}\left\{A_{xy}+\frac{a_{x-1:y}}{p_{x-1}}-\frac{a_{x:y-1}}{p_{y-1}}\right\}$	·1458
3	A^1_{xy}	$\frac{l_{x-1}}{l_x}\cdot\frac{l_{y-\frac{1}{2}}}{l_y}\cdot a_{x-1:y-\frac{1}{2}}-\frac{l_{y-\frac{1}{2}}}{l_y}\cdot a_{x:y-\frac{1}{2}}$	·1480
4	A^1_{xy}	$\frac{v^{\frac{1}{2}}}{l_x}\left\{l_{x-\frac{1}{2}}\cdot\bar{a}_{x-\frac{1}{2}:y}-l_{x+\frac{1}{2}}\cdot\bar{a}_{x+\frac{1}{2}:y}\right\}$	·1465
5	$\overline{A}^1_{xy}$	$\frac{1}{l_x}\left\{l_{x-\frac{1}{2}}\cdot\bar{a}_{x-\frac{1}{2}:y}-l_{x+\frac{1}{2}}\cdot\bar{a}_{x+\frac{1}{2}:y}\right\}$	·1494
29	$\overline{A}^1_{xy}$	$\mu_x\bar{a}_{xy}+\frac{1}{2}(\bar{a}_{x-1:y}-\bar{a}_{x+1:y})$	·1454
29*a*	$\overline{A}^1_{xy}$	$\frac{1}{2}\left\{\frac{\bar{a}_{x-1:y}}{p_{x-1}}-p_x\bar{a}_{x+1:y}\right\}$	·1451
30	$\overline{A}^1_{xy}$	$\mu_x(a_{xy}+\frac{1}{2})+\frac{1}{2}(a_{x-1:y}-a_{x+1:y})$	·1454
38	$\overline{A}^1_{xy}$	$\frac{1}{1+c^{y-x}}\cdot\overline{A}_{xy}+\frac{1-c^{y-x}}{1+c^{y-x}}\cdot\log_e s\,.\,\bar{a}_{xy}$	·1477
31	$\overline{A}^1_{xyz}$	$\mu_x\bar{a}_{xyz}+\frac{1}{2}(\bar{a}_{x-1:yz}-\bar{a}_{x+1:yz})$	·0764
41	$\overline{A}^1_{xyz}$	$\frac{c^x}{c^x+c^y+c^z}\cdot\overline{A}_{xyz}-\frac{c^y+c^z-2c^x}{c^x+c^y+c^z}\cdot\log_e s\,.\,\bar{a}_{xyz}$	·0767
32	$\overline{A}^1_{x:\overline{yz}}$	$\mu_x\bar{a}_{x:\overline{yz}}+\frac{1}{2}(\bar{a}_{x-1:\overline{yz}}-\bar{a}_{x+1:\overline{yz}})$	·1563
33	$\overline{A}^{\,1}_{\overline{xy}:z}$	$(\mu_x+\mu_y)\bar{a}_{xyz}+\frac{1}{2}(\bar{a}_{x-1:y-1:z}-\bar{a}_{x+1:y+1:z})$	·2288
34	$\overline{A}_{\overline{xy}:\underset{3}{z}}$	$\left\{\begin{array}{l}\mu_x(\bar{a}_{xz}-\bar{a}_{xyz})+\mu_y(\bar{a}_{yz}-\bar{a}_{xyz})\\ \quad+\frac{1}{2}(\bar{a}_{x-1:z}-\bar{a}_{x+1.z}+\bar{a}_{y-1:z}-\bar{a}_{y+1:z})\\ \quad-\frac{1}{2}(\bar{a}_{x-1:yz}-\bar{a}_{x+1:yz}+\bar{a}_{x:y-1:z}-\bar{a}_{x:y+1:z}\end{array}\right\}$	·0261
34*a*	$\overline{A}_{\overline{xy}:\underset{3}{z}}$	$\left\{\begin{array}{l}\mu_x(a_{xz}-a_{xyz})+\mu_y(a_{yz}-a_{xyz})\\ \quad+\frac{1}{2}(a_{x-1:z}-a_{x+1:z}+a_{y-1:z}-a_{y+1:z})\\ \quad-\frac{1}{2}(a_{x-1:y-1:z}-a_{x+1:y+1:z})\end{array}\right\}$	·0246

[44] To assist the student in recomputing the numerical examples for himself, the necessary annuity-values are here added. The values of the ordinary annuities were calculated by means of Chap. xii, Arts. 26 to 31, and from them the values of the corresponding continuous annuities were derived by formula 22 of Chap. ix.

Values of Annuities. 4 per-cent Interest.

Ages.	a	$\bar{a}$	Ages.	a	$\bar{a}$
TWO LIVES.					
30 & 44	12·565	13·060	31 & 45	12·334	12·829
29 „ 44½	12·522	13·017	29 „ 60	8·813	9·307
30 „ 44½	12·471	12·966	30 „ 60	8·798	9·292
29 „ 45	12·427	12·922	31 „ 60	8·781	9·275
29½ „ 45	12·406	12·901	44 „ 60	8·331	8·824
30 „ 45	12·384	12·879	45 „ 60	8·273	8·766
30½ „ 45	12·355	12·850	46 „ 60	8·210	8·703
THREE LIVES.					
29, 45, & 60	7·779	8·272	30, 44, & 60	7·816	8·309
30, 45, „ 60	7·764	8·257	30, 46, „ 60	7·713	8·206
31, 45, „ 60	7·753	8·246	31, 46, „ 60	7·699	8·192
29, 44, „ 60	7·831	8·324			

[45] It will be observed that, as in all foregoing formulas except Nos. 38 and 41, the contingent assurance is found from the difference between nearly equal annuities, tabulated only to three, or at most four, places of decimals, the results cannot be depended on to more than three places, and that there can be no confidence that even the third place is absolutely correct. Therefore, unless the annuities were tabulated to many more places than is ever the case, there can be no object in seeking rigid accuracy in the formulas, those formulas which are avowedly approximations, yielding results as trustworthy as the tables will admit of. For this reason formula 2 is not of practical use.

Seeing that in formulas 38 and 41 the annuity is multiplied by a small coefficient, these formulas give better results than any of the others, when, as is the case with the mortality table at the end of this volume, Makeham's formula has been used in the construction. Then even the fifth place of decimals is approximately correct.

[46] The reasoning by which formula 18 was deduced may be extended to continuous benefits; and for an assurance payable at the moment of the death of (x), if he die before (y) or within t years after the death of (y), we may write,

$$\overline{A}^{1}_{x:\overline{y(\overline{t|})}}=\overline{A}_x-\frac{D_{x+t}}{D_x}\left(\overline{A}_{x+t}-\overline{A}^{\;1}_{x+t:y}\right)\quad.\quad.\quad.\quad.\quad(42)$$

[47] In formula 42 we may substitute a status for either of the lives (x) or (y); so that, for example, we may have an assurance payable if (x) die before the survivor of (y) and (z), or within t years after the death of such survivor, the value of which benefit may be found by replacing $\overline{\mathrm{A}}_{\overset{1}{x+t}:y}$ by $\overline{\mathrm{A}}_{\overset{1}{x+t}:\overline{yz}}$.

[48] The formulas of approximate summation in Chap. xxiv, afford great facilities for calculating the values of complicated contingent assurances, and we now proceed to illustrate this branch of the subject. It will be sufficient here to take as example every case of simple survivorship involving three lives. Precisely the same methods are applicable for two lives, or for four or more lives, the only difference being the number of columns that appear in the calculations. In Chap. xv, compound survivorships will be dealt with. We shall take, as before, the ages $x=30$, $y=45$, and $z=60$; and we shall assume interest at 4 per-cent.

[49] In general, either Lubbock's formula, No. 17 of Chap. xxiv, may be employed, or one of Hardy's formulas,—preferably No. 39*a* of Chap. xxiv, as it is the shortest, and is at the same time surprisingly accurate. There is, however, one great advantage in using Hardy's formulas, namely that by them the annuity-value corresponding to the assurance is often obtainable with scarcely any additional trouble; whereas, by Lubbock's formula, the annuity-value must be found by an entirely separate operation. In practice the annual premiums are more frequently required than the single premiums, and therefore it is important to be able easily to get the annuity.

[50] To find $\overline{\mathrm{A}}^1_{xyz}$, the value of an assurance payable if (x) die before the failure of two joint lives, (y) and (z).

By Lubbock's formula we should have

$$\overline{\mathrm{A}}^1_{xyz}=\frac{(1+i)^{\frac{1}{2}}}{l_x l_y l_z}\cdot\Sigma(v^{t+1}.d_{x+t}.l_{y+\frac{1}{2}+t}.l_{z+\frac{1}{2}+t}) \quad . \quad . \quad (43)$$

In Table No. I the values of $l_{y+\frac{1}{2}+t}$, &c., or L_{y+t}, &c., are given, and their logarithms in Table No. II. Using these, and arranging the work in columns, the value sought can be found easily. As already remarked, however, the operation does not give the value of the annuity, $\bar{a}_{xyz}$. Also, in formula 43 it is assumed that ${}_{t-1|}q^1_{xyz}=\frac{1}{2}({}_{t-1}p_x-{}_tp_x)({}_{t-1}p_{yz}+{}_tp_{yz})$, and, while this is correct when only two lives are involved, and when a uniform distribution of deaths is assumed, it does not properly apply to three or more lives.

[51] By formula 28 of this chapter the work may be adapted to the continuous formulas. We have

$$\bar{A}^1_{xyz} = -\frac{1}{l_xl_yl_z}\cdot\int v^t.l_{y+t}.l_{z+t}.dl_{x+t}$$

$$= \frac{1}{l_xl_yl_z}\cdot\int v^t.l_{x+t}.l_{y+t}.l_{z+t}.\mu_{x+t}.dt \quad . \quad . \quad (44)$$

Here, by Chap. xii, Art. 57, the first portion of the work gives the annuity-value $\bar{a}_{xyz}$, from which $1+a_{xyz}$, the divisor for the annual premium, can at once be derived: afterwards, bringing in the column of μ_{x+t}, the contingent assurance is obtained. Taking formula 39*a* of Chap. xxiv, and making $n=6$, because the risk is determined at the first death, and (z), the oldest life, being 60, $7n=102-60=42$, we have, at 4 per-cent interest,

(1) t	(2) $\log v^t$	(3) $\log l_{x+t}$	(4) $\log l_{y+t}$	(5) $\log l_{z+t}$	(6) log coeff.	(7) (2)+&c.+(6)
0	0·00000	4·95272	4·89164	4·76969	$\bar{1}$·44716	14·06121
6	$\bar{1}$·89780	4·93143	4·85522	4·67372	0·20952	14·56769
18	$\bar{1}$·69340	4·87481	4·72714	4·26453	0·34242	13·90230
30	$\bar{1}$·48900	4·76969	4·40827	3·10483	0·20952	11·98131
36	$\bar{1}$·38680	4·67372	4·07806	1·85126	$\bar{1}$·74819	9·73803
	$\bar{2}$·46700	24·20237	22·96033	18·66403	$\bar{1}$·95681	64·25054

	(8) (7) + colog l_{xyz}	(9) $\log \mu_{x+t}$	(10) (8)+(9)	(11) Antilog (8) = Annuity Payments.	(12) Antilog (10) = Assurance Payments.
0	$\bar{1}$·44716	$\bar{3}$·88536	$\bar{3}$·33252	·28000	·002150
6	$\bar{1}$·95364	$\bar{3}$·94250	$\bar{3}$·89614	·89875	·007873
18	$\bar{1}$·28825	$\bar{2}$·14239	$\bar{3}$·43064	·19420	·002696
30	$\bar{3}$·36726	$\bar{2}$·46538	$\bar{5}$·83264	·00233	·000068
36	$\bar{5}$·12398	$\bar{2}$·66266	$\bar{7}$·78664	·00001	·000001
	$\bar{9}$·18029	$\bar{9}$·09829	$\overline{18}$·27858	1·37529	·012788
				6	6
				8·25174	·076728
				$=\bar{a}_{30:45:60}$	$=\bar{A}^1_{30:45:60}$

It will be seen from the table in Art. 43 that the value of the assurance thus found agrees exactly with that given by formula 41, which is correct in the fourth place of decimals.

[52] To find $\bar{A}^2_{\underset{1}{x}yz}$, the value of an assurance payable if (x) die after (y) and before (z). Arranging the work for Lubbock's formula, we should have

$$\bar{A}^2_{\underset{1}{x}yz} = \frac{(1+i)^{\frac{1}{2}}}{l_xl_yl_z}\Sigma v^{t+1}\, d_{x+t}(l_y - l_{y+\frac{1}{2}+t})l_{z+\frac{1}{2}+t} \quad . \quad . \quad . \quad (45)$$

As before, however, it will be more convenient to use the continuous method. The annual premium will be payable as long as (x) survives with (z), and we may therefore so arrange the work as to obtain $\bar{a}_{xz}$, from which to derive $1+a_{xz}$, the divisor. The most convenient formula will therefore be

$$\bar{A}^{2}_{\underset{1}{xyz}}=\frac{1}{l_x l_z}\int v^t.l_{x+t}.l_{z+t}.\mu_{x+t}(1-{}_tp_y)dt \quad . \quad . \quad . \quad (46)$$

instead of

$$\bar{A}^{2}_{\underset{1}{xyz}}=\frac{1}{l_x l_y l_z}\int v^t.l_{x+t}(l_y-l_{y+t})l_{z+t}.\mu_{x+t}.dt \quad . \quad . \quad (47)$$

Before commencing the other operations, it will be convenient to prepare the values of $\log(l_y-l_{y+t})-\log l_y=\log(1-{}_tp_y)$ for all the required values of t.

Taking as before formula 39*a* of Chap. xxiv, and, because the risk is determined on the death of z, making $n=6$, we have

(1) t	(2) $\log v^t$	(3) $\log l_{x+t}$	(4) $\log l_{z+t}$	(5) log coeff.	(6) (2)+(3)+(4)+(5)
0	0·00000	4·95272	4·76969	$\bar{1}$·44716	9·16957
6	$\bar{1}$·89780	4·93143	4·67372	0·20952	9·71247
18	$\bar{1}$·69340	4·87481	4·26453	0·34242	9·17516
30	$\bar{1}$·48900	4·76969	3·10483	0·20952	7·57304
36	$\bar{1}$·38680	4·67372	1·85126	$\bar{1}$·74819	5·65997
	$\bar{2}$·46700	24·20237	18·66403	$\bar{1}$·95681	41·29021

	(7) (6) + colog l_{xz}	(8) $\log(1-{}_tp_y)$	(9) $\log\mu_{x+t}$	(10) (7)+(8)+(9)	(11) Antilog (7) = Annuity Payments.	(12) Antilog (10) = Assurance Payments.
0	$\bar{1}$·44716	$-\infty$	$\bar{3}$·88536	$-\infty$	·28000	·00000000
6	$\bar{1}$·99006	$\bar{2}$·90542	$\bar{3}$·94250	$\bar{4}$·83798	·97737	·00068862
18	$\bar{1}$·45275	$\bar{1}$·49871	$\bar{2}$·14239	$\bar{3}$·09385	·28363	·00124120
30	$\bar{3}$·85063	$\bar{1}$·82699	$\bar{2}$·46538	$\bar{4}$·14300	·00709	·00013900
36	$\bar{5}$·93756	$\bar{1}$·92757	$\bar{2}$·66266	$\bar{6}$·52779	·00009	·00000337
	$\bar{8}$·67816	$\bar{2}$·15869 $-\infty$	$\bar{9}$·09829	$\overline{16}$·60262 $-\infty$	1·54818 6	·0020722 6
					9·28908 $=\bar{a}_{30:60}$	·0124332 $=\bar{A}^{2}_{\underset{1}{30:45:60}}$

[53] To find $\bar{A}^{2}_{xyz}$, the value of an assurance payable if (x) die second of the three lives. This problem might be solved by finding

independently $\bar{A}^2_{x\underset{1}{y}z}$ and $\bar{A}^2_{xy\underset{1}{z}}$, in the way described in Art. 52, and adding together the results. We should have $\bar{A}^2_{30:\underset{1}{45}:60}=\cdot0124332$ and $\bar{A}^2_{30:45:\underset{1}{60}}=\cdot0716368$: whence $\bar{A}^2_{30:45:60}=\cdot0840700$. A certain amount of labour may, however, be saved by performing the whole work in one operation by means of the formula

$$\begin{aligned}\bar{A}^2_{xyz}&=\int v^t\cdot{}_tp_x\cdot\mu_{x+t}\{{}_tp_z(1-{}_tp_y)+{}_tp_y(1-{}_tp_z)\}dt\\&=\int v^t\cdot{}_tp_x\cdot\mu_{x+t}({}_tp_y+{}_tp_z-2{}_tp_{yz})dt\ldots\quad\ldots\quad(48)\end{aligned}$$

It is not possible conveniently to arrange the work so as to form at the same time the corresponding annuity, $\bar{a}_{\overline{yz}:x}$, by which to calculate the annual premium. In using formula 39*a* of Chap. xxiv, we must in the present case take $n=8$, because the risk may remain undetermined until the death of (y), and therefore $7n=102-y$. The term for $t=0$ being zero, and not being required for the annuity, need not appear in the calculations.

(2) log $_tp_y$	(3) log $_tp_z$	(4) log $_tp_{yz}$ + log 2	(5) Antilog (4) $=2{}_tp_{yz}$	(6) Arith. Co. (5)	(7) $_tp_y$	(8) $_tp_z$	(9) (6)+(7)+(8)
$\bar{1}\cdot94856$	$\bar{1}\cdot86073$	0·11032	1·28920	$\bar{2}\cdot71080$	·88830	·72565	·32475
$\bar{1}\cdot71437$	$\bar{1}\cdot06562$	$\bar{1}\cdot08102$	0·12051	$\bar{1}\cdot87949$	·51805	·11631	·51385
$\bar{2}\cdot84738$	$\bar{5}\cdot83237$	$\bar{6}\cdot98078$	0·00001	$\bar{1}\cdot99999$	·07037	·00007	·07043
$\bar{3}\cdot67184$	$-\infty$	$-\infty$	0·00000	0·00000	·00470	·00000	·00470
$\bar{5}\cdot10836$	$-\infty$	$-\infty$	0·00000	0·00000	·00001	·00000	·00001
$\bar{9}\cdot29051$	$\bar{6}\cdot75872$	$\bar{6}\cdot17212$	1·40972	$\bar{2}\cdot59028$	1·48143	·84203	·91374
	$-\infty$	$-\infty$					

(10) log (9)	(11) log $_tp_x$	(12) log μ_{x+t}	(13) log v^t	(14) log coeff.	(15) (10)+...+(14)	(16) Antilog (15)
$\bar{1}\cdot51155$	$\bar{1}\cdot97088$	$\bar{3}\cdot96755$	$\bar{1}\cdot86373$	0·20952	$\bar{3}\cdot52323$	·0033360
$\bar{1}\cdot71084$	$\bar{1}\cdot87927$	$\bar{2}\cdot29003$	$\bar{1}\cdot59120$	0·34242	$\bar{3}\cdot81376$	·0065127
$\bar{2}\cdot84776$	$\bar{1}\cdot62680$	$\bar{2}\cdot80298$	$\bar{1}\cdot31867$	0·20952	$\bar{4}\cdot80573$	·0006393
$\bar{3}\cdot67210$	$\bar{1}\cdot31181$	$\bar{1}\cdot09771$	$\bar{1}\cdot18240$	$\bar{1}\cdot74819$	$\bar{5}\cdot01221$	·0000103
$\bar{5}\cdot00000$	$\bar{2}\cdot68105$	$\bar{1}\cdot40370$	$\bar{1}\cdot04613$	0·20952	$\bar{8}\cdot34040$	·0000000
$\overline{10}\cdot74225$	$\bar{3}\cdot46981$	$\bar{7}\cdot56197$	$\bar{3}\cdot00213$	0·71917	$\overline{21}\cdot49533$	·0104983
						8
						·0839864
						$=\bar{A}^2_{30:45:60}$

If we compare this result with ·0840700, the value obtained by calculating $\overline{A}^2_{x\underset{1}{y}z}$ and $\overline{A}^2_{xy\underset{1}{z}}$ separately, the difference is found to be only ·0000836. The great accuracy of the formula of approximation is thus again illustrated.

In the foregoing calculation of $\overline{A}^2_{xyz}$, the values of $\log {}_tp_x$ are formed by writing $\operatorname{colog} l_x$ at the foot of a card, which is moved down the column of $\log l_x$ as the work proceeds, and placed against the successive values of $\log l_{x+t}$. The values of $\log {}_tp_x$ thus formed by addition are at once entered on the working sheet. So, also, for the values of $\log {}_tp_y$ and $\log {}_tp_z$. The rest of the work explains itself.

[**54**] To find $\overline{A}^3_{xyz}$, the value of an assurance payable if (x) die last of the three lives. The most convenient formula is

$$\overline{A}^3_{xyz}=\frac{1}{l_xl_yl_z}\int v^t\cdot(l_y-l_{y+t})(l_z-l_{z+t})l_{x+t}\cdot\mu_{x+t}\cdot dt \quad . \quad (49)$$

As the annual premium will be payable during the whole of the life of (x), there is no need to consider the annuity in making the calculations. Using formula 39*a* of Chap. xxiv, we must in this case take $n=10$, because the risk continues as long as (x) survives, and $7n=102-x=72$. The term for $t=0$, being zero, need not be included in the calculations. After preparing separately the values of (l_y-l_{y+t}) and (l_z-l_{z+t}), we have

(1) t	(2) $\log v^t$	(3) $\log(l_y-l_{y+t})$	(4) $\log(l_z-l_{z+t})$	(5) $\log l_{x+t}$	(6) $\log \mu_{x+t}$	(7) log coeff.
10	$\bar{1}$·82967	4·05507	4·31942	4·91528	$\bar{3}$·99564	0·20952
30	$\bar{1}$·48900	4·71863	4·76019	4·76969	$\bar{2}$·46538	0·34242
50	$\bar{1}$·14833	4·89092	4·76969	4·14572	$\bar{1}$·17345	0·20952
60	$\bar{2}$·97800	4·89164	4·76969	3·10483	$\bar{1}$·55907	$\bar{1}$·74819
70	$\bar{2}$·80767	4·89164	4·76969	0·60206	$\bar{1}$·95117	0·20952
	$\bar{4}$·25267	23·44790	23·38868	17·53758	$\bar{5}$·14471	0·71917

	(8) (2)+&c.+(7)	(9) (8)+Colog l_{xyz}	(10) Antilog (9)
10	11·32460	$\bar{4}$·71055	·0005135
30	12·54531	$\bar{3}$·93126	·0085361
50	12·33763	$\bar{3}$·72358	·0052915
60	11·05142	$\bar{4}$·43737	·0002738
70	9·23175	$\bar{6}$·61770	·0000041
	56·49071	$\overline{17}$·42046	·0146190
			10
			·146190 $=\overline{A}^3_{30:45:60}$

[55] We may test the success of the approximations by adding together the values just found for $\bar{A}^1_{xyz}$, $\bar{A}^2_{xyz}$, and $\bar{A}^3_{xyz}$. The result, ·30690 should be $\bar{A}_x$. Finding $\bar{A}_x$ from the formula $\bar{A}_x=1-\delta\bar{a}_x$, it comes out ·30771, so that the error in the sum of the three approximations is only ·00081 in defect. This is quite accurate enough for practical purposes.

[56] To find $\bar{A}^{\ 1}_{\overline{xy}:z}$, the value of an assurance payable if the joint lives (x) and (y) fail before the death of (z). The formula for this benefit is

$$\bar{A}^{\ 1}_{\overline{xy}:z}=\frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t} . l_{z+t} . (\mu_{x+t}+\mu_{y+t}) . dt \quad . \quad . \quad (50)$$

The form is identical with the expression for $\bar{A}^1_{xyz}$, given in Art. 51, except that $(\mu_{x+t}+\mu_{y+t})$ is substituted for μ_{x+t}. It seems hardly necessary to give a type of the calculation.

[57] To find $\bar{A}^1_{x:\overline{yz}}$, the value of an assurance payable if (x) die before the survivor of (y) and (z); that is, payable if (x) die either first or second of the three lives. The formula for this benefit is

$$\bar{A}^1_{x:\overline{yz}}=-\int v^t . {}_tp_{\overline{yz}}\left(\frac{d}{dt}\,{}_tp_x\right)dt$$

$$=\int v^t({}_tp_{xy}+{}_tp_{xz}-{}_tp_{xyz})\mu_{x+t} . dt \quad . \quad . \quad (51)$$

The annual premium will be payable as long as (x) survives with either (y) or (z); and the annuity, $\int v^t({}_tp_{xy}+{}_tp_{xz}-{}_tp_{xyz})dt$ may easily be calculated at the same time as the assurance; and from it the divisor, $1+a_{x:\overline{yz}}$ may be derived. The values of $\log {}_tp_x$, $\log {}_tp_y$, and $\log {}_tp_z$ should first be calculated in the way described in Art. 53. We have, using formula 39a of Chap. xxiv, and taking $n=8$,

(1) t	(2) log $_tp_x$	(3) log $_tp_y$	(4) log $_tp_z$	(5) log $_tp_{xy}$	(6) log $_tp_{xz}$	(7) log $_tp_{xyz}$
0	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000
8	$\bar{1}$·97088	$\bar{1}$·94856	$\bar{1}$·86073	$\bar{1}$·91944	$\bar{1}$·83161	$\bar{1}$·78017
24	$\bar{1}$·87927	$\bar{1}$·71437	$\bar{1}$·06562	$\bar{1}$·59364	$\bar{2}$·94489	$\bar{2}$·65926
40	$\bar{1}$·62680	$\bar{2}$·84738	$\bar{5}$·83237	$\bar{2}$·47418	$\bar{5}$·45917	$\bar{6}$·30655
48	$\bar{1}$·31181	$\bar{3}$·67184	$-\infty$	$\bar{4}$·98365	$-\infty$	$-\infty$
	$\bar{2}$·78876	$\bar{4}$·18215	$\bar{6}$·75872	$\bar{6}$·97091	$\bar{6}$·23567	$\bar{8}$·74598
			$-\infty$		$-\infty$	$-\infty$

t	(8) $_tp_{xy}$	(9) $_tp_{xz}$	(10) $_tp_{xyz}$	(11) (8)+(9)−(10) $=_tp_{x:\overline{yz}}$	(12) log $_tp_{x:\overline{yx}}$	(13) log v^t	(14) log coe
0	1·00000	1·00000	1·00000	1·00000	0·00000	0·00000	$\bar{1}$·447
8	·83069	·67859	·60280	·90648	$\bar{1}$·95736	$\bar{1}$·86373	0·209
24	·39232	·08808	·04563	·43477	$\bar{1}$·63826	$\bar{1}$·59120	0·342
40	·02980	·00003	·00000	·02983	$\bar{2}$·47465	$\bar{1}$·31867	0·209
48	·00096	·00000	·00000	·00096	$\bar{4}$·98227	$\bar{1}$·18240	$\bar{1}$·748
	2·25377	1·76670	1·64843	2·37204	$\bar{5}$·05254	$\bar{3}$·95600	$\bar{1}$·956

t	(15) (12)+(13)+(14)	(16) log μ_{x+t}	(17) (15)+(16)	(18) Antilog (15)	(19) Antilog (17)
0	$\bar{1}$·44716	$\bar{3}$·88536	$\bar{3}$·33252	0·28000	·0021504
8	$\bar{0}$·03061	$\bar{3}$·96755	$\bar{3}$·99816	1·07303	·0099577
24	$\bar{1}$·57188	$\bar{2}$·29003	$\bar{3}$·86191	0·37315	·0072763
40	$\bar{2}$·00284	$\bar{2}$·80298	$\bar{4}$·80582	0·01007	·0006395
48	$\bar{5}$·91286	$\bar{1}$·09771	$\bar{5}$·01057	0·00008	·0000102
	$\bar{8}$·96535	$\bar{8}$·04363	$\overline{15}$·00898	1·73633	·0200341
				8	8
				13·89064 $=\bar{a}_{30:\overline{45:60}}$	·1602728 $=\bar{A}^1_{30:\overline{45:60}}$

Adding together $\bar{A}^1_{30:45:60}$ as found in Art. 51, and $\bar{A}^2_{30:45:60}$ as found in Art. 53, we have ·160714, which differs by only ·000441 from the value of $\bar{A}^1_{30:\overline{45:60}}$ as found in this article.

[**58**] To find $\bar{A}_{\overline{xy}:\underset{3}{z}}$, the value of an assurance payable on the death of the survivor of (x) and (y), if (z) be then alive. The value of this assurance might be found from the formula

$$\bar{A}_{\overline{xy}:\underset{3}{z}}=\frac{1}{l_xl_yl_z}\int v^t.l_{z+t}\{\mu_{x+t}.l_{x+t}(l_y-l_{y+t})+\mu_{y+t}.l_{y+t}(l_x-l_{x+t})\}dt \quad \ldots\ldots\ (52)$$

but by a simple transformation a much shorter formula may be found.

The assurance in question is evidently equal to an assurance payable at the death of (z) if he die third of the three lives, together with the value of the interest on 1 for such time as (z) may live after the death of the survivor of (x) and (y); because the assurances $\bar{A}_{\overline{xy}:\underset{3}{z}}$ and $\bar{A}_{xyz}^{\;\;3}$ depend on precisely the same probability, the difference in their values being due only to the fact that they are payable at different times. The value of a continuous annuity payable to (z) after the death of the survivor of (x) and (y) is $\bar{a}_z - \bar{a}_{z:\overline{xy}} = \bar{a}_z - \bar{a}_{xz} - \bar{a}_{yz} + \bar{a}_{xyz}$ and if we represent this annuity by the symbol $\bar{a}_{\overline{xy}|z}$, we have, seeing that δ is the annual interest, payable momently, on 1, $\bar{A}_{\overline{xy}:\underset{3}{z}} = \bar{A}_{xyz}^{\;\;3} + \delta\bar{a}_{\overline{xy}|z}$. In Art. 54 the formula is given for finding $\bar{A}_{xyz}^{\;\;3}$, and it can be used to give at the same time the annuity $\bar{a}_{\overline{xy}|z}$. Each payment of that annuity depends on (z) being alive to receive it, and on both (x) and (y) being dead. That is

$$\bar{a}_{\overline{xy}|z} = \frac{1}{l_x l_y l_z}\int v^t.(l_x - l_{x+t})(l_y - l_{y+t})l_{z+t}.dt \quad . \quad . \quad (53)$$

and in order to get $\bar{A}_{xyz}^{\;\;3}$ we have merely, by formula 49, to add to the expression for the annuity the factor μ_{z+t}. Taking the example for $x=30$, $y=45$, and $z=60$, and taking interest at 4 per-cent, we must in formula 39a of Chap. **xxiv**, make $n=6$ as the assurance will be determined on the death of z and therefore $7n=102-z=42$.

(1) t	(2) $\log v^t$	(3) $\log(l_x - l_{x+t})$	(4) $\log(l_y - l_{y+t})$	(5) $\log l_{z+t}$	(6) log coeff.	(7) (2)+ . . . +(6)
6	$\bar{1}$·89780	3·63246	3·79706	4·67372	0·20952	12·21056
18	$\bar{1}$·69340	4·16814	4·39035	4·26453	0·34242	12·85884
30	$\bar{1}$·48900	4·48916	4·71863	3·10483	0·20952	12·01114
36	$\bar{1}$·38680	4·62848	4·81921	1·85126	$\bar{1}$·74819	10·43394
	$\bar{2}$·46700	16·91824	17·72525	13·89434	0·50965	47·51448

	(8) (7)+Colog l_{xyz}	(9) $\log \mu_{z+t}$	(10) (8)+(9)	(11) Antilog (8)	(12) Antilog (10)
6	$\bar{3}$·59651	$\bar{2}$·66266	$\bar{4}$·25917	·003949	·0001816
18	$\bar{2}$·24479	$\bar{1}$·09771	$\bar{3}$·34250	·017571	·0022004
30	$\bar{3}$·39709	$\bar{1}$·55907	$\bar{4}$·95616	·002495	·0009040
36	$\bar{5}$·81989	$\bar{1}$·79387	$\bar{5}$·61376	·000066	·0000411
	$\overline{11}$·05828	$\bar{3}$·11331	$\overline{14}$·17159	·024081	·0033271
	$\delta\bar{a}_{\overline{xy}\|z} =$	·005667		6	6
	$\bar{A}_{xyz}^{\;\;3} =$	·019963		·144486	·0199626
	$\bar{A}_{\overline{xy}:\underset{3}{z}} =$	·025630		$=\bar{a}_{\overline{30:45}\|60}$	$=\bar{A}_{30:45:\overset{3}{60}}$

The annual premium will be payable as long as (z) survives with either (x) or (y), and therefore the annuity for the divisor is $1+a_{\overline{xy}:z}$, which is equal to $1+a_z-a_{\overline{xy}|z}$. We have seen that $\bar{a}_{\overline{xy}|z}=\bar{a}_z-\bar{a}_{xz}-\bar{a}_{yz}+\bar{a}_{xyz}$. Using formula 15 of Chap. ix, we have

$$
\begin{aligned}
&\bar{a}_z-\bar{a}_{xz}-\bar{a}_{yz}+\bar{a}_{xyz}\\
=&a_z+\tfrac{1}{2}-\tfrac{1}{12}(\mu_z+\delta)-a_{xz}-\tfrac{1}{2}+\tfrac{1}{12}(\mu_x+\mu_z+\delta)\\
&-a_{yz}-\tfrac{1}{2}+\tfrac{1}{12}(\mu_y+\mu_z+\delta)+a_{xyz}+\tfrac{1}{2}-\tfrac{1}{12}(\mu_x+\mu_y+\mu_z+\delta)\\
=&a_z-a_{xz}-a_{yz}+a_{xyz}.
\end{aligned}
$$

That is $a_{\overline{xy}|z}=\bar{a}_{\overline{xy}|z}$, and the divisor for the annual premium becomes $1+a_z-\bar{a}_{\overline{xy}|z}$ the last term of which is formed at the same time as the assurance.

[59] The remarks made in Chap. xii, Art. 63, apply also to the formulas in this chapter. If the values of $\log D_x$ or $\log D_{xy}$ are available, the arithmetical labour may be abridged. Thus, for example, formula 44 becomes

$$\bar{A}^1_{xyz}=\frac{1}{D_x.l_y.l_z}\int D_{x+t}.l_{y+t}.l_{z+t}.\mu_{x+t}.dt \quad . \quad . \quad . \quad (54)$$

$$=\frac{1}{D_{xy}.l_z}\int D_{x+t:y+t}.l_{z+t}.\mu_{x+t}.dt \quad . \quad . \quad . \quad . \quad (55)$$

CHAPTER XIV.

REVERSIONARY ANNUITIES.

1. An annuity to commence at the death of a person now aged y, and to continue thereafter so long as a person now aged x may live, is called a *Reversionary Annuity*, and is symbolized by $a_{y|x}$.

2. To find the value of $a_{y|x}$.

The payment of the annuity in the tth year depends on (x) surviving to the end of the year, (y) having died previously; and the value of the payment therefore is $v^t{}_tp_x(1-{}_tp_y)$. If in this expression we give to t every integral value from 1 upwards, and take the sum, we shall have the required value of the annuity. Thus

$$\begin{aligned} a_{y|x} &= \Sigma v^t{}_tp_x - \Sigma v^t{}_tp_{xy} \\ &= a_x - a_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

3. It was hardly necessary to go through the above algebraical process in order to find the value of the annuity, because the correctness of equation 1 is almost self-evident. The benefit required consists of all the payments of the annuity on (x) which will fall due after the death of (y); and therefore its value is found by deducting the value of the joint-life annuity from the value of the annuity on the life of (x) alone.

4. If the reversionary annuity is to be temporary or deferred, we have merely to insert temporary or deferred annuities, as the case may be, in formula 1, and write

$$|_n a_{y|x} = |_n a_x - |_n a_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$_{n|}a_{y|x} = {}_{n|}a_x - {}_{n|}a_{xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

5. It should, however, be noted that the similarity between a reversionary annuity and an immediate annuity does not go further, and that it is not correct to write ${}_{n|}a_{y|x}=v^n\,.\,{}_np_{xy}\,.\,a_{y+n|x+n}$, because, in order that payments under the deferred reversionary annuity may fall due, it is not necessary that both the lives should exist throughout the deferred term. Only (x) need survive the n years, and (y) may die at any time previously.

6. The reasoning by which equation 1 was established may be extended, for the annuity need not be payable yearly. If there be an annuity payable m times a year on the life of (x), and if all the payments which fall due during the joint lives of (x) and (y) be omitted, there will remain a reversionary annuity payable m times a year. That is

$$a^{(m)}_{y|x}=a^{(m)}_x-a^{(m)}_{xy}\quad\ldots\ldots\ldots\quad(4)$$

7. Using the values of $a^{(m)}_x$ and $a^{(m)}_{xy}$ given by formula 15 of Chap. ix, this becomes

$$a^{(m)}_{y|x}=a_x-a_{xy}+\frac{m^2-1}{12m^2}\mu_y\quad\ldots\ldots\quad(5)$$

8. The last term of equation 5 is small, and does not much affect the value, and it is usual in practice to omit it, and to write for all values of m

$$a^{(m)}_{y|x}=a_x-a_{xy}\quad\ldots\ldots\ldots\quad(6)$$

9. If the reversionary annuity is to be continuous, then we have

$$\begin{aligned}\bar{a}_{y|x}&=\bar{a}_x-\bar{a}_{xy}\\&=a_x-a_{xy}+\frac{\mu_y}{12}\quad\ldots\ldots\ldots\quad(7)\end{aligned}$$

but, as already mentioned, the term $\frac{\mu_y}{12}$ is usually omitted in practice.

10. So far the case has been considered of an annuity to a *life* (x) after another *life* (y); but the enquiry may be extended, and we may ask for the value of an annuity payable during the remainder of a *status* s, after the failure of a *status* z.

Evidently the value is

$$a_{z|s}=a_s-a_{sz}\quad\ldots\ldots\ldots\quad(8)$$

where s and z represent any combinations of lives, or of lives and terms-certain; because the annuity required is an annuity payable so long as the status s subsists, less an annuity payable so long as the status s subsists jointly with the status z.

11. By making various suppositions as to s and z, interesting results are arrived at. Thus, for example,

$$a_{\overline{n}|\,|x}=a_x-a_{x\overline{n}|}$$

which is the ordinary formula for a deferred annuity. In fact, $a_{\overline{n}|\,|x}$ represents an annuity to (x) after the expiry of a term of n years, the term-certain having taken the place of the life (y). Similarly

$$a_{x|\overline{n}|}=a_{\overline{n}|}-a_{x\overline{n}|}$$

This represents an annuity-certain to commence on the death of (x), but in no case to continue after n years from the present time.

12. Taking every case involving three lives, we have

$$\left.\begin{aligned}a_{z|xy}&=a_{xy}-a_{xyz}\\a_{yz|x}&=a_x\ -a_{xyz}\\a_{z|\overline{xy}}&=a_{\overline{xy}}-a_{\overline{xy}:z}\\a_{\overline{yz}|x}&=a_x\ -a_{x:\overline{yz}}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(9)$$

13. A few examples may be given here of more complicated statuses. The letters a, b, c, refer to the status s, and the letters x, y, z, to the status z.

An annuity payable during the joint lives of (a), (b), and (c), after the failure of the joint lives (x), (y), and (z), is

$$a_{xyz|abc}=a_{abc}-a_{abcxyz}$$

by equation 78 of Chap. vii.

An annuity payable during the joint lives of (a), (b), and (c), after the failure of the last survivor of the lives (x), (y), and (z), is

$$a_{\overline{xyz}|abc}=a_{abc}-a_{abc:\overline{xyz}},$$

and formula 79 of Chap. vii gives the value of the annuity $a_{abc:\overline{xyz}}$.

Similarly, $$a_{xyz|\overline{abc}}=a_{\overline{abc}}-a_{\overline{abc}:xyz}.$$

This is an annuity payable during the life of the survivor of (a), (b), and (c), but not to commence until the failure of the joint lives (x). (y), and (z).

Finally, $$\begin{aligned}a_{\overline{xyz}|\overline{abc}}&=a_{\overline{abc}}-a_{\overline{abc}:\overline{xyz}}\\&=a_{\overline{abcxyz}}-a_{\overline{xyz}},\end{aligned}$$

this last result being obtained by means of formula 80 of Chap. vii.

14. By means of the general formula, No. 8, the answers to many practical problems may be written down by inspection. The following are two examples.

15. Three persons, aged respectively x, y, and z, agree to purchase an annuity on the longest of their lives. It is to be shared equally between them while they are all living, and on the death of one it is to be equally divided between the two survivors, and the last survivor is to have the whole. What is the value of the share of each?

During the joint lives, (x) will have one-third of the annuity, and the value of this portion of his share is $\frac{1}{3}a_{xyz}$. He also will have half the annuity during his life jointly with that of (y) after the death of (z), or during his life jointly with that of (z) after the death of (y); and the value of this portion is $\frac{1}{2}(a_{z|xy}+a_{y|xz})$. Lastly, he will have the whole of the annuity after the death of the survivor of (y) and (z), the value of which is $a_{\overline{yz}|x}$. The whole value of the share of (x) is therefore $\frac{1}{3}a_{xyz}+\frac{1}{2}(a_{z|xy}+a_{y|xz})+a_{\overline{yz}|x}$, and this may be shown to be equal to $a_x-\frac{1}{2}(a_{xy}+a_{xz})+\frac{1}{3}a_{xyz}$. Similarly the value of the share of (y) is, $a_y-\frac{1}{2}(a_{xy}+a_{yz})+\frac{1}{3}a_{xyz}$; and the value of the share of (z) is, $a_z-\frac{1}{2}(a_{xz}+a_{yz})+\frac{1}{3}a_{xyz}$. The sum of the three shares is $a_x+a_y+a_z-a_{xy}-a_{xz}-a_{yz}+a_{xyz}$, or $a_{\overline{xyz}}$, as it should be.

16. Suppose, instead, that the annuity is to be shared in the following manner:—(x) and (y) are to share it equally during their joint lives; if (x) die first, then (y) and (z) are to share it equally during their joint lives, and the survivor of them is to have the whole; but if (y) die first, then (x) is to enjoy the whole during his life, and after his death it is to devolve wholly to (z). Required the values of their respective shares. It is not necessary to express the values of the shares in words, as it will be sufficient to write them down in symbols.

The value of the share of (x) is, $\frac{1}{2}a_{xy}+a_{y|x}=a_x-\frac{1}{2}a_{xy}$.

The value of the share of (y) is, $\frac{1}{2}a_{xy}+\frac{1}{2}a_{x|yz}+a_{\overline{xz}|y}=a_y-\frac{1}{2}a_{xy}-\frac{1}{2}a_{yz}+\frac{1}{2}a_{xyz}$.

The value of the share of (z) is, $\frac{1}{2}a_{x|yz}+a_{\overline{xy}|z}=a_z-a_{xz}-\frac{1}{2}a_{yz}+\frac{1}{2}a_{xyz}$. The sum of these is $a_{\overline{xyz}}$, as before.

17. Reversionary annuities may be looked at from a standpoint different from that hitherto occupied, and one which will enable us to deal effectually with a certain class of cases which occur in practice. We may consider the reversionary annuity to be an annuity to (x), deferred, not for a term-certain, but during the joint duration of the lives (x) and (y). If the failure of the joint lives take place in the

tth year by the death of (y), and if (x) live until the end of that year, he will be placed in possession of an annuity-due on his life at his then age, and the present value of this benefit is $v^t . {}_tp_x({}_{t-1}p_y - {}_tp_y)(1+a_{x+t})$ $= v^t \cdot \frac{l_{x+t} . d_{y+t-1}}{l_x l_y} \cdot (1+a_{x+t})$; and in order to find the present value of the reversionary annuity, this last expression must be summed for every integral value of t. That is,

$$a_{y|x} = \Sigma v^t \cdot \frac{l_{x+t} . d_{y+t-1}}{l_x l_y}(1+a_{x+t}) \quad . \quad . \quad . \quad . \quad (10)$$

$$= \Sigma \frac{d_{y+t-1} \mathrm{N}_{x+t-1}}{l_y \mathrm{D}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

18. It can easily be proved that expression 10 is identical with expression 1. Thus,

$$\Sigma v^t \cdot \frac{l_{x+t} . d_{y+t-1}}{l_x l_y}(1+a_{x+t})$$

$$= \frac{d_y}{l_y} \cdot vp_x(1+a_{x+1}) + \frac{d_{y+1}}{l_y} \cdot v^2 . {}_2p_x(1+a_{x+2}) + \&c.$$

$$= \frac{l_y - l_{y+1}}{l_y}(vp_x + v^2 . {}_2p_x + v^3 . {}_3p_x + \&c.)$$

$$+ \frac{l_{y+1} - l_{y+2}}{l_y}(v^2 . {}_2p_x + v^3 . {}_3p_x + v^4 . {}_4p_x + \&c.)$$

$$+ \frac{l_{y+2} - l_{y+3}}{l_y}(v^3 . {}_3p_x + v^4 . {}_4p_x + v^5 . {}_5p_x + \&c.)$$

$$+ \quad \&c. \qquad \&c. \qquad \&c. \qquad \&c.$$

$$= (vp_x + v^2 . {}_2p_x + v^3 . {}_3p_x + \&c.)$$

$$- \left(\frac{l_{y+1}}{l_y} \cdot vp_x + \frac{l_{y+2}}{l_y} \cdot v^2 . {}_2p_x + \frac{l_{y+3}}{l_y} \cdot v^3 . {}_3p_x + \&c.\right)$$

$$= a_x - a_{xy} .$$

19. The form of expression in equation 10, but not that in equation 10a, becomes useful when the lives are not to be subject to the same rate of mortality during the whole period of the annuity. For instance, a reversionary annuity may be granted while a married couple are in India, to be enjoyed by the wife after the death of her husband; and it may be known that so long as the husband lives, he and his wife will remain in India; but that on the death of the husband, his widow will return

to reside in Europe. In such case the factor $\frac{l_{x+t}\cdot d_{y+t-1}}{l_x l_y}$ in equation 10 must be calculated by a mortality table representing the value of life in India; whereas the factor $(1+a_{x+t})$ must have reference to the value of female life in Europe. If both factors be calculated by one mortality table, we shall have the ordinary value for the reversionary annuity, namely, $a_x - a_{xy}$; but when two mortality tables are combined, that value is altered.

20. The numerical computation of the value of such a benefit would be tedious if absolute exactitude were required, but in Arts. 36 to 44 of this chapter it will be shown how the formulas of approximate summation given in Chap. xxiv may be used for the purpose.

21. Equation 4 gives the value of a reversionary annuity in general form, but we must be careful to fix clearly in our minds the nature of the annuity represented. It is assumed that the annuity, $a_x^{(m)}$, is immediately put in operation, and that the reversionary annuity consists of all the payments of such annuity which will fall successively due after the death of (y). Therefore the first payment of the reversionary annuity will take place at the end of that mth part of the year in which (y) dies. Also, the annuity is curtate, and the last payment will take place at the beginning of that mth part of the year in which (x) dies, and there will not be a proportionate part payable up to the moment of the death of (x).

22. Corrections have been proposed, to be applied to the value of the annuity as given in formula 4, in order to make it suit varying circumstances.

23. Holmes Ivory, *J.I.A.*, iv, 299, gives the expression

$$a_{y|x}^{(m)} = a_x - a_{xy} - \frac{m-1}{2m} A_{\overset{1}{x}y} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This formula may be established and explained by the following reasoning. If (x) be alive at the end of the year in which (y) dies, the value of the annuity will then be that of an annuity-due upon a single life at the age to which (x) shall have then attained. Now the deduction from an ordinary yearly annuity-due to change it into an annuity-due payable m times a year, is $\frac{m-1}{2m}$; and it follows that the value of this correction, at the commencement of the transaction in the case of the reversionary annuity, is $\frac{m-1}{2m} A_{x\overset{1}{y}}$.

24. Although Holmes Ivory's formula, No. 11 above, gives the value of a reversionary annuity payable m times a year, yet the first payment is made at the *end of the year* of the death of (y), no matter at what intervals payments may afterwards be made. The annuity is also curtate. If $m=2$, that is, if the annuity be payable half-yearly, there will practically be a curtate annuity to be *entered on* at the moment of the death of (y), (the first payment being made six months after his death), and then to continue for the remainder of the lifetime of (x). That is,

$$a^{(2)}_{y|x}=a_x-a_{xy}-\tfrac{1}{4}\mathrm{A}_{x\overset{1}{y}}.$$

If now we wish to carry the formula further, and make the annuity complete, we must add $\frac{1}{4}$ to be received at the death of (x) if he die second, that is, $\frac{1}{4}\mathrm{A}_{\overset{2}{x}y}$, or $\frac{1}{4}(\mathrm{A}_x-\mathrm{A}_{\overset{1}{x}y})$. In this way we have

$$\begin{aligned}\mathring{a}^{(2)}_{y|x}&=a_x-a_{xy}-\tfrac{1}{4}\mathrm{A}_{x\overset{1}{y}}+\tfrac{1}{4}(\mathrm{A}_x-\mathrm{A}_{\overset{1}{x}y})\\&=a_x-a_{xy}+\tfrac{1}{4}\mathrm{A}_x-\tfrac{1}{4}(\mathrm{A}_{\overset{1}{x}y}+\mathrm{A}_{x\overset{1}{y}})\\&=a_x-a_{xy}-\tfrac{1}{4}(\mathrm{A}_{xy}-\mathrm{A}_x)\quad.\;.\;.\;.\;.\;.\;.\quad(12)\end{aligned}$$

This is a description of half-yearly annuity not uncommon in practice. In equation 15 below, a general formula is given for such annuities.

25. In order by Holmes Ivory's method to find a general formula for a complete reversionary annuity, we must in equation 11 add $\frac{1}{2m}\mathrm{A}_{\overset{2}{x}y}$ to the expression. The result will be, after reduction,

$$\mathring{a}^{(m)}_{y|x}=a_x-a_{xy}-\frac{1}{2m}(\mathrm{A}_{xy}-\mathrm{A}_x)-\frac{m-2}{2m}\mathrm{A}_{x\overset{1}{y}}\quad.\;.\;.\quad(13)$$

26. In formula 11 a probability is included which does not enter into the reversionary annuity, namely, the chance of both lives failing in the same year. In estimating the present value of the portion $\frac{m-1}{2m}$, all that requires to be taken into account is the chance of (x) being alive *at the end of the year* of the death of (y). For the value of the correction we therefore have, as a closer approximation,

$$\frac{m-1}{2m}\Sigma v^t({}_{t-1}p_y-{}_tp_y)\,{}_tp_x,$$

which, after reduction, becomes

$$\frac{m-1}{2m}\left(\frac{a_{x:y-1}}{p_{y-1}}-a_{xy}\right),$$

and we have for the reversionary annuity

$$a^{(m)}_{y|x}=a_x-a_{xy}-\frac{m-1}{2m}\left(\frac{a_{x:y-1}}{p_{y-1}}-a_{xy}\right) \quad . \quad . \quad . \quad (14)$$

Not only is formula 14 more correct than formula 11, but it lends itself more easily to calculation. The reader must not forget that both formulas 11 and 14 give the value of a *curtate* annuity payable m times a year, to (x) after (y), first payment at the *end of the year* of the death of (y).

27. In the case of all the annuities so far considered, the times of year at which payments are to take place when the annuity falls into possession, are determined at the outset, and are not left to depend on the time of year at which (y) dies. This, however, is not usually the arrangement in practice. Generally the annuity commences to run from the moment of the death of (y), and the first payment takes place at some determinate interval after that death. It is desirable to distinguish these two descriptions of annuity by difference of notation, and we shall therefore place the mark ^ over the letter a, writing $\hat{a}$, when the moment of the death of (y) determines the times of year at which the payments of the annuity are to be made.

28. Going back to equation 13, we can find from it an approximation to the value of a complete annuity to (x), payable m times a year, and to be entered on at the moment of the death of (y), so that the first payment shall take place $\frac{1}{m}$ of a year after such death. In the case of this annuity, $\frac{1}{2}$ will on the average be payable for the year of the death of (y) if (x) survive, because in that year the annuity will on the average run for half-a-year. But in the case of the annuity of formula 13, only $\frac{1}{m}$ is payable for the year of the death of (y), the first payment being made at the end of that year. Therefore to find the value of the annuity which we now seek we must, to the annuity of formula 13, add $\frac{1}{2}-\frac{1}{m}$, or $\frac{m-2}{2m}$, payable if (y) die before (x); the present value of

which is $\frac{m-2}{2m}\text{A}_{xy}^{\ 1}$. Hence the value of the annuity now required—that is, the value of a complete annuity to (x), payable m times a year, and to be entered on at the moment of the death of (y)—is

$$\mathring{a}_{y|x}^{(m)}=a_x-a_{xy}-\frac{1}{2m}(\text{A}_{xy}-\text{A}_x) \quad . \quad . \quad . \quad . \quad (15)$$

29. This last result may be reduced in a convenient manner. Since $\text{A}_{xy}=\frac{1-ia_{xy}}{1+i}$, and $\text{A}_x=\frac{1-ia_x}{1+i}$, therefore

$$\frac{1}{2m}(\text{A}_{xy}-\text{A}_x)=\frac{1}{2m}\cdot\frac{i}{1+i}\cdot(a_x-a_{xy})=\frac{d}{2m}(a_x-a_{xy}).$$

Therefore $$\mathring{a}_{y|x}^{(m)}=\left(1-\frac{d}{2m}\right)(a_x-a_{xy})\,. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

30. By another line of reasoning suggested by T. B. Sprague (*J.I.A.*, xv, 126), an expression for the annuity may be found, similar to that of equation 16. The formula, $a_x^{(m)}-a_{xy}^{(m)}$, assumes that the first payment of the annuity will be made at the end of that mth part of a year in which (y) dies, or, on the average, $\frac{1}{2m}$ of a year after his death. It also assumes that the last payment of the annuity will be made at the beginning of that mth part of a year in which (x) dies, or, on the average, $\frac{1}{2m}$ of a year before his death. If, however, the annuity is to be entered on at the moment of the death of (y), the first payment will be made $\frac{1}{m}$ of a year after his death, or $\frac{1}{2m}$ of a year later than supposed by the formula $a_x^{(m)}-a_{xy}^{(m)}$; and similarly all the other payments will be postponed by $\frac{1}{2m}$ of a year. The last payment being so postponed, the chance of receiving it will be reduced one-half; and as the amount of the payment is $\frac{1}{m}$, the deduction to be made on this account from the present value of the reversionary annuity is $\frac{1}{2m}\text{A}_{xy}^{\ 2}$. But if the reversionary annuity is to be complete, we must also add $\frac{1}{2m}\text{A}_{xy}^{\ 2}$, which is the value of the proportionate part to the day of the death of (x). It thus appears that

the deduction and the addition are equal, and cancel each other, because we deduct $\frac{1}{2m}A^2_{xy}$ on account of the last payment of the annuity being postponed, and add $\frac{1}{2m}A^2_{xy}$ on account of the reversionary annuity being made complete. Therefore it follows that the complete annuity to (x) payable m times a year, and to be entered on at the moment of the death of (y), may be derived from the formula $a^{(m)}_x - a^{(m)}_{xy}$, simply by postponing all the payments by $\frac{1}{2m}$ of a year. That is

$$\mathring{a}^{(m)}_{y|x} = v^{\frac{1}{2m}}(a^{(m)}_x - a^{(m)}_{xy}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

31. Seeing that $v^{\frac{1}{2m}} = \left(1 - \frac{d}{2m}\right)$ very nearly, and that $(a^{(m)}_x - a^{(m)}_{xy}) = (a_x - a_{xy})$ very nearly, equation **16** is practically the same as equation **17**.

32. From equation **17** another approximation to the value of the same annuity may be obtained. If the annuity be payable yearly, we have, $\mathring{a}_{y|x} = v^{\frac{1}{2}}(a_x - a_{xy})$. Now if the annuity be made payable m times a year, the same total amount will be paid as before, because the annuity is complete; but there will be a gain in interest by the payment in instalments of $\frac{1}{m}$ each throughout the year, instead of in a single sum of 1 at the end of the year. The annuity of 1 per annum payable m times in the year is therefore equal to an annuity of

$$\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}} + (1+i)^{\frac{m-2}{m}} + \&c. + (1+i)^{\frac{m-m}{m}}\right\}$$

payable once a year, and in Chap. ix, Art. **12**, this was shown to be equal to an annuity of $\frac{i}{j_{(m)}}$ payable yearly. Therefore

$$\mathring{a}^{(m)}_{y|x} = \frac{i\,v^{\frac{1}{2}}}{j_{(m)}}(a_x - a_{xy}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

[**33**] The values of reversionary annuities may be found by the application of the differential calculus.

Let θ denote the exact age which will be attained by (x) at the instant of the death of (y) if he be then alive. At that instant the value of the complete annuity on his life will be, according to formula **13** of Chap. xi, $\mathring{a}^{(m)}_{\theta}=\bar{a}_{\theta}\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}$. Now this value is in contingent reversion, and is to be entered on only in the event of (x) surviving (y); and the present value thereof is required so as to include a due consideration of all possible values of the variable age, θ, of survivorship. It is obvious that the present value of the continuous reversionary annuity, $\bar{a}_{\theta}$, taking into account all the values of θ, is $\bar{a}_x-\bar{a}_{xy}$. Also the present value of the other term, $\frac{\delta}{12m^2}$, is $\frac{\delta}{12m^2}\cdot\bar{A}^{\,1}_{xy}$. Hence the present value of the complete reversionary annuity payable m times a year, and to be entered on at the moment of the death of (y), is

$$\hat{\mathring{a}}^{(m)}_{y|x}=\left(1-\frac{\delta}{2m}\right)(\bar{a}_x-\bar{a}_{xy})+\frac{\delta}{12m^2}\cdot\bar{A}^{\,1}_{xy} \quad . \quad . \quad . \quad (19)$$

and this may be written

$$\hat{\mathring{a}}^{(m)}_{y|x}=\left(1-\frac{\delta}{2m}\right)\left(a_x-a_{xy}+\frac{\mu_y}{12}\right)+\frac{\delta}{12m^2}\bar{A}^{\,1}_{xy} \quad . \quad . \quad . \quad (20)$$

By equations 13 and 14 of Chap. xi, the last results may be written

$$\hat{\mathring{a}}^{(m)}_{y|x}=\mathring{a}^{(m)}_{x}-\mathring{a}^{(m)}_{xy}+\frac{\delta}{12m^2}\bar{A}^{\,1}_{xy} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The last term in equations 19, 20, and 21 is very small. When $m=2$, and interest is taken at 6 per-cent, it can never reach ·0012; and for larger values of m and lower rates of interest it is much smaller. Without sensible loss of accuracy $A^{\,1}_{xy}$ may be substituted for $\bar{A}^{\,1}_{xy}$.

34. When assurance companies grant reversionary annuities, they generally do so in consideration of periodical and not single premiums. The periodical premium is payable during the joint lives, and therefore the divisor, if the premium is to be payable yearly, is $1+a_{xy}$. Hence we have

$$\text{P}a_{y|x}=\frac{a_{y|x}}{1+a_{xy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

35. In the following tabular statement the various formulas are brought together for the purpose of comparison, and a numerical example of a quarterly annuity, calculated at 4 per-cent interest, is added, the age of (x) being taken as 30, and that of (y) as 45. It will be seen that the different corrections discussed in the preceding articles produce only trifling effects, and in practice it is usual to adopt

formula 6, notwithstanding that generally the annuities are like those of formulas 15 to 21, being complete, and running from the day of the death of (y). Formula 6 gives a result slightly in excess, and therefore errs on the safe side; but unless (y) be very much older than (x) the pence in the annual premium will hardly be affected. Formulas 11, 13, and 14 do not give annuities such as are often met with.

Number of Formula.	Formula.	Numerical Example, Interest 4 %, x=30, y=45, m=4.
4 & 5	$a_{y\|x}^{(m)} = a_x - a_{xy} + \frac{m^2-1}{12m^2}\mu_y$	4·772
6	$a_{y\|x}^{(m)} = a_x - a_{xy}$	4·771
7	$\bar{a}_{y\|x} = \bar{a}_x - \bar{a}_{xy}$	4·772
11	$a_{y\|x}^{(m)} = a_x - a_{xy} - \frac{m-1}{2m}\mathrm{A}_{xy}^{\;1}$	4·644
13	$\mathring{a}_{y\|x}^{(m)} = a_x - a_{xy} - \frac{1}{2m}(\mathrm{A}_{xy} - \mathrm{A}_x) - \frac{m-2}{2m}\mathrm{A}_{xy}^{\;1}$	4·664
14	$a_{y\|x}^{(m)} = a_x - a_{xy} - \frac{m-1}{2m}\left(\frac{a_{x:y-1}}{p_{y-1}} - a_{xy}\right)$	4·647
15	$\hat{\mathring{a}}_{y\|x}^{(m)} = a_x - a_{xy} - \frac{1}{2m}(\mathrm{A}_{xy} - \mathrm{A}_x)$	4·748
16	$\hat{\mathring{a}}_{y\|x}^{(m)} = \left(1 - \frac{d}{2m}\right)(a_x - a_{xy})$	4·748
17	$\hat{\mathring{a}}_{y\|x}^{(m)} = v^{\frac{1}{2m}}(a_x^{(m)} - a_{xy}^{(m)})$	4·749
18	$\hat{\mathring{a}}_{y\|x}^{(m)} = \frac{iv^{\frac{1}{2}}}{\dot{j}_{(m)}}(a_x - a_{xy})$	4·748
19, 20, & 21	$\hat{\mathring{a}}_{y\|x}^{(m)} = \left(1 - \frac{\delta}{2m}\right)\left(a_x - a_{xy} + \frac{\mu_y}{12}\right) + \frac{\delta}{12m^2}\bar{\mathrm{A}}_{xy}^{\;1}$	4·749

[36] Formula 10 is convenient for the application of the formulas of approximate summation of Chap. xxiv. In the form already given it is more specially suited to Lubbock's formula; but in the case of reversionary annuities we see by comparing formulas 6 and 7 that numerically there is but little difference in the results, whether continuous or yearly annuities be employed. This being so, it will be better to use the continuous method, because the arithmetical labour will be less.

$v^t\,{}_tp_y\,\mu_{x+t}\,a_{x+t}$

[37] Applying the differential and integral calculus, formula 10 becomes

$$\bar{a}_{y|x}=\frac{1}{l_x l_y}\int v^t.l_{x+t}.l_{y+t}.\mu_{y+t}.\bar{a}_{x+t}.dt \quad . \quad . \quad . \quad (23)$$

and in such a case as that mentioned in Art. 19, the values of l_x, l_y, l_{x+t}, l_{y+t}, and μ_{y+t}, would be taken from a mortality table applicable to lives resident in India, while $\bar{a}_{x+t}$ would be taken from a table of annuities on female lives in Europe. It will be sufficiently accurate to substitute $\frac{1}{2}+a_{x+t}$ for $\bar{a}_{x+t}$.

[38] As an example, let it be required to find at 4 per-cent interest the value of a reversionary annuity to a person aged 30, to be entered on at the death of a person aged 45. We shall use only one mortality table,—that at the end of the volume,—so that the result may be compared with the values in the table in Art. 35. Using formula 39*a* of Chap. xxiv, we shall have $7n=102-45$, and $n=8$, nearly.

(1) t	(2) log v^t	(3) log l_{x+t}	(4) log l_{y+t}	(5) log μ_{y+t}	(6) log $(\frac{1}{2}+a_{x+t})$	(7) log coeff.
0	0·00000	4·95272	4·89164	$\bar{2}$·08063	1·24687	$\bar{1}$·44716
8	$\bar{1}$·86373	4·92360	4·84020	$\bar{2}$·26316	1·20650	0·20952
24	$\bar{1}$·59120	4·83199	4·60601	$\bar{2}$·76738	1·07302	0·34242
40	$\bar{1}$·31867	4·57952	3·73902	$\bar{1}$·36506	0·83206	0·20952
48	$\bar{1}$·18240	4·26453	2·56348	$\bar{1}$·67626	0·65868	$\bar{1}$·74819
56	$\bar{1}$·04613	3·63377	0·00000	$\bar{1}$·99057	0·44871	0·20952
	$\bar{3}$·00213	27·18613	20·64035	$\bar{6}$·14306	5·46584	0·16633

	(8) (2)+ . . . +(7)	(9) (8)+Colog l_{xy}	(10) Antilog (9)
0	8·61902	$\bar{2}$·77466	·05952
8	9·30671	$\bar{1}$·46235	·28997
24	9·21202	$\bar{1}$·36766	·23316
40	8·04385	$\bar{2}$·19949	·01583
48	6·09354	$\bar{4}$·24918	·00018
56	3·32870	$\bar{7}$·48434	·00000
	44·60384	$\overline{15}$·53768	·59866
			8
			4·78928 $=\bar{a}_{45\|30}$

The result is only ·017 in excess of that given by formula 7.

[39] Formula 23 is useful when two rates of mortality, or two rates of interest, have to be brought into account, but in ordinary cases the reversionary annuity may be found with less arithmetical labour. We have

$$\bar{a}_{y|x} = \frac{1}{l_x l_y}\int v^t . l_{x+t}(l_y - l_{y+t})\,dt \quad . \quad . \quad . \quad . \quad (24)$$

and after the values of log $(l_y - l_{y+t})$ have been formed by a preliminary process, the work will be contained in seven columns instead of nine.

[40] The formulas of approximate summation may very easily be applied to find the values of reversionary annuities in cases where three or more lives are involved, and in general either of two forms may be employed, corresponding to equations 23 and 24 respectively. The work may also be so arranged as to find at the same time the divisor for the annual premium.

[41] Taking every case involving three lives (except those including compound survivorship probabilities, which will be discussed in Chap. xv) we have

$$\bar{a}_{z|xy} = \frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t}(l_z - l_{z+t})\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$= \frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t} . l_{z+t} . \mu_{z+t} . \bar{a}_{x+t:y+t} . dt \quad . \quad . \quad . \quad . \quad (25a)$$

$$\bar{a}_{yz|x} = \frac{1}{l_x l_y l_z}\int v^t . l_{x+t}(l_y l_z - l_{y+t} l_{z+t})\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$= \frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t} . l_{z+t} . (\mu_{y+t} + \mu_{z+t})\,\bar{a}_{x+t} . dt \quad . \quad . \quad . \quad (26a)$$

$$\bar{a}_{z|\overline{xy}} = \int v^t ({}_tp_x + {}_tp_y - {}_tp_{xy})(1 - {}_tp_z)\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\bar{a}_{\overline{yz}|x} = \frac{1}{l_x l_y l_z}\int v^t . l_{x+t}(l_y - l_{y+t})(l_z - l_{z+t})\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$= \frac{1}{l_x l_y l_z}\int v^t . l_{x+t}\{l_{y+t} . \mu_{y+t}(l_z - l_{z+t}) + l_{z+t} . \mu_{z+t}(l_y - l_{y+t})\}\,\bar{a}_{x+t} . dt \quad . \quad . \quad . \quad . \quad . \quad (28a)$$

[42] The value of $\bar{a}_{z|\overline{xy}}$ does not admit of being given in double form, as in order that the annuity may come into possession it is not necessary that both (x) and (y) should survive (z); and therefore it would not be correct to include $\bar{a}_{\overline{x+t:y+t}}$ in the formula.

[43] Formulas 25 and 28 will usually be found more convenient than formulas 25*a* and 28*a*, but the alternative forms are given to meet cases where different rates of mortality have to be brought into account. Formula 26*a*, on the other hand, is generally more easily applied than formula 26.

[44] In Art. 58 of Chap. xiii, a numerical example was given of formula 28 of this chapter, and in Chap. xv examples will be given of compound survivorship annuities. It is therefore unnecessary to multiply examples here.

CHAPTER XV.

COMPOUND SURVIVORSHIP ANNUITIES AND ASSURANCES.

1. The difference between simple and compound survivorships may be illustrated by means of a reversionary annuity to (x) after the failure of the survivor of (y) and (z). If no stipulation be made as to the order in which (y) and (z) shall die, the benefit is the ordinary reversionary annuity, $a_{\overline{yz}|x}$, of which the value was found in Chap. xiv, Art. 12. There the conditions include only one order of survivorship, namely, that (x) shall die third. But now, let the annuity be payable to (x) after the death of the survivor of (y) and (z), only if (y) be that survivor. Here there is a compound condition of survivorship. The failure of (y) before (x) is to follow the failure of (z) before (y); and unless the double event happen, the annuity will never come into possession.

2. Instead of an annuity to (x) after the death of the *survivor* of (y) and (z), there may be an annuity to (x) after the failure of the *joint lives* (y) and (z). If there be no stipulation as to which of the lives shall fail first, the annuity is the ordinary reversionary annuity, $a_{yz|x}$, of which the value was found in Chap. xiv, Art. 12. But the annuity becomes a compound survivorship if the condition be added, that the failure of the joint lives shall take place by the death of (z). The annuity then depends on a double event, namely, that (x) shall survive (z), and that (z) shall die before (y).

3. It will be convenient to use for compound survivorships a notation similar to that employed for contingent assurances and reversionary annuities, adding numerals to indicate the order in which the lives are to

fail. The numerals will be placed *above* the letter representing the life on the failure of which the benefit is to come immediately into possession, and *below* the letters representing the lives which are to fail previously in assigned orders. Thus, an annuity to (x) after the death of the survivor of (y) and (z), if (y) be that survivor, is $a_{\overset{2}{y}z|x}$, or $a_{\underset{1}{\overline{yz}}|x}$; and an annuity to (x), after the failure of the joint lives (y) and (z) caused by the death of (z), is $a_{y\overset{1}{z}|x}$.

4. If the value of $a_{y\overset{1}{z}|x}$ be known, the value of $a_{\overset{1}{y}z|x}$ can be found at once, because $a_{y\overset{1}{z}|x}+a_{\overset{1}{y}z|x}=a_{yz|x}$. Whence

$$a_{\overset{1}{y}z|x}=a_{yz|x}-a_{y\overset{1}{z}|x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the value of $a_{yz|x}$ was given in Chap. xiv, Art. 12.

5. An intimate relation subsists between the two compound survivorship annuities, $a_{y\overset{1}{z}|x}$ and $a_{\overset{2}{y}z|x}$. The annuity $a_{y\overset{1}{z}|x}$ comes into possession immediately on the death of (z), provided both (x) and (y) be then alive; but the annuity $a_{\overset{2}{y}z|x}$ is further deferred until the death of (y). The annuity $a_{y\overset{1}{z}|x}$ therefore includes all the payments of the annuity $a_{\overset{2}{y}z|x}$, and it includes, besides, an annuity during the joint lives of (x) and (y) after the death of (z). That is

$$a_{\overset{2}{y}z|x}=a_{y\overset{1}{z}|x}-a_{z|xy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of $a_{z|xy}$ was given in Chap. xiv, Art. 12.

6. To determine $a_{y\overset{1}{z}|x}$. The payment of the annuity at the end of the tth year will depend on (x) surviving to the end of that year, and on (z) having died previously leaving (y) surviving him. The present value of the payment therefore is, $v^t . {}_tp_x . {}_{|t}q_{y\overset{1}{z}}$, which is equal to

$$v^t . {}_tp_x\left(q_{y\overset{1}{z}}+{}_{1|}q_{y\overset{1}{z}}+{}_{2|}q_{y\overset{1}{z}}+\&\text{c.}+{}_{t-1|}q_{y\overset{1}{z}}\right).$$

Therefore each payment of $a_{y\overset{1}{z}|x}$ is composed of the corresponding payment of a_x, multiplied by a factor consisting of the sum of a series of probabilities, and which factor is different for each payment. Taking the sum of all the payments, we have

$$a_{y\overset{1}{z}|x}=\Sigma v^t . {}_tp_x . {}_{|t}q_{y\overset{1}{z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

7. If it be assumed that for all values of t, ${}_{|t}q_{y\overset{1}{z}}=Q_{y\overset{1}{z}} . {}_{|t}q_{yz}$, then equation 3 becomes

$$\begin{aligned} a_{y\overset{1}{z}|x}&=Q_{y\overset{1}{z}} . \Sigma v^t . {}_tp_x . {}_{|t}q_{yz} \\ &=Q_{y\overset{1}{z}} . a_{yz|x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

[8] Equation 4 is sometimes used to give an approximate value for $a_{y\overset{1}{z}|x}$; and the result is not far from the truth if (y) and (z) are advanced in age. It would be strictly accurate did Gompertz's law prevail. Thus, it was shown in Chap. xiii, Art. 38, that by Gompertz's law ($\log_e s$ according to that law being zero), $\int {}_tp_y \,.\, d_t p_z = \frac{1}{1+c^{y-z}} \cdot \int d_t p_{yz}$; or, changing signs, and taking the integral between the limits 0 and t, ${}_{|t}q_{y\overset{1}{z}} = \frac{1}{1+c^{y-z}} \cdot {}_{|t}q_{yz} = \mathrm{Q}_{y\overset{1}{z}} \,.\, {}_{|t}q_{yz}$ by formula 39 of Chap. xiii.

9. In Chap. xiv, Art. 17, the value of the ordinary reversionary annuity to (x) after (y) was found, by investigating the value of the benefit receivable on the death of (y) should (x) be then alive. By applying the same line of reasoning to the compound survivorship, a convenient formula may be derived, by means of which the numerical value may be approximately calculated. The life (x) will become possessed of an annuity-due on his own life at the end of the tth year, if he survive to the end of the year, and if (z) die in that year leaving (y) surviving him. The probability on which the receipt of the benefit depends therefore is, $\frac{1}{2}({}_{t-1}p_z - {}_tp_z)({}_{t-1}p_y + {}_tp_y){}_tp_x$, which may be written in terms of the l and d columns. Taking the sum for all values of t, we have the compound survivorship annuity

$$a_{y\overset{1}{z}|x} = \Sigma \frac{l_{x+t} \,.\, l_{y+t-\frac{1}{2}} \,.\, d_{z+t-1}}{l_x l_y l_z} \cdot v^t \,.\, (1 + a_{x+t})$$

$$= \Sigma \frac{l_{y+t-\frac{1}{2}} \,.\, d_{z+t-1}}{l_y l_z} \cdot \frac{\mathrm{N}_{x+t-1}}{\mathrm{D}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

[10] Lubbock's formula of approximate summation, No. 18 of Chap. xxiv, can be used with formula 5, but it is not entirely satisfactory, as the differences are usually irregular. In Arts. 18 to 38 it will be shown that others of the formulas of approximate summation may be employed more easily.

[11] To take an example, let it be required to find, at 4 per-cent interest, the value of an annuity to a life aged 30, to commence on the failure of a life aged 60, provided a life now aged 45 be then alive. Taking $n=7$ in Lubbock's formula, we have

(1) t	(2) $\log l_{y+t-\frac{1}{2}}$	(3) $\log d_{z+t-1}$	(4) $\log N_{x+t-1}$	(5) (2)+(3)+(4)	(6) (5) + Colog l_{yz} + Colog D_x
0	4·89421	3·22583	5·70070	13·82074	$\bar{1}$·71769
7	4·85159	3·34321	5·52278	13·71758	$\bar{1}$·61453
14	4·78767	3·39724	5·32797	13·51288	$\bar{1}$·40983
21	4·68343	3·30492	5·10809	13·09644	$\bar{2}$·99339
28	4·50284	2·91908	4·85014	12·27206	$\bar{2}$·16901
35	4·17783	1·96848	4·53267	10·67898	$\bar{4}$·57593
42	3·58001	0·00000	4·11899	7·69900	$\bar{7}$·59595
	31·47758	18·15876	35·16134	84·79768	$\overline{14}$·07633

	(7) Antilog (6)	Δ	Δ^2	Δ^3	Δ^4
0	·52202	− ·11037	− ·04434	+ ·04060	+ ·03786
7	·41165	− ·15471	− ·00374	+ ·07846	
14	·25694	− ·15845	+ ·07472		
21	·09849	− ·08373			
28	·01476				
35	·00038				
42	·00000				

1·30424
7
9·12968
2·13602
$6·99366 = a_{45:\overset{1}{60}|30}$

$\frac{n+1}{2}\cdot u_0 = 2·08808$
$c_1\Delta\, u_0 = 0·06307$
$c_4\Delta^4 u_0 = 0·00486$
2·15603
·02001
2·13602

$c_2\Delta^2 u_0 = ·01267$
$c_3\Delta^3 u_0 = ·00734$
·02001

12. Passing now to the compound survivorship assurance, to find the value of $A^{3}_{\underset{1}{xyz}}$, an assurance payable on the death of (x) if he die third of the three lives (x), (y), and (z), (z) having died first. The value may be expressed in terms of ordinary contingent assurances and of a compound survivorship annuity. If the assurance were payable at the death of (y) on his dying second of the three lives, (z) having died first, its value would be $A^{2}_{\underset{1}{xyz}}$. But the required assurance depends on the same probabilities, only payment is deferred until the death of (x); and therefore from $A^{2}_{\underset{1}{xyz}}$ must be deducted the value of the interest during the life of (x) after the death of (y) and (z), (z) having died first. The present value of this interest is $da^{2}_{yz|x}$; whence

$$A^{3}_{\underset{1}{xyz}} = A^{2}_{\underset{1}{xyz}} - da^{2}_{yz|x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

13. By Chap. xiii, formula 22, $\mathrm{A}^{2}_{x\underset{1}{y}z}=\mathrm{A}^{1}_{xy}-\mathrm{A}^{1}_{xyz}$; and by formula 2 of this chapter, $a^{2}_{yz|x}=a^{1}_{yz|x}-a_{z|xy}$. Also, because $a_{z|xy}=a_{xy}-a_{xyz}$, and $a_{xy}=\dfrac{v-\mathrm{A}_{xy}}{d}$, and $a_{xyz}=\dfrac{v-\mathrm{A}_{xyz}}{d}$; therefore $da^{2}_{yz|x}=da^{1}_{yz|x}+\mathrm{A}_{xy}-\mathrm{A}_{xyz}$, and

$$\begin{aligned}\mathrm{A}^{3}_{x\underset{1}{y}z}&=\mathrm{A}^{1}_{xy}-\mathrm{A}^{1}_{xyz}-\mathrm{A}_{xy}+\mathrm{A}_{xyz}-da^{1}_{yz|x}\\&=\mathrm{A}_{xyz}-\mathrm{A}^{1}_{xyz}-\mathrm{A}^{1}_{xy}-da^{1}_{yz|x}\quad\ldots\ldots\ldots\quad(7)\end{aligned}$$

In Chap. xii it was shown how to calculate the value of A_{xyz}; in Chap. xiii, how to calculate the values of A^{1}_{xyz} and A^{1}_{xy}; and in this chapter, how to calculate the value of $a^{1}_{yz|x}$. Therefore the value of $\mathrm{A}^{3}_{x\underset{1}{y}z}$ can be computed.

[**14**] Compound survivorship annuities and assurances can be dealt with in practice by means of the differential and integral calculus, more satisfactorily than by ordinary algebra.

[**15**] In Chap. xiv it was shown that in the case of ordinary reversionary annuities there is little difference in the numerical value whether the annuities be payable yearly or momently; that for all practical purposes $a_{y|x}=\bar{a}_{y|x}$, and $a_{yz|x}=\bar{a}_{yz|x}$. It does not numerically affect these approximate relations if another contingency be added, and we may write as a near approximate equation, $a^{1}_{yz|x}=\bar{a}^{1}_{yz|x}$. This fact may be made further evident by another line of reasoning. In the case of $a^{1}_{yz|x}$, when the annuity comes into possession the first payment will be made, on the average, six months after the death of (z); and therefore at the moment of the death of (z) the value of the benefit on which (x) will then enter (his age then being say $x+t$) will be, on the average, ${}_{\frac{1}{2}|}\mathbf{a}_{x+t}$. In the case of the continuous annuity, $\bar{a}^{1}_{yz|x}$, at the moment of the death of (z) (x) will enter on a continuous annuity on his life, and its value will then be $\bar{a}_{x+t}$. But in Chap. ix, Arts. 36 to 38, it was shown that, approximately, ${}_{\frac{1}{2}|}\mathbf{a}_{x+t}=\bar{a}_{x+t}$; and therefore, approximately, $a^{1}_{yz|x}=\bar{a}^{1}_{yz|x}$.

[**16**] The relations between the various compound survivorship annuities, and between the annuities and assurances, are the same whether the annuities be payable yearly or momently. Therefore

$$\bar{a}_{\overset{1}{y}z|x}=\bar{a}_{yz|x}-\bar{a}_{y\overset{1}{z}|x}\quad\ldots\ldots\ldots\quad(8)$$

$$\bar{a}^{2}_{yz|x}=\bar{a}^{1}_{yz|x}-\bar{a}_{z|xy}\quad\ldots\ldots\ldots\quad(9)$$

$$\bar{\mathrm{A}}^{3}_{x\underset{1}{y}z}=\bar{\mathrm{A}}^{2}_{x\underset{1}{y}z}-\delta\bar{a}^{2}_{yz|x}\quad\ldots\ldots\ldots\quad(10)$$

[17] In treating the matter algebraically, of the three benefits $a_{yz|x}^{\;1}$, $a_{yz|x}^{2}$, and $\mathrm{A}^{3}_{\underset{1}{xyz}}$, the value of $a_{yz|x}^{\;1}$ is the most easily found; and if the value of either of the others be required, recourse must be had to the relations established in equations 2 and 7, respectively. In following such a course, however, the arithmetical labour involved would be very considerable, because of its being necessary to compute also the values of simple reversionary annuities and contingent assurances involving three lives. This additional labour is avoided by using the differential and integral calculus, by means of which the value of either $\bar{a}_{yz|x}^{2}$ or $\bar{\mathrm{A}}^{3}_{\underset{1}{xyz}}$ may be directly found without first finding the value of $\bar{a}_{yz|x}^{\;1}$. Taking each of the benefits in its order:—

[18] To find the value of $\bar{a}_{yz|x}^{\;1}$. If both (x) and (y) be alive at the moment of the death of (z), (x) will then enter into possession of a continuous annuity on his life. The present value of the benefit therefore is

$$\bar{a}_{yz|x}^{\;1}=\frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t} . l_{z+t} . \mu_{z+t} . \bar{a}_{x+t} . dt \quad . \quad . \quad (11)$$

and by means of formula 39*a*, or another of the continuous approximate formulas of Chap. xxiv, the value may be computed. Comparing formula 11 of this chapter with formula 26*a* of Chap. xiv, it will be seen that only μ_{z+t} is included in the case of the compound survivorship annuity, whereas in the case of the reversionary annuity the factor $(\mu_{y+t}+\mu_{z+t})$ appears. In other respects the formulas are identical.

[19] The work of formula 11 may be conveniently arranged to give also the value of $\bar{a}_{xyz}$; and it is desirable to do so, because if the compound survivorship annuity is to be secured by an annual premium, the divisor will be $(1+a_{xyz})$.

[20] To find the value of $\bar{a}_{yz|x}^{2}$. At the moment of the death of (y), provided (z) have died previously, (x), if he be alive, will enter on a continuous annuity on his life. The present value of the benefit therefore is

$$\bar{a}_{yz|x}^{2}=\frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t} . (l_z - l_{z+t})\mu_{y+t} . \bar{a}_{x+t} . dt \quad . \quad (12)$$

[21] It will be seen that the labour of applying formula 12 is much less than in the case of formula 28*a* of Chap. xiv. It is easier to find the value of the compound survivorship annuity, $\bar{a}_{yz|x}^{2}$, than that of the reversionary annuity, $\bar{a}_{\overline{yz}|x}$. Moreover, in using formula 12, the work may, if so required, be arranged to find at the same time the value of $\bar{a}_{xy}$,

which will be useful in calculating the annual premium, the divisor for which is $(1+a_{xy})$.

[22] To find the value of $\bar{\mathrm{A}}^{3}_{x\underset{1}{y}z}$. At the moment of the death of (y), provided (z) have died previously, the assurance will become absolute on the life of (x) if he be then alive. Therefore the present value of the benefit is

$$\bar{\mathrm{A}}^{3}_{x\underset{1}{y}z}=\frac{1}{l_x l_y l_z}\int v^t . l_{x+t} . l_{y+t} . (l_z - l_{z+t}) \mu_{y+t} . \bar{\mathrm{A}}_{x+t} . dt \quad . \quad (13)$$

In using formula 13 it will be sufficiently accurate to write A_{x+t} for $\bar{\mathrm{A}}_{x+t}$, and to multiply the final result by $\left(1+\frac{i}{2}\right)$, or $(1+i)^{\frac{1}{2}}$.

[23] A numerical example of formulas 11, 12, and 13, may be useful to the student. As before, it will be convenient to take $x=30$, $y=45$, and $z=60$, and to assume interest at 4 per-cent. Also, formula 39a of Chap. xxiv is the best for the present purpose, because it is the shortest.

[24] To find at 4 per-cent interest the value of $\bar{a}_{45:\overset{1}{60}|30}$. Here, in using formula 39a of Chap. xxiv, n must be taken equal to 6, because the risk cannot be prolonged beyond the death of (z), and $7n=102-z=42$.

(1)	(2)	(3)	(4)	(5)	(6)	(7)
t	log v^t	log l_{x+t}	log l_{y+t}	log l_{z+t}	log coeff.	(2)+&c.+(6)
0	0·00000	4·95272	4·89164	4·76969	$\bar{1}$·44716	14·06121
6	$\bar{1}$·89780	4·93143	4·85522	4·67372	0·20952	14·56769
18	$\bar{1}$·69340	4·87481	4·72714	4·26453	0·34242	13·90230
30	$\bar{1}$·48900	4·76969	4·40827	3·10483	0·20952	11·98131
36	$\bar{1}$·38680	4·67372	4·07806	1·85126	$\bar{1}$·74819	9·73803
	$\bar{2}$·46700	24·20237	22·96033	18·66403	$\bar{1}$·95681	64·25054

	(8)	(9)	(10)	(11)	(12)	(13)
	(7)+Colog l_{xyz}	log μ_{z+t}	log $\bar{a}_{x+t}$	(8)+(9)+(10)	Antilog (8)	Antilog (11)
0	$\bar{1}$·44716	$\bar{2}$·46538	1·24687	$\bar{1}$·15941	·28000	·14435
6	$\bar{1}$·95364	$\bar{2}$·66266	1·21790	$\bar{1}$·83420	·89875	·68265
18	$\bar{1}$·28825	$\bar{1}$·09771	1·13319	$\bar{1}$·51915	·19420	·33048
30	$\bar{3}$·36726	$\bar{1}$·55907	0·99774	$\bar{3}$·92407	·00233	·00840
36	$\bar{5}$·12398	$\bar{1}$·79387	0·90488	$\bar{5}$·82273	·00001	·00007
	$\bar{9}$·18029	$\bar{5}$·57869	5·50058	$\bar{8}$·25956	1·37529	1·16595
					6	6
					8·25174	6·99570
					$=\bar{a}_{30:45:60}$	$=\bar{a}_{45:\overset{1}{60}\mid 30}$

It will be noticed that the value just found for the compound survivorship annuity payable momently, differs by only ·00204 from the value found by Lubbock's formula, in Art. 11, for the same annuity payable yearly.

[25] To find at 4 per-cent interest the value of $\bar{a}^{2}_{45:60|30}$. In using formula 39a of Chap. xxiv, we must take $n=8$, because the risk may remain undetermined until the death of (y), and $7n=102-y=57$. Also, because (l_z-l_{z+t}) vanishes when $t=0$, the first term need not be calculated.

(1) t	(2) log v^t	(3) log l_{x+t}	(4) log l_{y+t}	(5) log (l_z-l_{z+t})	(6) log μ_{y+t}	(7) log $\bar{a}_{x+t}$
8	$\bar{1}$·86373	4·92360	4·84020	4·20798	$\bar{2}$·26316	1·20650
24	$\bar{1}$·59120	4·83199	4·60601	4·71599	$\bar{2}$·76738	1·07302
40	$\bar{1}$·31867	4·57952	3·73902	4·76964	$\bar{1}$·36506	0·83206
48	$\bar{1}$·18240	4·26453	2·56348	4·76969	$\bar{1}$·67626	0·65868
56	$\bar{1}$·04613	3·63377	0·00000	4·76969	$\bar{1}$·99057	0·44871
	$\bar{3}$·00213	22·23341	15·74871	23·23299	$\bar{4}$·06243	4·21897

	(8) (2)+&c.+(7)	(9) (8)+Colog l_{xyz}	(10) log coeff.	(11) (9)+(10)	(12) Antilog (11)
8	13·30517	$\bar{2}$·69112	0·20952	$\bar{2}$·90064	·07955
24	13·58559	$\bar{2}$·97154	0·34242	$\bar{1}$·31396	·20604
40	12·60397	$\bar{3}$·98992	0·20952	$\bar{2}$·19944	·01583
48	11·11504	$\bar{4}$·50099	$\bar{1}$·74819	$\bar{4}$·24918	·00018
56	7·88887	$\bar{7}$·27482	0·20952	$\bar{7}$·48434	·00000
	58·49864	$\overline{15}$·42839	0·71917	$\overline{14}$·14756	·30160
					8
					2·41280 $=\bar{a}^{2}_{45:60\|30}$

[26] To find at 4 per-cent interest the value of $\overline{A}^{3}_{30:45:\underset{1}{60}}$. In using formula 39$a$ of Chap. xxiv, we must again take $n=8$, because $7n=102-y=57$. Also, as in the last preceding article, the term for $t=0$ need not be calculated.

(1) t	(2) $\log v^t$	(3) $\log l_{x+t}$	(4) $\log l_{y+t}$	(5) $\log (l_z - l_{z+t})$	(6) $\log \mu_{y+t}$	(7) $\log A_{x+t}$
8	$\bar{1}$·86373	4·92360	4·84020	4·20798	$\bar{2}$·26316	$\bar{1}$·55870
24	$\bar{1}$·59120	4·83199	4·60601	4·71599	$\bar{2}$·76738	$\bar{1}$·72076
40	$\bar{1}$·31867	4·57952	3·73902	4·76964	$\bar{1}$·36506	$\bar{1}$·85704
48	$\bar{1}$·18240	4·26453	2·56348	4·76969	$\bar{1}$·67626	$\bar{1}$·90609
56	$\bar{1}$·04613	3·63377	0·00000	4·76969	$\bar{1}$·99057	$\bar{1}$·94088
	$\bar{3}$·00213	22·23341	15·74871	23·23299	$\bar{4}$·06243	$\bar{2}$·98347

	(8) (2)+&c.+(7)	(9) (8)+Colog l_{xyz}	(10) log coeff.	(11) (9)+(10)	(12) Antilog (11)
8	11·65737	$\bar{3}$·04332	0·20952	$\bar{3}$·25284	·0017899
24	12·23333	$\bar{3}$·61928	0·34242	$\bar{3}$·96170	·0091559
40	11·62895	$\bar{3}$·01490	0·20952	$\bar{3}$·22442	·0016766
48	10·36245	$\bar{5}$·74840	$\bar{1}$·74819	$\bar{5}$·49659	·0000314
56	7·38104	$\bar{8}$·76699	0·20952	$\bar{8}$·97651	·0000001
	53·26314	$\overline{20}$·19289	0·71917	$\overline{20}$·91206	·0126539
					8
					·1012312
					$=A^{3}_{\underset{1}{30:45:60}}$

$$\bar{A}^{3}_{\underset{1}{30:45:60}} = \left(1+\frac{i}{2}\right) A^{3}_{\underset{1}{30:45:60}} = \cdot 10326$$

[27] It has been shown in Arts. 12 to 16 that all possible compound survivorship annuities and assurances involving three lives are so connected with each other, that each can be made to depend on one compound survivorship annuity of the form $a_{y\overset{1}{z}|x}$. It can likewise be proved that all possible compound survivorship annuities and assurances involving four lives can be expressed in terms of annuities of the forms $a_{xy\overset{1}{z}|w}$ and $a_{y\overset{1}{z}|wx}$. The resulting formulas are, however, very lengthy, and the amount of arithmetical labour which would be required to apply them, renders them useless in practice.

[28] By applying the continuous method, working formulas like Nos. 11, 12, and 13, may be derived, which render the calculations comparatively brief and manageable. These formulas do not require much either of demonstration or illustration, and the numerous examples given in the present and previous chapters of the method of using the formulas of approximate summation, have rendered the operations familiar to the student.

[29] To find the value of $\bar{a}_{xy\overset{1}{z}|w}$. At the moment of the death of (z), if all the other lives still exist, (w) will come into possession of a

continuous annuity on his own life. Therefore, using for conciseness the symbols ${}^{t}l_x$, &c., for l_{x+t}, &c.,

$$\bar{a}_{xy\overset{1}{z}}|_{w}=\frac{1}{l_{wxyz}}\int v^t.{}^{t}l_w.{}^{t}l_x.{}^{t}l_y.{}^{t}l_z.{}^{t}\mu_z.{}^{t}\bar{a}_w.dt\quad .\quad .\quad .\quad (14)$$

Similarly

$$\bar{a}_{y\overset{1}{z}}|_{wx}=\frac{1}{l_{wxyz}}\int v^t.{}^{t}l_w.{}^{t}l_x.{}^{t}l_y.{}^{t}l_z.{}^{t}\mu_z.{}^{t}\bar{a}_{wx}.dt\quad .\quad .\quad (15)$$

Also $$\bar{a}_{y\overset{1}{z}}|_{\overline{wx}}=\bar{a}_{y\overset{1}{z}}|_{w}+\bar{a}_{y\overset{1}{z}}|_{x}-\bar{a}_{y\overset{1}{z}}|_{wx}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (16)$$

where the values of the annuities on the right side of the equation must be calculated separately by formulas 11 and 15.

[30] To find the value of $\bar{a}_{x\overset{2}{y}\underset{1}{z}}|_{w}$. At the moment of the death of (y), if (z) have died previously, and if (w) and (x) be still alive, (w) will come into possession of a continuous annuity on his life. Therefore

$$\bar{a}_{x\overset{2}{y}\underset{1}{z}}|_{w}=\frac{1}{l_{wxyz}}\int v^t.{}^{t}l_w.{}^{t}l_x.{}^{t}l_y(l_z-{}^{t}l_z).{}^{t}\mu_y.{}^{t}\bar{a}_w.dt\quad .\quad .\quad (17)$$

Similarly

$$\bar{a}_{\overset{2}{y}z}|_{wx}=\frac{1}{l_{wxyz}}\int v^t.{}^{t}l_w.{}^{t}l_x.{}^{t}l_y(l_z-{}^{t}l_z).{}^{t}\mu_y.{}^{t}\bar{a}_{wx}.dt\quad .\quad (18)$$

Also $$\bar{a}_{\overset{2}{y}z}|_{\overline{wx}}=\bar{a}_{\overset{2}{y}z}|_{w}+\bar{a}_{\overset{2}{y}z}|_{x}-\bar{a}_{\overset{2}{y}z}|_{wx}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (19)$$

where the values of the annuities on the right side of the equation must be calculated separately by formulas 12 and 18.

[31] To find the value of $\bar{a}_{x\overset{2:3}{y}\underset{1}{z}}|_{w}$, an annuity on the life of (w), to commence at the death of (y), provided (z) die before him in the life-time of (x). At the moment of the death of (z), if all the other lives survive, (w) will become entitled to a reversionary annuity payable to him if he survive (y). Therefore

$$\bar{a}_{x\overset{2:3}{y}\underset{1}{z}}|_{w}=\frac{1}{l_{wxyz}}\int v^t.{}^{t}l_w.{}^{t}l_x.{}^{t}l_y.{}^{t}l_z.{}^{t}\mu_z({}^{t}\bar{a}_w-{}^{t}\bar{a}_{wy})dt\quad .\quad (20)$$

[32] Lastly, to find the value of $\bar{a}_{\overset{3}{x}\underset{2}{y}\underset{1}{z}}|_{w}$. At the moment of the death of (y), if (z) have died previously, and if (x) be still living, (w) will become entitled to a reversionary annuity payable to him if he survive (x). Therefore

$$\bar{a}_{\overset{3}{x}\underset{2}{y}\underset{1}{z}}|_{w}=\frac{1}{l_{wxyz}}\int v^t.{}^{t}l_w.{}^{t}l_x.{}^{t}l_y(l_z-{}^{t}l_z).{}^{t}\mu_y.({}^{t}\bar{a}_w-{}^{t}\bar{a}_{wx})dt\ .\ (21)$$

[33] The formulas for compound survivorship assurances are very similar to those for the annuities. We have

$$\bar{\mathrm{A}}^{2}_{\underset{1}{wxyz}}=\frac{1}{l_{wxyz}}\int v^t.{}^tl_w.{}^tl_x.{}^tl_y(l_z-{}^tl_z).{}^t\mu_w.dt \quad . \quad . \quad (22)$$

[34] Let $\bar{\mathrm{A}}^{2:3:4}_{\underset{1}{w:xyz}}$ represent the value of an assurance payable at the death of (w) if (z) die first of the four lives. Immediately on the death of (z), if he die first, the assurance will become absolute on the life of (w). Therefore

$$\bar{\mathrm{A}}^{2:3:4}_{\underset{1}{w:xyz}}=\frac{1}{l_{wxyz}}\int v^t.{}^tl_w.{}^tl_x.{}^tl_y.{}^tl_z.{}^t\mu_z.{}^t\bar{\mathrm{A}}_w.dt \quad . \quad . \quad (23)$$

[35] To find the value of $\bar{\mathrm{A}}^{3}_{\underset{21}{wxyz}}$. Immediately on the death of (y), if (z) have died previously, and if (w) and (x) both survive, the assurance will become an assurance on the life of (w) contingent on his dying before (x). Therefore

$$\bar{\mathrm{A}}^{3}_{\underset{21}{w:xyz}}=\frac{1}{l_{wxyz}}\int v^t.{}^tl_w.{}^tl_x.{}^tl_y(l_z-{}^tl_z).{}^t\mu_y.{}^t\bar{\mathrm{A}}^{1}_{wx}.dt \quad . \quad (24)$$

[36] In the case of $\bar{\mathrm{A}}^{3:4}_{\underset{21}{w:xyz}}$, the formula is the same as for $\bar{\mathrm{A}}^{3}_{\underset{21}{wxyz}}$, except that on the death of (y) the assurance will become absolute on the life of (w), and not contingent. That is

$$\bar{\mathrm{A}}^{3:4}_{\underset{21}{w:xyz}}=\frac{1}{l_{wxyz}}\int v^t.{}^tl_w.{}^tl_x.{}^tl_y(l_z-{}^tl_z).{}^t\mu_y.{}^t\bar{\mathrm{A}}_w.dt \quad . \quad (25)$$

[37] Lastly, to find the value of $\bar{\mathrm{A}}^{4}_{\underset{321}{wxyz}}$. At the moment of the death of (y), (z) having died previously, and (w) and (x) being still alive, the assurance will become an assurance on the life of (w), contingent on his surviving (x). Therefore

$$\bar{\mathrm{A}}^{4}_{\underset{321}{wxyz}}=\frac{1}{l_{wxyz}}\int v^t.{}^tl_w.{}^tl_x.{}^tl_y(l_z-{}^tl_z).{}^t\mu_y.({}^t\bar{\mathrm{A}}_w-{}^t\bar{\mathrm{A}}^{1}_{wx})dt. \quad (26)$$

[38] By means of equations 14 to 26, the value of every possible compound survivorship annuity or assurance involving four lives may be found. Thus, immediately from equation 26,

$$\bar{\mathrm{A}}^{4}_{\underset{1}{wxyz}}=\frac{1}{l_{wxyz}}\int v^t.{}^tl_w.{}^tl_x.{}^tl_y(l_z-{}^tl_z)\{{}^t\mu_x({}^t\bar{\mathrm{A}}_w-{}^t\bar{\mathrm{A}}^{1}_{wy})+{}^t\mu_y({}^t\bar{\mathrm{A}}_w-{}^t\bar{\mathrm{A}}^{1}_{wx})\}dt.$$

[39] In using formulas 24 and 26, the values of contingent assurances involving two lives are required. If the basis of calculation is to be the Carlisle mortality table at 3 per-cent interest, a complete table of these

values is given by Gray, Smith, and Orchard in their work, "Assurance and Annuity Tables."

If the Carlisle mortality table at other rates of interest is to be used, Chisholm's Commutation Tables afford an easy means of obtaining the required values of the contingent assurances. For other mortality tables, some of the formulas of Chap. xiii must be employed.

[40] In many cases the values of compound survivorships involving five or more lives may be found by formulas similar to those which are applicable to benefits embracing only four lives. Sometimes, however, one of the factors under the sign of integration will itself be a compound survivorship, in which event a process of double integration must be resorted to. Thus, for example, if the value be sought of $\bar{a}^{4}_{\underset{321}{wxyz}|u}$, an annuity to (u), to commence on the death of (w), if (x) die before (w), and (y) before (x), and (z) before (y), it will be observed that on the death of (y), (z) having died first, (u) will become entitled to a compound survivorship annuity payable to him after the death of (w) if (x) die first. Throwing the benefit into the form of an integral, it becomes

$$\bar{a}^{4}_{\underset{321}{wxyz}|u} = \frac{1}{l_{uwxyz}}\int v^t . l_{u+t:w+t:x+t:y+t}(l_z - l_{z+t})\mu_{y+t} . \bar{a}^{2}_{w+t:x+t|u+t} . dt \quad (27)$$

[41] If, in the evaluation of this integral, formula 39*a* of Chap. xxiv is to be employed, we must first of all find, by a separate summation, the value of $\bar{a}^{2}_{w+t:x+t|u+t}$ at each of the epochs $t=h$, $t=3h$, $t=5h$, and $t=6h$, where h is so taken that u_{7h} falls just beyond the limit of the mortality table.

[42] This process would, however, be tedious, and one much shorter may be derived by using another of the formulas of Chap. xxiv, which involves equidistant values of u_n. Of these, formula 33 is the best for present purposes. In that formula, making $n=5$, we have $\int_0^\infty u_x dx = 10\left(u_5 + u_{15} + u_{25} + \&c. + \frac{u_{-5} - u_5}{24}\right)$. Also, in equation 27 of this chapter, expressing the compound survivorship annuity in form of an integral, and substituting D_u for l_u and ${}^tD_u$ for $v^t l_{u+t}$, we have

$$\bar{a}^{4}_{\underset{321}{wxyz}|u} = \frac{1}{D_u . l_{wxyz}}\int_0^\infty {}^tD_u . {}^tl_{wxy}(l_z - {}^tl_z){}^t\mu_y \cdot \frac{1}{{}^tD_u {}^tl_{wx}}\int_t^\infty {}^tD_u . {}^tl_{wx} . {}^t\mu_x . {}^t\bar{a}_{w|u} . dt . dt$$

$$= \frac{1}{D_u . l_{wxyz}}\int_0^\infty {}^tl_y(l_z - {}^tl_z){}^t\mu_y . \int_t^\infty {}^tD_u . {}^tl_{wx} . {}^t\mu_x({}^ta_u - {}^ta_{uw})dt . dt \quad . \quad (28)$$

[43] In applying formula 28 numerically, the expression under the second integral must first be computed for the values of $t = -10$, 0, 10,

20, &c., which will give the values of this integral for $t=-5, 5, 15, 25$, &c., and these then are the data for evaluating the entire integral. The successive values of the expression under the second integral may be computed by continued summation, much as N_x is formed from D_x.

[44] As a numerical example, let u, w, x, y, and z, be respectively 30, 40, 50, 60, and 70, and let interest be taken at 4 per-cent. Using four-figure logarithms, which are quite sufficient for the purpose, the calculations are as follows. The denominator is $D_{30}.l_{40:50:60:70}$, and its logarithm is 23·5683.

(1) t	(2) $\log D_{u+t}$	(3) $\log l_{w+t}$	(4) $\log l_{x+t}$	(5) $\log \mu_{x+t}$	(6) $\log \bar{a}_{w+t\rvert u+t}$	(7) (2)+ . . . +(6)	(8) $\frac{1}{10^{10}}$ Antilog (7)
−10	4·6419	4·9527	4·9153	$\bar{3}$·9956	0·5456	13·0511	1124·8
0	4·4417	4·9153	4·8621	$\bar{2}$·1881	0·5912	12·9984	996·3
10	4·2340	4·8621	4·7697	$\bar{2}$·4654	0·6386	12·9698	932·8
20	4·0104	4·7697	4·5795	$\bar{2}$·8030	0·6635	12·8261	670·1
30	3·7477	4·5795	4·1457	$\bar{1}$·1735	0·6428	12·2892	194·6
40	3·3872	4·1457	3·1048	$\bar{1}$·5591	0·5550	10·7518	5·6
	24·4629	28·2250	26·3771	$\bar{8}$·1847	3·6367	74·8864	3924·2

(9) t	(10) 10Σ(8)	(11) −10Δ(8)	(12) (10)+$\frac{1}{24}$(11)	(13) log (12)+10	(14) $\log l_{y+t}$	(15) $\log \mu_{y+t}$	(16) $\log (l_z-l_{z+t})$
−5	27,994	1,285	28,048	14·4480	4·8233	$\bar{2}$·3174	(−)4·0543
5	18,031	635	18,057	14·2567	4·6929	$\bar{2}$·6285	4·0926
15	8,703	2,627	8,812	13·9451	4·4083	$\bar{2}$·9855	4·5118
25	2,002	4,755	2,200	13·3424	3·7390	$\bar{1}$·3651	4·5780
35	56	1,890	135	12·1303	2·1106	$\bar{1}$·7546	4·5795
	56,786	11,192	57,252	68·1225	19·7741	$\bar{5}$·0511	21·8162

t	(17) (13)+ . . . +(16)	(18) (17)−log Denom.	(19) 10 Antilog (18)
−5	(−)21·6430	(−)$\bar{2}$·0747	(−)·1188
5	21·6707	$\bar{2}$·1024	·1266
15	21·8507	$\bar{2}$·2824	·1916
25	21·0245	$\bar{3}$·4562	·0286
35	18·5750	$\bar{5}$·0067	·0001
	104·7639	$\overline{14}$·9224	·3469
			·0102

$$\bar{a}^{4}_{40:\underset{3}{50}:\underset{2}{60}:\underset{1}{70}\rvert 30} = \cdot 3367$$

[45] In England it is only very seldom that the values of complicated benefits, such as those discussed in this chapter, are required; but in Scotland the necessity for them frequently arises in connection with disentail transactions. Sometimes five or more lives, and several orders of survivorship, are involved, but in all such cases the principles here laid down afford the means of making the calculations with facility.

[46] Sometimes, besides probabilities of life, probabilities of marriage or remarriage, or of the birth of issue, have to be brought into account. When the necessary tables of such probabilities are available, these further contingencies may be included by adding suitable columns to the calculations.

CHAPTER XVI.

Commutation Columns, and their Application to Varying Benefits, and to Returns of Premium.

1. In Chap. vii a good deal of space has already been devoted to commutation columns, and they have been studied in their relation to simple benefits, such as ordinary annuities, assurances, and premiums, and temporary and deferred benefits. We shall now go into the question more fully, and show to what extended uses commutation columns may be put. It is desirable, in order to bring all the subject together, to recapitulate what has gone before, even at the cost of a certain amount of repetition; but it is not necessary to investigate again first principles.

2. Columns headed D, N, C, and M, have already been introduced; and now two more must be added, S and R; the nature of which will immediately appear.

3. The following equations explain the construction of the columns:—

N_{x+1}

$$\left.\begin{aligned}
D_x &= v^x l_x \\
N_x &= D_{x+1} + D_{x+2} + \&c. \\
&= \Sigma D_{x+1} \\
S_x &= N_x + N_{x+1} + \&c. \\
&= D_{x+1} + 2D_{x+2} + 3D_{x+3} + \&c. \\
&= \Sigma N_x \\
&= \Sigma^2 D_{x+1} \\
C_x &= v^{x+1} d_x \\
&= v^{x+1}(l_x - l_{x+1}) = vD_x - D_{x+1} \\
M_x &= C_x + C_{x+1} + \&c. \\
&= \Sigma C_x \\
R_x &= M_x + M_{x+1} + \&c. \\
&= C_x + 2C_{x+1} + 3C_{x+2} + \&c. \\
&= \Sigma M_x \\
&= \Sigma^2 C_x
\end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

the symbol Σ^2 denoting the sum of the sum of the function under it.

4. It was shown in Chap. vii that for immediate and deferred and temporary annuities, the expressions contain values from only the N column in the numerator, and a value from the D column in the denominator. This fact explains the symbols that have been adopted to represent the columns; the letter N being the first of the word *Numerator*, and D the first of the word *Denominator*. The letter S was doubtless chosen as the first of the word *Sum*, the column S being formed by summing the column N. It is only in the case of the most simple benefits that the letters retain these relative positions in the expressions; because D may frequently occur in the numerator, and N, or in fact the letter of any other column, in the denominator; but it was from the simplest benefits that the columns took their names. When the assurance columns were added to the tables, it was natural to adopt for them the letters C, M, R; these being next in the alphabet to those of the corresponding annuity columns.

5. De Morgan gave the name *Commutation* to the columns; his reason being, that they supply the means of *commuting* one benefit for another. Thus, for example, seeing that at age 30 the value of an

paid to [illegible]

endowment payable at 34 is $\frac{D_{34}}{D_{30}}$; therefore a person could afford to give up a sum of money, D_{34} (or at 3 per-cent, 31,797), due at age 30, in exchange for a sum of money, D_{30} (or at 3 per-cent, 36,949), due at age 34. Or a person might now give up an annuity of N_{34} (=600,400) per annum, to be entered on at age 30, to receive in return another annuity of N_{30} (=735,104) per annum, to be entered on at age 34 if he should live so long. These properties are independent of the age of the life, and show that the most simple information given by the tables is, the proportion in which a benefit due at one age ought to be changed, or commuted, so as to retain the same value and be due at another age.

6. The following equations display the connection between the assurance and the annuity columns.

$$\left.\begin{aligned} C_x&=vD_x-D_{x+1}\\ M_x&=vN_{x-1}-N_x\\ &=D_x-dN_{x-1}\\ R_x&=vS_{x-1}-S_x\\ &=N_{x-1}-dS_{x-1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$= vN_x - N_{x+1}$

$D_x - dN_x$

7. The symbol $a_{x\overline{n|}}$ has been used to represent a temporary annuity on the life of (x); and it has been shown that $a_{x\overline{n|}}=\frac{N_x-N_{x+n}}{D_x}$. We may therefore conveniently write $N_{x\overline{n|}}=N_x-N_{x+n}$. Extending the notation to the other columns, we have

$$\left.\begin{aligned} N_{x\overline{n|}}&=N_x-N_{x+n}\\ S_{x\overline{n|}}&=S_x-S_{x+n}\\ M_{x\overline{n|}}&=M_x-M_{x+n}\\ R_{x\overline{n|}}&=R_x-R_{x+n}\end{aligned}\right\} \quad \cdots\cdots\cdots \quad (3)$$

8. The following equations have already been investigated in Chap. vii.

$$a_x=\frac{N_x}{D_x} \quad ✓$$

temp

$${}_{|n}a_x=\frac{N_x-N_{x+n}}{D_x}=\frac{N_{x\overline{n|}}}{D_x}$$

~~temp~~

$${}_{n|}a_x=\frac{N_{x+n}}{D_x}$$

$$A_x=\frac{M_x}{D_x} \quad ✓ \qquad \cdots\cdots \quad (4)$$

temp

$${}_{|n}A_x=\frac{M_x-M_{x+n}}{D_x}=\frac{M_{x\overline{n|}}}{D_x}$$

deferred

$${}_{n|}A_x=\frac{M_{x+n}}{D_x}$$

9. If, in the equation ${}_{n|}a_x=\frac{N_{x+n}}{D_x}$, a negative value be given to n, what is the interpretation of the result? In order to find the answer to the question, let the values of N and D be written in terms of the l_x column. We then have

$$\begin{aligned} {}_{-n|}a_x&=\frac{v^{x-n+1}l_{x-n+1}+v^{x-n+2}l_{x-n+2}+\&c.+v^xl_x+v^{x+1}l_{x+1}+\&c.}{v^xl_x}\\ &=\frac{(1+i)^{n-1}l_{x-n+1}+(1+i)^{n-2}l_{x-n+2}+\&c.+l_x+vl_{x+1}+\&c.}{l_x}\\ &=\frac{1}{l_x}[\{(1+i)^{n-1}l_{x-n+1}+(1+i)^{n-2}l_{x-n+2}+\&c.+l_x\}+l_xa_x]\end{aligned}$$

If l_{x-n} annuities had been set up n years ago, on l_{x-n} live then aged $x-n$, and if the yearly payments had been invested at compound interest, instead of being drawn when due, a fund of $\{(1+i)^{n-1}l_{x-n+1}+(1+i)^{n-2}l_{x-n+2}+\&c.+l_x\}$ would have been thereby created; and, in addition, there would have now still remained l_x current annuities on lives now aged x. Therefore $\frac{N_{x-n}}{D_x}$ represents the share of each of the l_x survivors in such a fund; that is, it represents the sum which (x) should receive in respect of the surrender of a whole-life annuity granted n years ago, but never drawn, and which would have been entirely lost had he died within the n years.

10. Or, from another point of view, $\frac{N_{x-n}}{D_x}$ is the sum which (x) should now pay, in consideration of having enjoyed for n years an unbought annuity, and of having the right to continue to enjoy it during the remainder of his life.

11. From what has just been said, we see that $\frac{N_{x-n}-N_x}{D_x}$ represents the share of (x) in a fund created by the investment during n years of the payments under l_{x-n} temporary life annuities granted under the conditions that the annuities should be forborne during the n years, and that the accumulations should be divided among the survivors at the end of that time. Such a fund is called a *Tontine Fund*, from an Italian named Tonti, who established one of a similar character about the middle of the 17th century.

12. The accumulations of a life annuity as thus found are greater than the accumulations of an annuity-certain for the same period, because the benefit of survivorship is given to those members of the fund who live to the end of the period. These receive in equal shares the accumulations of the payments of deceased members as well as the accumulations of their own payments; and it may therefore be briefly said that $\frac{N_{x-n}-N_x}{D_x}$ represents the accumulations of an annuity, *with benefit of survivorship.*

13. If annuities-due be substituted for ordinary annuities, the expression becomes $\frac{N_{x-n-1}-N_{x-1}}{D_x}$.

14. A very similar line of reasoning may be applied to assurances. If l_{x-n} persons forming a society agree that at the end of n years the l_x survivors shall contribute equally to create a fund amounting to 1 in respect of each of the $l_{x-n}-l_x$ deceased members, with compound interest from the end of the year of his death; the total sum to be provided at the end of the n years will be

$$(1+i)^{n-1}d_{x-n}+(1+i)^{n-2}d_{x-n+1}+\&c.+d_{x-1},$$

which is equal to

$$\frac{1}{v^x}\{C_{x-n}+C_{x-n+1}+\&c.+C_{x-1}\}, \quad \text{or,} \quad l_x\frac{M_{x-n}-M_x}{D_x},$$

and the contribution to be paid by each of the l_x survivors will be $\frac{M_{x-n}-M_x}{D_x}$.

eserve

15. Combining now the results as regards annuities and assurances respectively, we are able to say what sum an office will have in reserve (expenses being left out of account), against each of l_x policies remaining on the books, out of l_{x-n} policies of 1 each granted n years ago at premiums of P_{x-n} on lives then aged $x-n$. If the accumulated premiums and the accumulated claims respectively be apportioned amongst the survivors, we have the required reserve, namely,

$$\frac{P_{x-n}(N_{x-n-1}-N_{x-1})-(M_{x-n}-M_x)}{D_x}.$$

This is a very important application of the principle, and will be developed more fully in Chap. xviii on Policy-Values.

reasing

16. To find the value of a life annuity commencing at k per annum, and increasing or decreasing h per annum throughout life. Writing $(\text{v}a)_x$ for the value of a varying annuity, we have the equation

$$(\text{v}a)_x=\frac{1}{D_x}\{kN_x\pm h(D_{x+2}+2D_{x+3}+3D_{x+4}+\&c.)\}$$

$$=\frac{kN_x\pm hS_{x+1}}{D_x} \quad \ldots\ldots\ldots\ldots \quad (5)$$

by formula 1.

17. If in the last result we make both k and h equal to unity, and take the positive sign, we have the value of an annuity commencing at unity, and increasing 1 per annum throughout life. This increasing annuity may be symbolized by $(\text{I}a)_x$; and we have

$$(\text{I}a)_x=\frac{N_x+S_{x+1}}{D_x}$$

$$=\frac{S_x}{D_x} \quad \ldots\ldots\ldots\ldots \quad (6)$$

18. In the same way, using the symbols $(\mathrm{vA})_x$ and $(\mathrm{IA})_x$ for the corresponding assurances, we have

$$(\mathrm{vA})_x = \frac{k\mathrm{M}_x \pm h\mathrm{R}_{x+1}}{\mathrm{D}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$(\mathrm{IA})_x = \frac{\mathrm{R}_x}{\mathrm{D}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

19. When the negative sign is taken in equations 5 and 7, there is in practice a limit to the magnitude of h as compared with k. The annuity is one commencing at k, and decreasing h per annum throughout life; and if (x) live long enough the total of the decrease may exceed the original yearly rent of the annuity. Thus, if $k=mh$, we shall have for the annuity

$$(\mathrm{v}a)_x = \frac{mh\mathrm{D}_{x+1} + (m-1)h\mathrm{D}_{x+2} + \&\mathrm{c.} + h\mathrm{D}_{x+m} + 0\mathrm{D}_{x+m+1}}{\mathrm{D}_x}$$

$$- \frac{h\mathrm{D}_{x+m+2} + 2h\mathrm{D}_{x+m+3} + \&\mathrm{c.}}{\mathrm{D}_x}$$

Here (x) will receive an annuity gradually diminishing, until at the end of the $(m+1)$th year he will receive nothing, and at the end of each year thereafter he will receive a *negative* amount, increasing annually; that is, he will have to *pay* instead of *receive.* Theoretically this case presents no difficulties; but in practice, if an office were to grant such an annuity, security would have to be taken for the punctual payment by the annuitant of the instalments due by him after the $(m+1)$th year: otherwise, after the office had discharged its part of the bargain, it might find itself deprived of a portion of the consideration in exchange for which it had made its payments.

20. So far we have assumed the benefit to vary during the whole of life, but there may be a benefit varying for a limited number of years, and then remaining constant. Thus, there may be an annuity commencing at k, and increasing or decreasing h per annum until after n years, there being $n-1$ increments or decrements. The value of such an annuity, which may be written $(\mathrm{v}_{\overline{n|}}a)_x$, will be

$$(\mathrm{v}_{\overline{n|}}a)_x = \frac{k\mathrm{N}_x \pm h(\mathrm{S}_{x+1} - \mathrm{S}_{x+n})}{\mathrm{D}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Similarly, for the corresponding assurance,

$$(\mathrm{v}_{\overline{n|}}\mathrm{A})_x = \frac{k\mathrm{M}_x \pm h(\mathrm{R}_{x+1} - \mathrm{R}_{x+n})}{\mathrm{D}_x} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

21. If k and h both be equal to unity we have, taking the positive sign, and substituting I for v in the symbol,

$$\left.\begin{aligned}(\mathrm{I}_{\overline{n}|}a)_x &= \frac{\mathrm{S}_x-\mathrm{S}_{x+n}}{\mathrm{D}_x} = \frac{\mathrm{S}_{x\overline{n}|}}{\mathrm{D}_x}\\ (\mathrm{I}_{\overline{n}|}\mathrm{A})_x &= \frac{\mathrm{R}_x-\mathrm{R}_{x+n}}{\mathrm{D}_x} = \frac{\mathrm{R}_{x\overline{n}|}}{\mathrm{D}_x}\end{aligned}\right\} \quad .\ .\ .\ .\ \ (11)$$

22. If the whole benefit is to cease after n years, formulas 9 to 11 must be correspondingly modified. In formulas 9 and 10 the payments at the end of the nth year and each year thereafter will be $k\pm(n-1)h$; and in order to find the values of temporary benefits we must deduct from the values in these equations the value of a deferred benefit of $k\pm(n-1)h$. If we write $(\mathrm{v}a)_{x\overline{n}|}$ and $(\mathrm{vA})^{1}_{x\overline{n}|}$ for the values of temporary varying annuities and assurances respectively, we have

$$\begin{aligned}(\mathrm{v}a)_{x\overline{n}|} &= \frac{k\mathrm{N}_x\pm h(\mathrm{S}_{x+1}-\mathrm{S}_{x+n})-(k\pm\overline{n-1}h)\mathrm{N}_{x+n}}{\mathrm{D}_x}\\ &= \frac{k(\mathrm{N}_x-\mathrm{N}_{x+n})\pm h(\mathrm{S}_{x+1}-\mathrm{S}_{x+n}-\overline{n-1}\mathrm{N}_{x+n})}{\mathrm{D}_x}\\ &= \frac{k(\mathrm{N}_x-\mathrm{N}_{x+n})\pm h(\mathrm{S}_{x+1}-\mathrm{S}_{x+n+1}-n\mathrm{N}_{x+n})}{\mathrm{D}_x} \quad .\ .\ .\ \ (12)\end{aligned}$$

Similarly, for the assurance

$$(\mathrm{vA})^{1}_{x\overline{n}|} = \frac{k(\mathrm{M}_x-\mathrm{M}_{x+n})\pm h(\mathrm{R}_{x+1}-\mathrm{R}_{x+n+1}-n\mathrm{M}_{x+n})}{\mathrm{D}_x} \quad .\ \ (13)$$

Also, when $k=h=1$,

$$\left.\begin{aligned}(\mathrm{I}a)_{x\overline{n}|} &= \frac{\mathrm{S}_x-\mathrm{S}_{x+n}-n\mathrm{N}_{x+n}}{\mathrm{D}_x}\\ (\mathrm{IA})^{1}_{x\overline{n}|} &= \frac{\mathrm{R}_x-\mathrm{R}_{x+n}-n\mathrm{M}_{x+n}}{\mathrm{D}_x}\end{aligned}\right\} \quad .\ .\ .\ .\ \ (14)$$

23. By means of the relation between R_x and S_x given in formula 2, equation 8 may be written in the form

$$(\mathrm{IA})_x = \frac{v\mathrm{S}_{x-1}-\mathrm{S}_x}{\mathrm{D}_x} = \frac{\mathrm{N}_{x-1}-d\mathrm{S}_{x-1}}{\mathrm{D}_x} \quad .\ .\ .\ .\ \ (15)$$

and thus the R_x column may be dispensed with. This brings us back to the connection between an increasing assurance and an increasing annuity spoken of in Chap. vii, Art. 48. The expression now given, being interpreted, becomes $(\mathrm{IA})_x = v(\mathrm{Ia})_x - (\mathrm{I}a)_x = v\{1 + a_x + (\mathrm{I}a)_x\} - (\mathrm{I}a)_x$ as before.

The values of R may in a similar way be eliminated from equations 7, 10, 11, 13, and 14.

24. Values of varying benefits deferred t years may be obtained simply by writing $x+t$ for x in the numerators of the expressions which have been deduced, the denominators being left unchanged.

25. If an annuity or an assurance commence at k, and increase or decrease h at intervals of t years, there being $m-1$ such increments or decrements, we have

$$(\mathrm{v}a)_x = \frac{k\mathrm{N}_x \pm h(\mathrm{N}_{x+t} + \mathrm{N}_{x+2t} + \ldots + \mathrm{N}_{x+\overline{m-1}.t})}{\mathrm{D}_x} \quad . \quad (16)$$

$$(\mathrm{vA})_x = \frac{k\mathrm{M}_x \pm h(\mathrm{M}_{x+t} + \mathrm{M}_{x+2t} + \ldots + \mathrm{M}_{x+\overline{m-1}.t})}{\mathrm{D}_x} \quad . \quad (17)$$

26. When the negative sign is taken, care must be exercised as to the limit of h, for the reason pointed out in Art. 19.

27. The subject of varying premiums may now be considered.

No matter how the premiums may be payable, at the outset the present value of the benefit must be exactly equal to the present value of the premiums by which the benefit is to be secured; and hence, an equation exists by means of which the premium may be found. In the simplest case, $\mathrm{A}_x = \mathrm{P}_x(1 + a_x)$; or $\mathrm{M}_x = \mathrm{P}_x\mathrm{N}_{x-1}$. The first member of the equation may conveniently be called the *Benefit Side*, and the second member the *Payment Side*. In complicated cases the solution of the problem will be much facilitated if the Benefit Side and the Payment Side, respectively, be written down, and then equated. Seeing that when commutation symbols are used, D_x is generally found as a common denominator on both sides of the equation, it may be omitted; but when that course is adopted, care must be taken to add D_x as a coefficient to any term that would not have it as a denominator if it were not omitted. Thus, in the example already given, we cannot write $\mathrm{M}_x = \mathrm{P}_x(1 + \mathrm{N}_x)$, but rather $\mathrm{M}_x = \mathrm{P}_x(\mathrm{D}_x + \mathrm{N}_x)$.

28. In dealing with increasing and decreasing scales of premiums, two distinct problems are met with, which, however, have sometimes been confounded. The increment of the premium (using the word increment in its algebraical sense to denote either increment or decrement), may depend solely on the sum assured; or it may depend on the unknown periodical premium itself: in symbols, the increment may be k per unit of the sum assured; or it may be $\kappa\pi$, where π is the premium payable for the first year. We shall take up these problems in their order.

29. Let the annual premium be π for the first year; and let it increase or decrease every t years by k for each unit assured; and after mt years let it remain constant for the rest of life. To find π. Here we have

$$\text{Benefit Side } = \mathrm{A}_x$$

$$\text{Payment Side} = \pi(1+a_x) \pm k({}_{t-1|}a_x + {}_{2t-1|}a_x + \&c. + {}_{mt-1|}a_x)$$

or, in commutation symbols,

$$\text{Benefit Side } = \mathrm{M}_x$$

$$\text{Payment Side} = \pi \mathrm{N}_{x-1} \pm k(\mathrm{N}_{x+t-1} + \mathrm{N}_{x+2t-1} + \&c. + \mathrm{N}_{x+mt-1})$$

Whence, equating the two sides,

$$\pi = \frac{\mathrm{M}_x \mp k(\mathrm{N}_{x+t-1} + \mathrm{N}_{x+2t-1} + \&c. + \mathrm{N}_{x+mt-1})}{\mathrm{N}_{x-1}} \quad . \quad . \quad (18)$$

30. Instead of deciding on k beforehand, and leaving π to be determined, we may fix on the value of π, and seek the value of k. The Benefit and Payment Sides will be the same as before, and we shall have

$$k = \frac{\mathrm{M}_x - \pi \mathrm{N}_{x-1}}{\mathrm{N}_{x+t-1} + \mathrm{N}_{x+2t-1} + \&c. + \mathrm{N}_{x+mt-1}} \quad . \quad . \quad . \quad (19)$$

31. In equation 19, k will be negative, that is a decrement, when $\pi \mathrm{N}_{x-1}$ is greater than M_x, that is, when π is greater than P_x. In practical questions care must be taken what values are assigned to k and π in equations 18 and 19. In equation 18, when the premium is to be an increasing one, k must be fixed so that on the one hand π may not be less than the premium for a temporary assurance for t years; and on the other hand, so that $\pi + mk$ may not be greater than the whole-life premium at age $x+mt$. If π were to be less than the term premium at the outset, the assured might keep up the policy for the first t years, and then drop it, as he cannot be compelled to continue paying premiums. The result in such event would be, that he would be assured for t years for less than the premium for a temporary assurance for the term, that is, for a premium less than that required to cover the risk; and the office would be the loser. If, again, k were such that $\pi + mk$ were greater than P_{x+mt}, then, if the life assured remained still in good health at the end of mt years, he might drop the policy, and effect a new one at the premium P_{x+mt}; and once more the office would be the loser. In both these cases the payment of part of the consideration for the current risk would be deferred; and as the company does not take security for future premiums, it cannot afford to enter on the arrangement.

32. If the premium be a decreasing one, care must also be taken that π is always not less than mk: otherwise a *negative* premium would be payable at one stage of the transaction; that is, the assured would become entitled to an annuity instead of having to pay a premium; and that is not intended to be part of the contract.

33. Instead of one only of the quantities π and k being fixed upon, both may be decided on at the outset, if there be left as the unknown quantity k' the mth increment. This will give

$$\text{Benefit Side} = M_x$$

$$\text{Payment Side} = \pi N_{x-1} \pm k(N_{x+t-1} + N_{x+2t-1} + \&c. + N_{x+\overline{m-1}.t-1}) \pm k' N_{x+mt-1}$$

Whence

$$\pm k' = \frac{M_x - \pi N_{x-1} \mp k(N_{x+t-1} + N_{x+2t-1} + \&c. + N_{x+\overline{m-1}.t-1})}{N_{x+mt-1}} \quad (20)$$

34. We may also have a premium varying by arbitrary and not uniform differences, being π for the first t years, π_1 for the second t years, and so on, and finally, after mt years, π_m constant for the remainder of life. Here, to find π_m, we have

$$\text{Benefit Side} = M_x$$

$$\text{Payment Side} = \pi N_{x-1} + (\pi_1 - \pi) N_{x+t-1} + (\pi_2 - \pi_1) N_{x+2t-1} + \&c. + (\pi_m - \pi_{m-1}) N_{x+mt-1}$$

Whence

$$\pi_m = \pi_{m-1} + \frac{M_x - \{\pi N_{x-1} + (\pi_1 - \pi) N_{x+t-1} + \&c. + (\pi_{m-1} - \pi_{m-2}) N_{x+\overline{m-1}.t-1}\}}{N_{x+mt-1}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

35. Taking up now the second of the two problems mentioned in Art. 28, let the increment of the premium be $\kappa\pi$. The Benefit Side will be M_x as before; and for the Payment Side we shall have

$$\pi N_{x-1} \pm \kappa\pi (N_{x+t-1} + N_{x+2t-1} + \&c. + N_{x+mt-1});$$

Whence,

$$\pi = \frac{M_x}{N_{x-1} \pm \kappa (N_{x+t-1} + N_{x+2t-1} + \&c. + N_{x+mt-1})} \quad . \; . \quad (22)$$

and $$\kappa = \frac{\pm(M_x - \pi N_{x-1})}{\pi(N_{x+t-1} + N_{x+2t-1} + \&c. + N_{x+mt-1})} \quad . \quad . \quad . \quad (23)$$

36. The same limitations as before must be imposed as to the relative values of π and κ. It is not possible beforehand to say what the limits are, and they must be determined by trial in each case.

37. A very common form of increasing premium is the "Half Premium Scale," published in the prospectuses of many offices. According to this system, the premium for the first five or seven years is exactly half what it will be for the remainder of life. The general expression is

$$\pi = \frac{M_x}{N_{x-1} + N_{x+t-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Formula 24 is derived from formula 22 simply by making both κ and m equal to unity.

38. The varying benefits so far considered have been of a comparatively simple character, but by an extended application of the principles of the commutation columns the values may be found, in theory at any rate, of almost all benefits which vary according to ascertained law, although in practice the numerical solution of the problems may frequently involve great, if not prohibitive, arithmetical labour.

39. We have already seen that

$$S_x = N_x + N_{x+1} + \&c. = D_{x+1} + 2D_{x+2} + 3D_{x+3} + \&c.,$$

so that S_x represents an annuity, the successive payments of which are in the ratio 1, 2, 3, &c. If, now, we sum S_x exactly as N_x was summed, and write ΣS_x for the result of the summation, we shall have

$$\begin{aligned} \Sigma S_x &= S_x + S_{x+1} + S_{x+2} + \&c. \\ &= D_{x+1} + 3D_{x+2} + 6D_{x+3} + 10D_{x+4} + \&c. \quad . \quad . \quad (25) \end{aligned}$$

We can proceed in the same way with successive summations as far as we please, forming columns which may be denoted by $\Sigma^2 S_x$, $\Sigma^3 S_x$, &c., and we shall find that

$$\left.\begin{aligned} \Sigma^2 S_x &= D_{x+1} + 4D_{x+2} + 10D_{x+3} + 20D_{x+4} + \&c. \\ \Sigma^3 S_x &= D_{x+1} + 5D_{x+2} + 15D_{x+3} + 35D_{x+4} + \&c. \\ \&c. & \qquad \&c. \qquad \&c. \end{aligned}\right\} \quad . \quad (26)$$

Thus ΣS_x will represent an annuity, the successive payments of which are in the ratio 1, 3, 6, 10, &c.; $\Sigma^2 S_x$ an annuity, the successive payments of which are in the ratio 1, 4, 10, 20, &c., and so on.

40. The assurance columns may be treated in precisely the same manner, and we may have

$$\left.\begin{aligned} R_x &= C_x + 2C_{x+1} + 3C_{x+2} + 4C_{x+3} + \&c. \\ \Sigma R_x &= C_x + 3C_{x+1} + 6C_{x+2} + 10C_{x+3} + \&c. \\ \Sigma^2 R_x &= C_x + 4C_{x+1} + 10C_{x+2} + 20C_{x+3} + \&c. \\ \Sigma^3 R_x &= C_x + 5C_{x+1} + 15C_{x+2} + 35C_{x+3} + \&c. \\ \&c. \qquad & \qquad \&c. \qquad\qquad \&c. \end{aligned}\right\} \quad . \; . \quad (27)$$

41. If, now, there be a series, the terms of which are u_0, u_1, u_2, &c., and if the successive terms be expressed by means of the initial term and its differences, by formula 8 of Chap. xxii, we shall have

$$\begin{aligned} u_0 &= u_0 \\ u_1 &= u_0 + \Delta u_0 \\ u_2 &= u_0 + 2\Delta u_0 + \Delta^2 u_0 \\ u_3 &= u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0 \\ u_4 &= u_0 + 4\Delta u_0 + 6\Delta^2 u_0 + 4\Delta^3 u_0 + \Delta^4 u_0 \\ \&c. \qquad & \qquad \&c. \qquad\qquad \&c. \end{aligned}$$

Here the coefficients of Δu_0 are the same as the coefficients of D in S_x; the coefficients of $\Delta^2 u_0$ the same as the coefficients of D in ΣS_x, and so on.

42. From the last preceding article we see that if there be a varying annuity, the payments of which are u_0 in the first year, u_1 in the second year, u_2 in the third year, &c., we may substitute for it a series of annuities, the first being a uniform annuity of u_0 per annum; the second an annuity deferred one year, and thereafter having for its successive payments Δu_0, $2\Delta u_0$, $3\Delta u_0$, &c.; the third an annuity deferred two years, and thereafter having for its successive payments $\Delta^2 u_0$, $3\Delta^2 u_0$, $6\Delta^2 u_0$, &c., and so on for the other annuities.

43. The extended commutation columns suggested in Arts. 39 and 40 afford the means of finding the value of the series of annuities. The value of the first annuity of the series is, $\frac{N_x}{D_x} u_0$. If the second annuity were to be entered on at once its value would be, $\frac{S_x}{D_x} \Delta u_0$; but as it is deferred a year, its value is, $\frac{S_{x+1}}{D_x}\Delta u_0$. In the same way the value of the third annuity of the series is, $\frac{\Sigma S_{x+2}}{D_x}\Delta^2 u_0$; and that of the fourth, $\frac{\Sigma^2 S_{x+3}}{D_x}\Delta^3 u_0$; and so on. Therefore, writing $(\mathrm{v}a)_x$ for the value of the varying annuity, the payments of which are u_0, u_1, u_2, &c., we have

$$(\mathrm{v}a)_x = \frac{1}{D_x}(N_x u_0 + S_{x+1}\Delta u_0 + \Sigma S_{x+2}\Delta^2 u_0 + \&c.) \quad . \quad . \quad (28)$$

44. Similarly, writing $(\mathrm{vA})_x$ for the varying assurance under which u_0 is to be paid if death occur in the first year, u_1 if in the second year, u_2 if in the third year, &c., we shall have

$$(\mathrm{vA})_x = \frac{1}{D_x}(M_x u_0 + R_{x+1}\Delta u_0 + \Sigma R_{x+2}\Delta^2 u_0 + \&c.) \quad . \quad (29)$$

45. To take examples:—Let it be required to find the value of an annuity, the successive payments of which are 4, 7, 12, 19, &c. Here $u_0=4$; $\Delta u_0=3$; $\Delta^2 u_0=2$; and the higher orders of differences vanish. Therefore the value of the annuity is $\frac{1}{D_x}(4N_x + 3S_{x+1} + 2\Sigma S_{x+2})$.

Again:—An assurance is to be granted on the life of (x) for 100 the first year; 101 the second year; 104 the third year; and, generally, for $100+(n-1)^2$ should death occur in the nth year. The annual premium is always to bear to the sum assured the ratio κ. To find κ. In this question, $u_0=100$; $\Delta u_0=1$; and $\Delta^2 u_0=2$. Therefore

$$\text{Benefit Side} = 100M_x + R_{x+1} + 2\Sigma R_{x+2}$$

$$\text{Payment Side} = \kappa(100N_{x-1} + S_x + 2\Sigma S_{x+1})$$

Whence
$$\kappa = \frac{100M_x + R_{x+1} + 2\Sigma R_{x+2}}{100N_{x-1} + S_x + 2\Sigma S_{x+1}}.$$

46. In the foregoing investigations the benefits have been supposed to be for the whole of life. If they be temporary, then corresponding sections of the commutation columns must be summed. Thus, if there be sought the value of an assurance for n years only, u_0 the first year, u_1 the second year, and so on, the simplest course will be to form a column of n values of C, from C_x to C_{x+n-1} inclusive, and to sum it from the bottom upwards to form a column of the values of $M_{x\overline{n}|}$, $M_{x+1:\overline{n-1}|}$, &c. This second column must then be summed from the

bottom upwards to give the corresponding column of R for temporary assurances, and so on. It is true that the ordinary columns as constructed for the whole of life might be used in intricate cases; but the terminal values required in the formulas would be of a character so complicated as to be practically unworkable.

47. Benefits *with return of premium* are an important class of varying benefits.

48. To find the single premium for an endowment, the premium to be returned in the event of the nominee dying before the age at which the endowment is payable.

Here the benefit consists of two portions;—1st, the endowment payable if the life survive n years; and 2nd, a temporary assurance for n years, the sum assured being the single premium paid. Therefore, writing E for the single premium, we have

$$\text{Benefit Side} = \text{D}_{x+n} + E(\text{M}_x - \text{M}_{x+n})$$

$$\text{Payment Side} = E\text{D}_x.$$

Whence $$E = \frac{\text{D}_{x+n}}{\text{D}_x - (\text{M}_x - \text{M}_{x+n})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

49. If the benefit be a deferred annuity instead of an endowment, in the numerator of the last equation, N_{x+n} will take the place of D_{x+n}.

50. If the question be to find the annual premiums for the foregoing benefits, the assurance portion will be an increasing temporary assurance consisting of the annual premiums, and we shall have, when π is the annual premium,

$$\text{Benefit Side} = \text{D}_{x+n} + \pi(\text{R}_x - \text{R}_{x+n} - n\text{M}_{x+n})$$

$$\text{Payment Side} = \pi(\text{N}_{x-1} - \text{N}_{x+n-1})$$

Whence $$\pi = \frac{\text{D}_{x+n}}{\text{N}_{x-1} - \text{N}_{x+n-1} - (\text{R}_x - \text{R}_{x+n} - n\text{M}_{x+n})} \quad . \quad . \quad (31)$$

51. The corresponding premium for a deferred annuity may be found by substituting N_{x+n} for D_{x+n} in the numerator of the last expression.

52. Formulas 30 and 31 give expressions for premiums which are to be returned if the benefit be never entered upon. We now proceed to find expressions for premiums which are to be returned along with the benefit contracted for; so that only the interest on the premiums will remain to the office as consideration for the benefit.

53. To find the single premium for a whole-life assurance, the premium to be returned along with the sum assured.

Writing $\mathcal{A}$ for the single premium required, we have

$$\text{Benefit Side} = \mathrm{A}_x(1+\mathcal{A})$$

$$\text{Payment Side} = \mathcal{A}$$

Whence $$\mathcal{A} = \frac{\mathrm{A}_x}{1-\mathrm{A}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The same result is given in Chap. vii, Art. 60.

Using commutation symbols, we have

$$\text{Benefit Side} = \mathrm{M}_x(1+\mathcal{A})$$

$$\text{Payment Side} = \mathrm{D}_x\mathcal{A}$$

Whence $$\mathcal{A} = \frac{\mathrm{M}_x}{\mathrm{D}_x-\mathrm{M}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

54. From equation 37 of Chap. vii, we see that the single premium now found is equal to $\frac{\mathrm{P}_x}{d}$: that is, the single premium for an assurance with return of premium is the same as the annual premium to secure a perpetuity the first payment of which is to be made at the end of the year of death.

55. If, instead of a whole-life assurance there be an endowment assurance, and if the premium is to be returned along with the sum assured at whatever time that may be paid, the equations are,

$$\text{Benefit Side} = (\mathrm{M}_x-\mathrm{M}_{x+n}+\mathrm{D}_{x+n})(1+\mathcal{A})$$

$$\text{Payment Side} = \mathrm{D}_x\mathcal{A}$$

Whence $$\left.\begin{aligned} \mathcal{A} &= \frac{\mathrm{M}_x-\mathrm{M}_{x+n}+\mathrm{D}_{x+n}}{\mathrm{D}_x-(\mathrm{M}_x-\mathrm{M}_{x+n}+\mathrm{D}_{x+n})} \\ &= \frac{\mathrm{A}_{x\overline{n}|}}{1-\mathrm{A}_{x\overline{n}|}} \\ &= \frac{\mathrm{P}_{x\overline{n}|}}{d} \end{aligned}\right\} \quad . \quad . \quad . \quad (34)$$

The last equation is perfectly analogous to the results obtained for whole-life assurances.

56. If the premium is to be returned only in the event of (x) dying within the n years, and not if he survive to receive the endowment, then we have

$$\text{Benefit Side} = (\mathrm{M}_x - \mathrm{M}_{x+n})(1+A) + \mathrm{D}_{x+n}$$

$$\text{Payment Side} = \mathrm{D}_x A$$

Whence
$$A = \frac{\mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n}}{\mathrm{D}_x - (\mathrm{M}_x - \mathrm{M}_{x+n})} \quad . \; . \; . \; . \; . \; . \; . \quad (35)$$

57. If the premium is to be returned only in the event of (x) surviving the n years, then

$$\text{Benefit Side} = \mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n}(1+A)$$

$$\text{Payment Side} = \mathrm{D}_x A$$

Whence
$$A = \frac{\mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n}}{\mathrm{D}_x - \mathrm{D}_{x+n}} \quad . \; . \; . \; . \; . \; . \; . \quad (36)$$

The three cases considered in equations 34, 35, and 36, respectively, must be carefully distinguished.

58. To find the annual premium for a whole-life assurance, all the premiums paid to be returned along with the sum assured. If π be the annual premium required, the benefit will consist of an ordinary assurance of 1, and an assurance commencing at π, and increasing by π yearly throughout life. Therefore

$$\text{Benefit Side} = \mathrm{M}_x + \pi \mathrm{R}_x$$

$\pi \ddot{a}_x = A_x + \pi(IA)_x$

$$\text{Payment Side} = \pi \mathrm{N}_{x-1}$$

Whence
$$\pi = \frac{\mathrm{M}_x}{\mathrm{N}_{x-1} - \mathrm{R}_x} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (37)$$

59. In the case of an endowment assurance, all the premiums to be returned along with the sum assured at whatever time that may be paid, the benefit consists of an ordinary endowment assurance of 1, a temporary assurance commencing at π and increasing π per annum, and an endowment of $n\pi$. Therefore

$$\text{Benefit Side} = \mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n} + \pi\{\mathrm{R}_x - \mathrm{R}_{x+n} + n(\mathrm{D}_{x+n} - \mathrm{M}_{x+n})\}$$

$$\text{Payment Side} = \pi(\mathrm{N}_{x-1} - \mathrm{N}_{x+n-1})$$

$\pi \ddot{a}_{x\overline{n|}} = A_{x\overline{n|}} + \pi(IA)^{1}_{x\overline{n|}} + n\pi A$

Whence

$$\pi = \frac{\mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n}}{\mathrm{N}_{x-1} - \mathrm{N}_{x+n-1} - \{\mathrm{R}_x - \mathrm{R}_{x+n} + n(\mathrm{D}_{x+n} - \mathrm{M}_{x+n})\}} \quad . \quad (38)$$

60. If the premiums are to be returned only in the event of (x) dying within the n years, and not in the event of his surviving to the end of the term, then we have

$$\text{Benefit Side} = \mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n} + \pi(\mathrm{R}_x - \mathrm{R}_{x+n} - n\mathrm{M}_{x+n})$$

$$\text{Payment Side} = \pi(\mathrm{N}_{x-1} - \mathrm{N}_{x+n-1})$$

$\pi\, a_{x\overline{n}|}$ $A_{x\overline{n}|}) + \pi(IA$

Whence

$$\pi = \frac{\mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n}}{\mathrm{N}_{x-1} - \mathrm{N}_{x+n-1} - (\mathrm{R}_x - \mathrm{R}_{x+n} - n\mathrm{M}_{x+n})} \quad . \quad . \quad (39)$$

61. If, on the other hand, the premiums are to be returned only in the event of (x) surviving the n years, then

$$\text{Benefit Side} = \mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n}(1 + n\pi)$$

$$\text{Payment Side} = \pi(\mathrm{N}_{x-1} - \mathrm{N}_{x+n-1})$$

$\pi\, a_{x\overline{n}|} = A^{1}_{x\overline{n}|} + A_{x\overline{n}|}(1$

Whence

$$\pi = \frac{\mathrm{M}_x - \mathrm{M}_{x+n} + \mathrm{D}_{x+n}}{\mathrm{N}_{x-1} - \mathrm{N}_{x+n-1} - n\mathrm{D}_{x+n}} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Again there are three cases which must be distinguished.

62. Instead of retaining the whole of the interest as consideration for the benefit, the office might agree to pay away part of it along with the sum assured and the returned premiums; and this might be done in either one of two ways. The office might grant simple interest on the premiums received, and retain the excess of compound over simple interest; or it might allow compound interest, but at a lower rate than that which it realizes on its funds.

63. To find the single premium for a whole-life assurance, the premium to be returned with simple interest at rate i along with the sum assured. Let the required single premium be denoted by A. The benefit consists of three portions:—1st, an assurance of 1; 2nd, a uniform assurance of A in respect of the single premium to be returned; and 3rd, an increasing assurance in respect of the interest, commencing at iA, and increasing iA yearly throughout life. Therefore we have

$$\text{Benefit Side} = \mathrm{M}_x(1 + A) + \mathrm{R}_x iA$$

$$\text{Payment Side} = \mathrm{D}_x A$$

Whence

$$A = \frac{\mathrm{M}_x}{\mathrm{D}_x - \mathrm{M}_x - i\mathrm{R}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

64. To find the annual premium for a whole-life assurance, all premiums paid to be returned with simple interest at rate i along with the sum assured.

Here, as in the last example, the benefit consists of three portions:—1st, an assurance of 1; 2nd, an increasing assurance in respect of the annual premium π, commencing at π, and increasing π yearly throughout life; and 3rd, an assurance in respect of the interest on the annual premiums. With regard to this third portion, the interest to be returned to the representatives of the assured in respect of the first premium will form an increasing assurance, commencing at $i\pi$, and increasing $i\pi$ yearly throughout life. In respect of the second premium, there will be a similar increasing assurance deferred one year. In respect of the third premium there will also be a similar increasing assurance deferred two years, and so on. These several increasing assurances taken together, will form an assurance of $i\pi$ if death occur in the first year, $3i\pi$ if death occur in the second year, $6i\pi$ if death occur in the third year, and so on. But we have seen in equation 27, that a column of ΣR_x will give the value of such an assurance. Therefore we have

$$\text{Benefit Side} = M_x + \pi R_x + i\pi \Sigma R_x$$

$$\text{Payment Side} = \pi N_{x-1}$$

Whence

$$\pi = \frac{M_x}{N_{x-1} - R_x - i\Sigma R_x} \quad \ldots\ldots \quad (42)$$

65. For endowment assurances the expressions become very complicated if ordinary commutation tables prepared for whole-life benefits be used. But by preparing beforehand special commutation tables, the expressions take exactly the forms of those in equations 41 and 42. If for the final value of M in the special table we take $C_{x+n-1} + D_{x+n}$ (the assurance being payable at age $x+n$ or previous death); for the value at the next younger $C_{x+n-2} + C_{x+n-1} + D_{x+n}$, and so on; that is, if for the special M column the ordinary C column be summed only up to C_{x+n-1}, and D_{x+n} be added to each of the values so formed; we shall have a table possessing, for endowment assurances, properties precisely the same as those possessed by the ordinary M column for whole-life assurances. This special M column may then be summed to form the special R column, which, in its turn, may be summed to form the special ΣR column. By the aid then of these special columns, equations 41 and 42 will apply to endowment assurances.

66. The following problem is a curious one.

An assurance is to be effected on (x), at an annual premium of π, subject to the condition that interest for each year on the premiums paid, up to and including the year of death, is to be allowed by the office at rate i, which is the rate realized by the office on its investments.

To find π.

It might be thought that, as the office pays on the premiums received interest at the rate it realizes on its funds, therefore the premium must be calculated independent of the rate of interest, and account must be taken only of the number of premiums which, on the average of such transactions, will be paid. From this point of view we should have

$$\text{Benefit Side}=1$$

$$\text{Payment Side}=\pi(1+e_x)$$

Whence
$$\pi=\frac{1}{1+e_x}.$$

A little consideration will, however, show that this result is erroneous. It assumes that the office will retain in hand the whole of the premiums, and invest them at interest, so as to earn the interest which it has contracted to pay to the assured. But such is not the case. Out of premiums received, the office will have to pay current claims, and therefore the funds it will have in hand will always be less than the premiums received, and will not be sufficient to earn the interest which has to be paid away. Therefore part of the interest paid away must be provided out of the premiums themselves, and the value found for π is too small. A correct solution must be sought. The benefit consists of three portions:—1st, an ordinary assurance on the life of (x); 2nd, an annuity, payable to (x) at the end of each year he survives, of the interest on the premiums, the annuity therefore commencing at $i\pi$, and increasing $i\pi$ per annum; and 3rd, an assurance consisting of the interest for the year of the death of (x), which assurance therefore also commences at $i\pi$ and increases $i\pi$ per annum. Therefore we have

$$\text{Benefit Side}=\mathrm{M}_x+i\pi(\mathrm{S}_x+\mathrm{R}_x)$$

$$\text{Payment Side}=\pi\mathrm{N}_{x-1}$$

Whence
$$\pi=\frac{\mathrm{M}_x}{\mathrm{N}_{x-1}-i\mathrm{S}_x-i\mathrm{R}_x}=\frac{\mathrm{M}_x}{\mathrm{N}_{x-1}-i\mathrm{S}_x-vi\mathrm{S}_{x-1}+i\mathrm{S}_x}$$

$$=\frac{\mathrm{M}_x}{\mathrm{N}_{x-1}-(1-v)\mathrm{S}_{x-1}}=\frac{\mathrm{M}_x}{v\mathrm{S}_{x-1}-\mathrm{S}_x}$$

$$=\frac{\mathrm{M}_x}{\mathrm{R}_x}\quad\dots\dots\dots\dots\dots\dots\quad(43)$$

Or the question may be looked at in another way, which gives the answer more simply. Since the office pays away full interest on the premiums, the effect is the same as if the premiums were receivable without interest at the same time as the claim is paid. Hence the payment to the office is equivalent to an assurance commencing at π and increasing π per annum. That is the Payment Side is πR_x, and the Benefit Side is M_x. Whence $\pi = \frac{M_x}{R_x}$.

67. Expressions may now be briefly considered for premiums for whole-life assurances, under the conditions that all premiums paid are to be returned, with compound interest at rate j, along with the sum assured, the premiums to be calculated at rate i; when of necessity i is greater than j.

68. Taking up first the case of the single premium, the benefit consists of two portions:—1st, the ordinary assurance $\frac{M_x}{D_x}$; and 2nd, an assurance in respect of the return of premium. With regard to this second portion, if death occur in the first year, the amount to be paid will be $A(1+j)$, the single premium required being A; and the present value thereof is $A.\frac{d_x}{l_x}\cdot\frac{(1+j)}{(1+i)}$. Similarly, the value of the return if death take place in the second year is $A.\frac{d_{x+1}}{l_x}\cdot\frac{(1+j)^2}{(1+i)^2}$; and so on.

Let $\frac{1+j}{1+i} = \frac{1}{1+J}$; so that $J = \frac{i-j}{1+j}$. Then the value of that portion of the benefit which consists of return of premium with interest becomes $A\left\{\frac{d_x}{l_x(1+J)} + \frac{d_{x+1}}{l_x(1+J)^2} + \frac{d_{x+2}}{l_x(1+J)^3} + \&c.\right\} = A\frac{M'_x}{D'_x}$, where M'_x and D'_x are calculated at rate of interest J. To find A we therefore have the equation

$$\text{Benefit Side} = \frac{M_x}{D_x} + A\frac{M'_x}{D'_x}$$

$$\text{Payment Side} = A$$

Whence
$$A = \frac{M_x \div D_x}{1 - M'_x \div D'_x} = \frac{A_x}{1 - A'_x} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where A'_x is calculated at rate of interest J.

69. To find the annual premium, π, under the same conditions, a very similar process must be followed. The value of the return in respect of the first annual premium will be $\pi\frac{M'_x}{D'_x}$; and in respect of the second,

$\pi w\dfrac{M'_{x+1}}{D'_x}$; and in respect of the third, $\pi w^2\dfrac{M'_{x+2}}{D'_x}$, and so on, when we write w for $\dfrac{1}{1+j}$; and the total value of the return will be $\pi\dfrac{M'_x+wM'_{x+1}+w^2M'_{x+2}+\&c.}{D'_x}$.

We therefore have

$$\text{Benefit Side}=\frac{M_x}{D_x}+\pi\frac{M'_x+wM'_{x+1}+w^2M'_{x+2}+\&c.}{D'_x}$$

$$\text{Payment Side}=\pi\frac{N_{x-1}}{D_x}$$

$$\text{Whence}\qquad \pi=\frac{M_x\div D_x}{\dfrac{N_{x-1}}{D_x}-\dfrac{M'_x+wM'_{x+1}+w^2M'_{x+2}+\&c.}{D'_x}}\quad .\ .\ (45)$$

70. In practice J will never be such a rate of interest as is to be found in ordinary tables, and therefore in formula 44 the value of A'_x will always have to be specially computed, and the most convenient course will be to use a formula of approximate summation. Any of the formulas of approximate summation given in Chap. xxiv may be employed. In the case of formula 45, the negative term in the denominator can be computed by constructing the commutation column of M'_x. This term may, however, be put in the form $\dfrac{C'_x+C'_{x+1}(1+a_{\overline{1}|})+C'_{x+2}(1+a_{\overline{2}|})+\&c.}{D'_x}$, to the calculation of which Lubbock's formula may be applied.

71. So far, questions have been discussed in which only the net premiums are to be returned, the *Net Premium* being that premium which, according to the mortality table and rate of interest adopted, will exactly provide the benefit contracted for.

72. But in all practical operations, an addition, usually called *Loading*, is made to the net premium. A fund must be provided for unavoidable expenses. A margin must be allowed in order to meet possible adverse fluctuations in the rates of mortality and interest. Moreover, it is generally desired to make a profit on the business. The net premium increased by the loading is called the *Office Premium*.

73. The loading may be computed in various ways. It may be purely arbitrary, and have no defined relation to the premium; but under such circumstances we have no data on which to base investigations. For the purpose of mathematical analysis the loading must be assumed, either directly or indirectly, to be a function of the premium. It may be

a percentage of the premium payable; or it may be a constant depending on the sum assured. Thus, if P_x be the net premium and P'_x the office premium, we may have $P'_x = P_x(1+\kappa)$; or $P'_x = P_x + c$. The most general case is, however, when the loading consists partly of a percentage and partly of a constant; when $P'_x = P_x(1+\kappa)+c$; as this case includes both the others. We shall assume this third method of loading, as either of the others may be derived from it, by making c or κ, as the case may be, equal to zero.

74. In practice, when a contract is made to return the premiums along with the sum assured, it is understood that the *office* premiums are to be returned, and not merely the *net* premiums; and in consequence of this change in the conditions, the formulas must also be modified. It will be convenient, as hitherto, to arrange the problem in the form of an equation, the Benefit Side and the Payment Side. By definition, the *net* premium is that premium which, according to the mortality table and rate of interest assumed, will exactly provide for the benefit; and it is therefore the *net* premium which must first be found, the *office* premium being afterwards derived therefrom. The *net* premium only, therefore, must appear on the Payment Side of the equation. On the other hand, the return of the *office* premium is part of the benefit; and therefore the *office* premium only must appear on the Benefit Side of the equation. For the purpose of solving the equation, the office premium must, however, be expressed in terms of the net premium and the loading.

75. It will be convenient to take up again, each in its order, the problems already solved for net premiums.

76. To find the single premium for an endowment, the *office* premium, E', to be returned in the event of the nominee dying before the age at which the endowment is payable; (*see* Art. 48). Let $E' = E(1+\kappa)+c$. Here we have

R'

$$\text{Benefit Side} = D_{x+n} + \{E(1+\kappa)+c\}(M_x - M_{x+n})$$

$$\text{Payment Side} = E D_x.$$

Whence
$$E = \frac{D_{x+n} + c(M_x - M_{x+n})}{D_x - (1+\kappa)(M_x - M_{x+n})} \quad . \quad . \quad . \quad . \quad (46)$$

and
$$E' = E(1+\kappa)+c.$$

77. If the question be to find the annual premium for the foregoing benefit, let the annual *net* premium be π, and the annual *office* premium π'; and let $\pi' = \pi(1+\kappa)+c$. Then we have, (*see* Art. 50),

$$\text{Benefit Side}=D_{x+n}+\{\pi(1+\kappa)+c\}\{R_x-R_{x+n}-nM_{x+n}\}$$

$$\text{Payment Side}=\pi(N_{x-1}-N_{x+n-1})$$

Whence

$$\pi=\frac{D_{x+n}+c(R_x-R_{x+n}-nM_{x+n})}{N_{x-1}-N_{x+n-1}-(1+\kappa)(R_x-R_{x+n}-nM_{x+n})} \quad . \quad (47)$$

and

$$\pi'=\pi(1+\kappa)+c.$$

78. If the benefit be a deferred annuity, N_{x+n} must take the place of D_{x+n} in the numerators of formulas 46 and 47.

79. To find the single premium for a whole-life assurance, the *office* premium, A', to be returned along with the sum assured. Let $A'=A(1+\kappa)+c$. Then we have, (*see* Art. 53),

$$\text{Benefit Side}=M_x\{1+A(1+\kappa)+c\}$$

$$\text{Payment Side}=AD_x$$

Whence

$$\left.\begin{aligned} A&=\frac{M_x(1+c)}{D_x-M_x(1+\kappa)} \\ &=\frac{A_x(1+c)}{1-A_x(1+\kappa)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and

$$A'=A(1+\kappa)+c.$$

80. To find the corresponding single premium for an endowment assurance we must, in equation 48, substitute $M_x-M_{x+n}+D_{x+n}$ or $A_{x\overline{n}|}$ for M_x or A_x respectively.

81. To find the annual premium for a whole-life assurance, all the *office* premiums paid to be returned along with the sum assured.

Let π' be the office premium, and π the net premium, and let $\pi'=\pi(1+\kappa)+c$. Then, (*see* Art. 58),

$$\text{Benefit Side}=M_x+\{\pi(1+\kappa)+c\}R_x$$

$$\text{Payment Side}=\pi N_{x-1}$$

Whence

$$\pi=\frac{M_x+cR_x}{N_{x-1}-(1+\kappa)R_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and

$$\pi'=\pi(1+\kappa)+c.$$

82. In the case of an endowment assurance, all the *office* premiums to be returned along with the sum assured at whatever time that may be paid, we have, (*see* Art. 59),

Benefit Side

$$=M_x-M_{x+n}+D_{x+n}+\{\pi(1+\kappa)+c\}\{R_x-R_{x+n}+n(D_{x+n}-M_{x+n})\}$$

Payment Side

$$=\pi(N_{x-1}-N_{x+n-1})$$

Whence

$$\pi=\frac{M_x-M_{x+n}+D_{x+n}+c\{R_x-R_{x+n}+n(D_{x+n}-M_{x+n})\}}{N_{x-1}-N_{x+n-1}-(1+\kappa)\{R_x-R_{x+n}+n(D_{x+n}-M_{x+n})\}}\ . \quad (50)$$

and $$\pi'=\pi(1+\kappa)+c.$$

83 To find the single premium for a whole-life assurance, the *office* premium to be returned with simple interest at rate i along with the sum assured. Let A' be the office single premium, and A the net single premium, and let $A'=A(1+\kappa)+c$. Then, (*see* Art. 63),

$$\text{Benefit Side}=M_x\{1+A(1+\kappa)+c\}+R_x i\{A(1+\kappa)+c\}$$

$$\text{Payment Side}=AD_x$$

Whence $$A=\frac{M_x(1+c)+R_x ic}{D_x-(1+\kappa)(M_x+R_x i)}\ . \ . \ . \ . \ . \ . \quad (51)$$

and $$A'=A(1+\kappa)+c.$$

84. To find the annual premium for a whole-life assurance, all the *office* premiums paid to be returned with simple interest at rate i along with the sum assured. Let π' be the office premium, and π the net premium, and let $\pi'=\pi(1+\kappa)+c$. Then, (*see* Art. 64),

$$\text{Benefit Side}=M_x+\{\pi(1+\kappa)+c\}\{R_x+i\Sigma R_x\}$$

$$\text{Payment Side}=\pi N_{x-1}$$

Whence $$\pi=\frac{M_x+c(R_x+i\Sigma R_x)}{N_{x-1}-(1+\kappa)(R_x+i\Sigma R_x)}\ . \ . \ . \ . \ . \ . \quad (52)$$

and $$\pi'=\pi(1+\kappa)+c.$$

85. To obtain for endowment assurances equations corresponding to Nos. 51 and 52 special commutation tables must be prepared, as explained in Art. 65.

86. To find the single and annual premiums for a whole-life assurance, all the *office* premiums paid to be returned with compound interest at rate j along with the sum assured, we must proceed in the way explained in Art. 68. The equation for the single premium is,

$$\text{Benefit Side} = \frac{\mathrm{M}_x}{\mathrm{D}_x} + \{A(1+\kappa)+c\}\frac{\mathrm{M}'_x}{\mathrm{D}'_x}$$

$$\text{Payment Side} = A$$

$$\text{Whence} \qquad A = \frac{\dfrac{\mathrm{M}_x}{\mathrm{D}_x} + c\dfrac{\mathrm{M}'_x}{\mathrm{D}'_x}}{1-(1+\kappa)\dfrac{\mathrm{M}'_x}{\mathrm{D}'_x}} = \frac{\mathrm{A}_x + c\mathrm{A}'_x}{-(1+\kappa)\mathrm{A}'_x} \quad . \quad . \quad . \quad . \quad (53)$$

$$\text{and} \qquad A' = A(1+\kappa)+c,$$

where A'_x is the ordinary whole-life assurance taken at rate of interest J, and where A' and A are respectively the office and the net single premiums sought.

87. The equation for the annual premium is

$$\text{Benefit Side} = \frac{\mathrm{M}_x}{\mathrm{D}_x} + \{\pi(1+\kappa)+c\}\frac{\mathrm{M}'_x + w\mathrm{M}'_{x+1} + w^2\mathrm{M}'_{x+2} + \&c.}{\mathrm{D}'_x}$$

$$\text{Payment Side} = \pi\frac{\mathrm{N}_{x-1}}{\mathrm{D}_x}$$

$$\text{Whence} \qquad \pi = \frac{\dfrac{\mathrm{M}_x}{\mathrm{D}_x} + c\,\dfrac{\mathrm{M}'_x + w\mathrm{M}'_{x+1} + w^2\mathrm{M}'_{x+2} + \&c.}{\mathrm{D}'_x}}{\dfrac{\mathrm{N}_{x-1}}{\mathrm{D}_x} - (1+\kappa)\dfrac{\mathrm{M}'_x + w\mathrm{M}'_{x+1} + w^2\mathrm{M}'_{x+2} + \&c.}{\mathrm{D}'_x}} \quad . \quad . \quad (54)$$

$$\text{and} \qquad \pi' = \pi(1+\kappa)+c,$$

where M′ and D′ are calculated at rate of interest J; and where π' and π are respectively the office and the net annual premiums sought.

88. All the formulas which have been investigated in this chapter are applicable to joint-life annuities and assurances when the proper joint-life commutation columns have been prepared.

89. In Arts. 17 and 18 of Chap. xiii, special commutation columns for contingent assurances were described. By adding a column R^1_{xy}, formed from M^1_{xy} exactly as R_x is formed from M_x, a variety of practical problems may be solved. David Chisholm supplies such a column for the Carlisle Mortality Table in his book of "Commutation Tables."

90. To find the single and annual premiums for a contingent assurance payable if (x) die before (y), the whole of the premiums paid to be returned if (y) die before (x).

If A and π be the single and annual premiums respectively, we have for the single premium

$$\text{Benefit Side} = \mathrm{M}_{\overset{1}{x}y} + A\,\mathrm{M}_{x\overset{1}{y}}$$

$$\text{Payment Side} = A\,\mathrm{D}_{xy}$$

Whence
$$A = \frac{\mathrm{M}_{\overset{1}{x}y}}{\mathrm{D}_{xy} - \mathrm{M}_{x\overset{1}{y}}} = \frac{\mathrm{A}_{\overset{1}{x}y}}{1 - \mathrm{A}_{x\overset{1}{y}}} \quad \dots\dots\dots \quad (55)$$

and for the annual premium

$$\text{Benefit Side} = \mathrm{M}_{\overset{1}{x}y} + \pi\mathrm{R}_{x\overset{1}{y}}$$

$$\text{Payment Side} = \pi\mathrm{N}_{x-1:y-1}$$

Whence
$$\pi = \frac{\mathrm{M}_{\overset{1}{x}y}}{\mathrm{N}_{x-1:y-1} - \mathrm{R}_{x\overset{1}{y}}} = \frac{\mathrm{A}_{\overset{1}{x}y}}{1 + a_{xy} - (\mathrm{IA})_{x\overset{1}{y}}} \quad \dots \quad (56)$$

In the absence of tables of $\mathrm{R}_{x\overset{1}{y}}$ the value of $(\mathrm{IA})_{x\overset{1}{y}}$ may be found by means of Lubbock's formula of approximate summation, No. 17 of Chap. xxiv.

91. If A' and π', the *office* premiums, are to be returned, we have

$$A = \frac{\mathrm{M}_{\overset{1}{x}y} + c\mathrm{M}_{x\overset{1}{y}}}{\mathrm{D}_{xy} - (1+\kappa)\mathrm{M}_{x\overset{1}{y}}} = \frac{\mathrm{A}_{\overset{1}{x}y} + c\mathrm{A}_{x\overset{1}{y}}}{1 - (1+\kappa)\mathrm{A}_{x\overset{1}{y}}} \quad \dots \quad (57)$$

and
$$A' = A(1+\kappa) + c$$

also
$$\pi = \frac{\mathrm{M}_{\overset{1}{x}y} + c\mathrm{R}_{x\overset{1}{y}}}{\mathrm{N}_{x-1:y-1} - (1+\kappa)\mathrm{R}_{x\overset{1}{y}}} = \frac{\mathrm{A}_{\overset{1}{x}y} + c(\mathrm{IA})_{x\overset{1}{y}}}{1 + a_{xy} - (1+\kappa)\,(\mathrm{IA})_{x\overset{1}{y}}} \quad \dots \quad (58)$$

and
$$\pi' = \pi(1+\kappa) + c.$$

92. To find the single premium, A, for a reversionary annuity to (x) after (y), the premium to be returned if (x) die before (y).

We have

$$\text{Benefit Side} = a_x - a_{xy} + A\,\mathrm{A}_{\overset{1}{x}y}$$

$$\text{Payment Side} = A$$

Whence
$$A = \frac{a_x - a_{xy}}{1 - \mathrm{A}_{\overset{1}{x}y}} \quad \dots\dots\dots \quad (59)$$

93. If the *office* premium A' is to be returned, we have

$$\text{Benefit Side} = a_x - a_{xy} + \{A(1+\kappa) + c\}\mathrm{A}_{\overset{1}{x}y}$$

$$\text{Payment Side} = A$$

Whence
$$A=\frac{a_x-a_{xy}+cA^1_{xy}}{1-(1+\kappa)A^1_{xy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and
$$A'=A(1+\kappa)+c.$$

94. To find the annual premium, π, for a reversionary annuity to (x) after (y), all the premiums paid to be returned if (x) die before (y). We have

$$\text{Benefit Side}=a_x-a_{xy}+\pi(\text{IA})^1_{xy}$$

$$\text{Payment Side}=\pi(1+a_{xy})$$

Whence
$$\pi=\frac{a_x-a_{xy}}{1+a_{xy}-(\text{IA})^1_{xy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

95. If the *office* premium, π', is to be returned, we have

$$\text{Benefit Side}=a_x-a_{xy}+\{\pi(1+\kappa)+c\}(\text{IA})^1_{xy}$$

$$\text{Payment Side}=\pi(1+a_{xy})$$

Whence
$$\pi=\frac{a_x-a_{xy}+c(\text{IA})^1_{xy}}{1+a_{xy}-(1+\kappa)(\text{IA})^1_{xy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and
$$\pi'=\pi(1+\kappa)+c.$$

96. Equations 59 to 62 may be translated into commutation symbols, remembering that $A^1_{xy}=\dfrac{M^1_{xy}}{D_{xy}}$, and $(\text{IA})^1_{xy}=\dfrac{R^1_{xy}}{D_{xy}}$.

97. It follows from equations 10, 11 and 12, of Chap. xiii, that M^1_{xy} and R^1_{xy} may be expressed in terms of N_{xy} and S_{xy} respectively. Thus, when the tables are in Davies's form,
and $x>y$,

$$\left.\begin{aligned}M^1_{xy}&=\tfrac{1}{2}\{v(N_{x-1:y-1}+N_{x-1:y})-(N_{x:y-1}+N_{xy})\}\\ R^1_{xy}&=\tfrac{1}{2}\{v(S_{x-1:y-1}+S_{x-1:y})-(S_{x:y-1}+S_{xy})\}\end{aligned}\right] \quad . \quad (63)$$

or $x<y$,

$$\left.\begin{aligned}M^1_{xy}&=\tfrac{1}{2}\{v(N_{x-1:y-1}-N_{x:y-1})+N_{x-1:y}-N_{xy}\}\\ R^1_{xy}&=\tfrac{1}{2}\{v(S_{x-1:y-1}-S_{x:y-1})+S_{x-1:y}-S_{xy}\}\end{aligned}\right] \quad . \quad (64)$$

And when the tables are in De Morgan's form, then in all cases

$$\left.\begin{aligned}M^1_{xy}&=\tfrac{1}{2}\{vN_{x-1:y-1}-N_{xy}+v^{\frac{1}{2}}(N_{x-1:y}-N_{x:y-1})\}\\ R^1_{xy}&=\tfrac{1}{2}\{vS_{x-1:y-1}-S_{xy}+v^{\frac{1}{2}}(S_{x-1:y}-S_{x:y-1})\}\end{aligned}\right] \quad . \quad (65)$$

98. It must be pointed out that in the case of benefits with return of premium, the single and the annual premiums are independent of each other, and must be independently calculated.

If no return of premium is to be made, then there is the universal equation $P(1+a)=A$, from which, if we have the values of A and a, we may at once derive P. But this relation does not hold for benefits with return of premium. Thus, for example, to take the case of a whole-life assurance, we found, equation 33, that $A=\frac{M_x}{D_x-M_x}$; and, equation 37, that $\pi=\frac{M_x}{N_{x-1}-R_x}$; and if we compare these results we at once see that A is *not* equal to $\pi(1+a_x)$. The reason is, that the benefit with return of *Single* premium is quite different from the benefit with return of *Annual* premium. In the case of the single premium the return is constant throughout life; but in the case of the annual premium the benefit is a constantly increasing one. If death occur early, only a small amount will be returned; but if death be long deferred, the return will be large.

CHAPTER XVII.

Successive Lives.

1. In the case of all the benefits which have been hitherto discussed, it is assumed that at the moment of entering into the contract all the lives involved are in existence, and are of the ages stated in the question. Thus, in speaking of a reversionary annuity to (y) after (x), $a_{x|y}$, the assumption is, that both (y) and (x) are alive at the time of the grant of the annuity, and that the annuity will become payable only if (y) survive (x). Similarly, in speaking of an assurance payable on the death of (y) if he die after (x), $\mathrm{A}_{x\overset{2}{y}}$, it is assumed that both (x) and (y) are alive, and of known ages, at the time of the grant of the policy.

2. There may, however, be an annuity to commence at the death of (x), and to continue during the lifetime of a person to be then nominated, and whose age is then to be y; or there may be an assurance payable on the death of a person who is to be nominated of age y at the death of (x). Benefits of this description are said to be on *Successive Lives.*

3. Instead of there being only two lives, there may be an unlimited number, each life to be nominated on the failure of the life immediately preceding it.

4. Let $\mathrm{A}_{\overline{0(1)(2)\ldots(n)}}$, or $\mathrm{A}_{\overline{x(y)(z)\&c.}}$, denote the value of an assurance to be paid on the death of the nth life to be nominated in succession after the life at present in possession.

Let $a_{\overline{0(1)(2)\ldots(n-1)}|(n)}$, or $a_{\overline{x(y)\&c}|(z)}$, denote the value of an annuity on the nth life to be nominated in succession after the life at present in possession.

Let $a_{0|\overline{(1)(2)\ldots(n)}}$, or $a_{x|\overline{(y)(z)\&c.}}$, denote the value of an annuity on all the n lives to be nominated in succession after the life at present in possession.

Let $a_{\overline{0(1)(2)\ldots(n)}}$, or $a_{\overline{x(y)(z)\&c.}}$, denote the value of an annuity on all the $n+1$ successive lives, including the life at present in possession.

Alternative symbols are adopted, so that when only the order of succession is to be indicated, the numerals may be used in the suffixes; while if the ages of the lives are to be more particularly designated, the letters may be used. The symbol 0, or x, will represent the life in possession, and it will not be placed in brackets. The symbols representing the succeeding lives will be placed in brackets, to show that the lives are to be nominated one after the other, and not at the time of entering on the transaction.

5. To find the value of an assurance payable at the end of the year in which the nth successive life to be nominated after the life in possession shall fail.

If n be equal to unity, that is, if only one life, to be aged y, is to be nominated after the life in possession at present aged x, the value of the reversion at the end of the year of the death of (x) will be A_y; because (y) is then to be nominated, and the sum assured will become payable at his death. Therefore the present value of the reversion in this case is $A_x \times A_y$. Similarly, if there be two lives (y) and (z), to be nominated in succession after the life in possession, the value of the reversion at the death of (y) will be A_z; and therefore its present value is $A_x \times A_y \times A_z$. Precisely the same reasoning applies whatever may be the number of successive lives, and we therefore have the general formula,

$$A_{\overline{0(1)(2)\ldots(n)}} = A_0 \times A_{(1)} \times A_{(2)} \ldots A_{(n)} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

6. If all the lives to be nominated in succession after the life in possession are to be aged y at the time of nomination, then formula 1 becomes

$$A_{\overline{0(1)(2)\ldots(n)}} = A_x \times (A_y)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

7. If a fine of 1 be payable at the end of the year in which the life in possession fails, and a like fine at the end of the year in which each successive life fails, there being $(n+1)$ fines in all,—to find the present value of all the fines.

The value sought is evidently the sum of the values of a series of $n+1$ assurances, the first payable on the failure of the life in possession, the second payable on the failure of the first succeeding life, and so on. Therefore if F represent the value of all the fines, we have, by formula 1,

$$F = A_0 + A_0 \times A_{(1)} + A_0 \times A_{(1)} \times A_{(2)} + \ldots + A_0 \times A_{(1)} \ldots A_{(n)} \quad . \quad (3)$$

8. Unless at the outset some supposition be made as to the ages of the lives to be nominated, it is not possible to express formula 3 in simpler terms. But in practice an average age for the successive lives is assumed, that average age being determined according to the particular circumstances of the case in hand. We then have $A_{(1)}=A_{(2)}=\&c.=A_{(n)}=A_y$, say; and the expression in equation 3 becomes a geometrical progression of which the first term is A_0 or A_x, the common ratio A_y, and the number of terms $n+1$; and it admits of easy summation. Therefore

$$F=A_x\frac{1-(A_y)^{n+1}}{1-A_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

9. If n be indefinitely great, $(A_y)^{n+1}$ vanishes, because A_y must always be less than unity; and the value of the fines in perpetuity is

$$F=\frac{A_x}{1-A_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

10. From the foregoing investigation it follows that the value of the nth and succeeding fines in perpetuity is

$$F=A_x\frac{(A_y)^{n-1}}{1-A_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

11. The most common forms of transaction in which fines on the failure of successive lives occur, arise in connection with copyhold estates, and leases on lives.

12. As defined in *Williams' Principles of the Law of Real Property*, "Copyhold lands are lands holden by Copy of Court Roll; " that is, the muniments of title to such lands are *copies* of the *roll* or " book in which an account is kept of the proceedings in the *Court* of the " Manor to which the lands belong. For all copyhold lands belong to " and are parcel of some Manor. * * * Copyholds are also said to " be holden according to the custom of the Manor to which they belong."

13. The customs of Manors differ; but with all Manors, on every change of tenancy, whether occasioned by the death of the tenant or by the sale of the land, fines of greater or less amount become payable to the lord. Sometimes, also, on the death of the lord a fine is payable to his successor by every tenant. In former times these fines were usually arbitrary in amount, but with the majority of Manors, they have now, by custom, become fixed; and even in the case of those Manors where the fines are still arbitrary, they are limited to two years' improved value of the land.

14. If the fines were payable to the lord, only on a change of tenancy arising through death, their value could be found by means of the foregoing formulas. In equations 4, 5, and 6, we should only have to assume a suitable value for y, such value depending on the average age of a tenant on his succeeding by inheritance to the property; and it would thus be comparatively easy to calculate the amount that should be paid down by a tenant in order to free the land from the burden of future fines. But seeing that fines are also payable when a change of tenancy occurs through alienation, the calculation cannot be made on this basis. The average duration of tenancies varies very much with different Manors, and depends to a great extent on the situation of the Manor; and therefore attention must be particularly given to the circumstances of each individual case. As, however, by statutes of 1852 and 1858 the copyholder was empowered to call upon the lord for enfranchisement of the land, it became important to establish some guide for the valuation of the lord's interest; and in 1855 a table was published by the Copyhold Commissioners, which, although not obligatory, is very often deferred to. (*See J.I.A.*, xxi, 385.)

15. In some parts of England the custom prevails of granting leases on lives. Land is held on lease for the life of a person named, not necessarily, nor usually, the life of the tenant himself; and in addition to an annual rent, which is sometimes nominal, the tenant, on failure of that life, pays to the landlord a fine of stipulated amount; and on doing so, has the right to insert a new life in the lease. Sometimes, instead of there being only one life at a time in the lease, there are several; and a fine is payable to the landlord on failure of any one of the lives, or on failure of the last survivor, for the privilege of inserting the name of a new nominee. As, under this custom, fines are payable only on the deaths of the persons named in the lease, and are not payable on alienation of the property by sale, the value of future fines can be calculated on the principles which have been investigated in this chapter. If the lease be held on only one series of lives, formulas 4 to 6 are applicable. If several series of lives are included in the lease, these formulas may be extended as follows:—

16. An estate is held on a certain number of lives in possession, x_1, x_2, x_3, &c., and each life is renewable at the end of the year of failure on payment of a fine of 1. Required the present value of all the future fines.

Let the successive lives to follow the life x_1, be denoted by y_1, z_1, &c.,

those to follow the life x_2, by y_2, z_2, &c.; and so on, then by formula 3, if F be the value of all the fines,

$$\begin{aligned} \mathrm{F} = {} & \mathrm{A}_{x_1} + \mathrm{A}_{x_1} \times \mathrm{A}_{y_1} + \mathrm{A}_{x_1} \times \mathrm{A}_{y_1} \times \mathrm{A}_{z_1} + \&c. \\ & + \mathrm{A}_{x_2} + \mathrm{A}_{x_2} \times \mathrm{A}_{y_2} + \mathrm{A}_{x_2} \times \mathrm{A}_{y_2} \times \mathrm{A}_{z_2} + \&c. \\ & + \quad \&c. \qquad\qquad \&c. \qquad\qquad \&c. \end{aligned}$$

17. In order to get a workable expression for the value, a uniform age must be assumed for the lives to succeed those in possession. If therefore we make $y_1 = z_1 = \&c. = y_2 = z_2 = \&c. = y_3 = z_3 = \&c. = \mathbf{x}$, we have, by formula 5,

$$\mathrm{F} = \frac{\mathrm{A}_{x_1} + \mathrm{A}_{x_2} + \mathrm{A}_{x_3} + \&c.}{1 - \mathrm{A}_{\mathbf{x}}} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

18. It is easy to express the assurances that occur in formula 7 in terms of the corresponding annuities. Assuming that there are n series of lives in the lease, then because $1 - \mathrm{A}_x = d(1 + a_x)$ we have from formula 7,

$$\begin{aligned} \mathrm{F} &= \frac{n(1-d) - d(a_{x_1} + a_{x_2} + a_{x_3} + \&c.)}{d(1 + a_{\mathbf{x}})} \\ &= \frac{\frac{n}{i} - (a_{x_1} + a_{x_2} + a_{x_3} + \&c.)}{1 + a_{\mathbf{x}}} \quad . \; . \; . \; . \; . \; . \; . \quad (8) \end{aligned}$$

19. In either equation, 7 or 8, we have an expression for the sum which the tenant should pay down to the landlord in order to be relieved of all future fines. As it is to the interest of the tenant to have the best possible lives inserted in the lease, it is usual for $\mathbf{x}$ in the above formula to be taken at the age at which the annuity is greatest by the table on which the valuation is made. The Carlisle Table has often been used for this purpose, and by it $a_{\mathbf{x}}$ is greatest at age 7. It is evident, however, that the assumed age is not likely always to be that of the lives selected. In fact, it is very usual to insert in leases the names of the members of the Royal Family, or of other prominent persons, so that there may be no difficulty in ascertaining whether or not a fine has become due by the failure of a life.

20. Passing now to annuities on successive lives, let it be required to find the value of an annuity on a life (y) to be nominated on failure of the life in possession, at present aged x.

It will be convenient to assume that the first payment of the annuity on the life succeeding (x) is to be made at the end of the year of the death of (x), at which moment also the life (y) is to be nominated. On this supposition, the value of the annuity on the nominated life will be $1+a_y$ at the end of the year of the death of (x); and therefore its present value is $A_x(1+a_y)$, which may be written $A_x\dfrac{1-A_y}{d}$. We therefore have the equation

$$a_{x|(y)}=A_x\frac{1-A_y}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

21. Similar reasoning applies in the case of a third life (z), to be nominated after the death of (y); and we have $a_{\overline{x(y)}|(z)}=A_xA_y\dfrac{1-A_z}{d}$; and generally, for the value of an annuity on the nth successive life

$$a_{\overline{0(1)(2)\ldots(n-1)}|(n)}=A_0A_{(1)}A_{(2)}\ldots A_{(n-1)}\frac{1-A_{(n)}}{d} \quad . \quad . \quad . \quad (10)$$

22. To find the value of the whole annuity on the $n+1$ lives, including the life in possession, we must in equation 10 give to n successively every value from 1 up to n, and add the sum of the results to the value of the annuity on the life in possession, which is $\dfrac{1-A_0}{d}-1$. It will be found that all the intermediate terms cancel one another, and that as a final result we have

$$a_{\overline{0(1)(2)\ldots(n)}}=\frac{1-A_0A_{(1)}A_{(2)}\ldots A_{(n)}}{d}-1$$

$$=\frac{1-A_{\overline{0(1)(2)\ldots(n)}}}{d}-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

23. It thus appears that the same relation holds between the annuity and the assurance on successive lives, as between the annuity and the assurance on a single life, or on joint lives. We might have arrived at this conclusion by simple reasoning, and without the aid of an algebraical demonstration. As in the case of a single life, the annuity is 1 per annum, and the assurance is certainly payable one year after the last payment of the annuity. The verbal demonstrations of the formulas, which were given in Chap. vii, Arts. 30 and 41 to 44, connecting A_x and a_x, are equally applicable to the benefits discussed in this chapter, and as a consequence the Conversion Tables of Chap. viii may be used. From the value of an annuity on all the successive lives we can at once find, by reference to the table, the value of the corresponding assurance.

24. In practice, as already mentioned, it is usual to assume that all the successive lives nominated after the life in possession will be of uniform age. If that age be y, and if the life in possession be aged x, formula 10 will become

$$a_{\overline{x(y)(y)\text{ to }(n-1)}|(y)}=\mathrm{A}_x(\mathrm{A}_y)^{n-1}\frac{1-\mathrm{A}_y}{d} \quad . \quad . \quad . \quad . \quad (12)$$

and formula 11 will become

$$a_{\overline{x(y)(y)\text{ to }(n)}}=\frac{1-\mathrm{A}_x(\mathrm{A}_y)^n}{d}-1 \quad . \quad . \quad . \quad . \quad (13)$$

25. To find the total present value of an annuity on any n lives that follow each other at intervals of t lives; that is, the sum of the values of the annuities on the mth, $(m+t)$th, $(m+2t)$th, &c., $(m+\overline{n-1}.t)$th lives, we must in equation 10 give to n successively the values m, $m+t$, $m+2t$, &c., $m+\overline{n-1}.t$, and take the sum. The result is

$$\begin{aligned}\frac{1}{d}\mathrm{A}_{\overline{0(1)(2)\text{ to }(m-1)}}\{(1-\mathrm{A}_{(m)})&+(1-\mathrm{A}_{(m+t)})\mathrm{A}_{(m)}\mathrm{A}_{(m+1)}\ \cdots\ \mathrm{A}_{(m+t-1)}\\ &+(1-\mathrm{A}_{(m+2t)})\mathrm{A}_{(m)}\mathrm{A}_{(m+1)}\ \cdots\ \mathrm{A}_{(m+2t-1)}\\ &+\quad \&c. \qquad \&c. \qquad \&c.\\ &+(1-\mathrm{A}_{(m+\overline{n-1}.t)})\mathrm{A}_{(m)}\mathrm{A}_{(m+1)}\ \cdots\ \mathrm{A}_{(m+\overline{n-1}.t-1)}\}\end{aligned}$$

If it be assumed as usual that the life in possession is aged x, and that all the successive lives will at the time of nomination be aged y, the last expression may be very much simplified. The assurance $\mathrm{A}_{\overline{0(1)(2)\text{ to }(m-1)}}$ becomes $\mathrm{A}_x(\mathrm{A}_y)^{m-1}$. Each of the factors $(1-\mathrm{A}_{(m+t)})$, $(1-\mathrm{A}_{(m+2t)})$, &c., becomes $(1-\mathrm{A}_y)$; and the coefficients of these factors become, respectively, 1, $(\mathrm{A}_y)^t$, $(\mathrm{A}_y)^{2t}$, &c., $(\mathrm{A}_y)^{\overline{n-1}\cdot t}$, thus constituting a geometrical progression of n terms, of which the first term is unity, and the common ratio is $(\mathrm{A}_y)^t$, and the sum of which is therefore $\frac{1-(\mathrm{A}_y)^{nt}}{1-(\mathrm{A}_y)^t}$. Under these circumstances the present value of the annuity on the n lives following each other at intervals of t lives, is therefore $\frac{1}{d}\mathrm{A}_x(\mathrm{A}_y)^{m-1}(1-\mathrm{A}_y)\frac{1-(\mathrm{A}_y)^{nt}}{1-(\mathrm{A}_y)^t}$.

26. As an example of the foregoing formulas, let it be required to find the value of the fines in the following case:—

An estate is held in perpetuity on successive lives, on the condition that from the end of the year in which each successive life drops a period of t years is to elapse before a new life is nominated, and a fine of 1 is to be paid at the beginning and at the end of each such period of t years.

Assuming that the life in possession is aged x, and that the succeeding lives will be nominated at age y, the values of the fines will be as follows:—1st fine A_x; 2nd fine v^tA_x; 3rd fine $v^tA_xA_y$; 4th fine $v^{2t}A_xA_y$; 5th fine $v^{2t}A_x(A_y)^2$; and so on. In order that this series may be summed it may conveniently be divided into two portions, the first, third, fifth, &c., fines being included in the first portion, and the second, fourth, sixth, &c., fines in the second portion. The former portion constitutes an infinite geometrical series, having A_x for its first term and v^tA_y for its common ratio; and therefore its sum is $\frac{A_x}{1-v^tA_y}$. The latter portion also constitutes an infinite geometrical series, having likewise v^tA_y for its common ratio, but having v^tA_x for its first term; and therefore its sum is $\frac{v^tA_x}{1-v^tA_y}$. By adding the two portions together we have $\frac{A_x(1+v^t)}{1-v^tA_y}$ for the value of all the fines.

27. As a second example we may find the total value of the annuities on n lives, each of which is to be nominated t years after the death of its predecessor.

It can easily be seen that the values of the successive annuities are $A_xv^t(1+a_y)$, $A_xv^{2t}A_y(1+a_y)$, $A_xv^{3t}(A_y)^2(1+a_y)$, &c. These form a geometrical progression of n terms of which the first term is $A_xv^t(1+a_y)$, and the common ratio v^tA_y. Therefore the total value of the annuities is $A_xv^t(1+a_y)\,\frac{1-v^{nt}(A_y)^n}{1-v^tA_y}$.

28. An important application of the doctrine of successive lives occurs in connection with Church Patronage. An advowson, as defined in *Williams' Principles of the Law of Real Property*, "is a perpetual "right of presentation to an ecclesiastical benefice. The owner of the "advowson is termed the patron of the benefice, but as such he has no "property or interest in the glebe or tithes, which belong to the "incumbent. As patron, he simply enjoys a right of nomination from "time to time, as the living becomes vacant. * * * The sale of an "advowson will not include the right to the next presentation unless "made when the Church is full; that is, before the right to present has "actually arisen by the death, resignation, or deprivation of the former "incumbent."

29. When the Church is full the next presentation may be sold independently of the advowson, but the sale of the right to present to a vacancy already existing, or of the next presentation if the present incumbent be in mortal sickness, would fall within the offence of simony, an offence which consists in the buying or selling of holy orders or of an ecclesiastical benefice.

30. Although the patron of a living, unless himself a clergyman, cannot derive a personal benefit from his property, yet as he may present his son, or nephew, or other relative or person, if duly qualified, the property is of pecuniary value; and advowsons and next presentations have very frequently been subjects of sale and purchase. In order that the income may be received, certain services must be rendered, the pecuniary remuneration of which may suitably be estimated at the payment that would have to be made to a "curate in charge" should it be necessary for the incumbent, from ill-health or other cause, to absent himself from his post. The stipend of a curate, in such a case, is not the smallest sum for which one can be found to perform the duties, but is regulated by law, and depends upon the population of the parish, and to a certain extent on the value of the living.* Remuneration for the services rendered is considered to be part of the unavoidable expenses connected with the property, and it is only the net income of the living after this has been provided for that should be taken into account in the valuation. Provision must also be made for "first fruits" which, in respect of certain benefices, have to be paid by the incumbent when he enters into possession; and for fees; and also for probable expenses from dilapidation, &c., because, although these last are, by the Ecclesiastical Dilapidations Act, 1871, primarily chargeable on the estate of the *late* incumbent, yet, as he may die worth very little or nothing, and any unsecured balance is, by the same Act, a debt due from the new incumbent which may be charged upon the benefice, therefore some allowance must generally be made under this head. These

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* The stipend of the curate in charge is regulated by statute, 1 & 2 Vict., c. 106, secs. LXXXV. & *seq.*, and the following is the scale:—

Lowest	£80
If the population is from 300 to 500	100
„ „ 500 „ 750	120
„ „ 750 „ 1,000	135
, „ 1,000 or over	150

But if the value of the living be large in proportion to the population, the Bishop has discretionary power to increase the stipend beyond the amounts above given; and in certain special cases the Bishop, with the concurrence of the Archbishop, may reduce the amounts.

charges must be estimated according to the circumstances of each case; and their aggregate amount is of the nature of a fine payable on the failure of each successive life.

31. As the income from a benefice is apportionable, the incumbent receiving it up to the day of his death, complete annuities should be used in the calculations. Moreover, by law a clergyman cannot be appointed to a living until he is at least twenty-four years of age, and therefore 24 is the minimum age that may be assumed for y, as the common age of the successive lives.

32. To find the value of the next presentation to a living, of which the annual income is s, the stipend of a curate in charge is κ, and the expenditure at entry is f.

At the end of the year of the death of (x), the present incumbent, the new incumbent will receive, on the average, half a year's income, out of which the charges mentioned in Art. 30 must be met; and he will then enjoy the whole of the free income as a complete annuity on his life. The present value of the next presentation, which may be written (NP), is therefore

$$(\mathrm{NP})=\mathrm{A}_x\left\{\left(\tfrac{1}{2}+\mathring{a}_y\right)(s-\kappa)-f\right\} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

33. On similar principles, the value of the next n presentations, which may be written $(\mathrm{NP})^{(n)}$, is

$$(\mathrm{NP})^{(n)}=\mathrm{A}_x(s-\kappa)\left\{\frac{1-(\mathrm{A}_y)^n}{d}-\tfrac{1}{2}\right\}-f\mathrm{A}_x\frac{1-(\mathrm{A}_y)^n}{1-\mathrm{A}_y} \quad . \quad (15)$$

and the value of the advowson, which may be written $(\mathrm{NP})^{(\infty)}$, is

$$(\mathrm{NP})^{(\infty)}=\mathrm{A}_x(s-\kappa)\left(\tfrac{1}{2}+a_\infty\right)-\frac{f\mathrm{A}_x}{1-\mathrm{A}_y} \quad . \quad . \quad . \quad (16)$$

34. It will be noticed that in obtaining formula 15, no account is taken of the proportion of income up to the day of the death of the last of the incumbents; but the value of this is very small, and in practical calculations may be discarded.

[35] In formulas 14, 15, and 16, it is assumed that the new incumbent will be presented at the end of the year of the death of his predecessor; that he will then receive half a year's income; and that he will enter on a complete yearly annuity. These assumptions are not in strict accordance with fact, and a better approximation will be obtained by supposing the

new incumbent to be presented at the moment of the death of his predecessor, and then to enter on a continuous annuity on his life. Modifying the formulas accordingly, we have for the next presentation,

$$(\mathrm{NP})=\bar{\mathrm{A}}_x\{(s-\kappa)\bar{a}_y-f\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

[36] For the next n presentations, we have

$$\begin{aligned}(\mathrm{NP})^{(n)}&=(\mathrm{NP})\{1+\bar{\mathrm{A}}_y+(\bar{\mathrm{A}}_y)^2+\ldots+(\bar{\mathrm{A}}_y)^{n-1}\}\\&=(\mathrm{NP})\,\frac{1-(\bar{\mathrm{A}}_y)^n}{1-\bar{\mathrm{A}}_y}\\&=\bar{\mathrm{A}}_x\left\{(s-\kappa)\,\bar{a}_y\frac{1-(\bar{\mathrm{A}}_y)^n}{1-\bar{\mathrm{A}}_y}-f\frac{1-(\bar{\mathrm{A}}_y)^n}{1-\bar{\mathrm{A}}_y}\right\}\end{aligned}$$

But $1-\bar{\mathrm{A}}_y=\delta\bar{a}_y$. Therefore

$$(\mathrm{NP})^{(n)}=\bar{\mathrm{A}}_x\left\{(s-\kappa)\,\frac{1-(\bar{\mathrm{A}}_y)^n}{\delta}-f\frac{1-(\bar{\mathrm{A}}_y)^n}{1-\bar{\mathrm{A}}_y}\right\} \quad . \quad . \quad (18)$$

[37] And lastly, making n infinite in equation 18, we have for the value of the advowson,

$$(\mathrm{NP})^{(\infty)}=\frac{\bar{\mathrm{A}}_x(s-\kappa)}{\delta}-\frac{\bar{\mathrm{A}}_x f}{1-\bar{\mathrm{A}}_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

38. It is very common, although by no means universal, for a residence to be attached to a benefice; and theoretically the annual value of the house, that is, the rent it would bring in if let on a repairing lease, should be added to the income of the benefice. In practice, however, it is usual, when a residence is attached to the benefice, not to take any account of it; and when there is no residence, to deduct from the income the rent of a house for the clergyman.

39. As an example of the foregoing, let it be required to find the values of the respective shares of A and B in an advowson which is held by them and their heirs as tenants in common, they having the right of alternate presentation.

Assuming that the net income of the benefice is unity, and that the expenses at entry of each incumbent are f, the value at entry of the interest of each incumbent is $(\frac{1}{2}+\mathring{a}_y-f)$. If A have the right of next presentation, the value of his interest in the advowson is therefore

$$(\tfrac{1}{2}+\mathring{a}_y-f)\{\mathrm{A}_x+\mathrm{A}_x(\mathrm{A}_y)^2+\mathrm{A}_x(\mathrm{A}_y)^4+\&c.\}\ \text{ or }\ (\tfrac{1}{2}+\mathring{a}_y-f)\,\frac{\mathrm{A}_x}{1-(\mathrm{A}_y)^2}.$$

The share of B is equal to that of A, except that in his case the right of presentation is on each occasion deferred one life; and therefore the value of his interest is $(\frac{1}{2}+\mathring{a}_y-f)\,\frac{A_xA_y}{1-(A_y)^2}$.

40. The correctness of these results may be proved by adding them together. Thus

$$
\begin{aligned}
&(\tfrac{1}{2}+\mathring{a}_y-f)\,\frac{A_x}{1-(A_y)^2}+(\tfrac{1}{2}+\mathring{a}_y-f)\,\frac{A_xA_y}{1-(A_y)^2}\\
&=(\tfrac{1}{2}+\mathring{a}_y-f)\,\frac{A_x(1+A_y)}{1-(A_y)^2}\\
&=(\tfrac{1}{2}+\mathring{a}_y-f)\,\frac{A_x}{1-A_y}\\
&=A_x\frac{\frac{1}{2}+a_y+\frac{1}{2}A_y}{1-A_y}-\frac{fA_x}{1-A_y}\\
&=A_x\frac{\frac{1}{2}+a_y+\frac{1}{2}-\frac{1}{2}d(1+a_y)}{d(1+a_y)}-\frac{fA_x}{1-A_y}\\
&=A_x\frac{1-\frac{1}{2}d}{d}-\frac{fA_x}{1-A_y}\\
&=A_x(\tfrac{1}{2}+a_\infty)-\frac{fA_x}{1-A_y},
\end{aligned}
$$

which is the value of the whole advowson as given in formula 16, when we write $s-\kappa=1$.

[41] If in solving the above example, the continuous formulas of Arts. 35 to 37 are used, we have for the share of A, $\bar{A}_x\frac{\bar{a}_y-f}{1-(\bar{A}_y)^2}$; and for the share of B, $\bar{A}_x\bar{A}_y\frac{\bar{a}_y-f}{1-(\bar{A}_y)^2}$. Adding these results together, we have for the value of the advowson $\bar{A}_x\frac{\bar{a}_y-f}{1-\bar{A}_y}$, or $\frac{\bar{A}_x}{\delta}-\frac{\bar{A}_xf}{1-\bar{A}_y}$, which agrees with formula 19 when $s-\kappa=1$.

CHAPTER XVIII.

POLICY-VALUES.

1. At the outset the present value of the benefit secured under a policy is, from the nature of the case, exactly equal to the present value of the premiums. With the lapse of time, however, the conditions alter. The date at which the benefit will become payable approaches, and the value of the benefit increases; and on the other hand the number of premiums still remaining to be paid diminishes, and the value of the premiums decreases. Therefore the present value of the benefit becomes greater than that of the premiums, and the difference constitutes the *Value of the Policy*.

2. Let ${}_nV_x$ represent the value of an ordinary whole-life policy of 1, effected exactly n years ago, on a life then aged x, at an annual premium of P_x. At the outset the value of the benefit was A_x, and the value of the premiums $P_x(1+a_x)$; and there existed the equation $A_x=P_x(1+a_x)$. Therefore at that moment the policy had no value. Now, however, after n years, the life assured being aged $x+n$, and the $(n+1)$th premium being just due, the value of the benefit has increased to A_{x+n}, and the value of the premiums has diminished to $P_x(1+a_{x+n})$; and

$$ {}_nV_x=A_{x+n}-P_x(1+a_{x+n}) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

3. The value of ~~the~~ policy is thus seen to be the sum which the Assurance Office must have in hand to provide for that portion of the liability under the contract which the future premiums will not cover; and the only source from which it can be derived is the accumulations of the balance of past premiums not absorbed by the risk already incurred.

4. In deducing equation 1, attention was fixed exclusively on the future; but the same result will be arrived at by considering only the

past. Applying the reasoning of Chap. xvi, Arts. 9 to 15, the value of the policy consists of the portion assignable to that particular contract of the difference between the accumulated premiums and the accumulated claims in respect of all the entrants. That is

$$ {}_nV_x = \frac{P_x(N_{x-1}-N_{x+n-1})-(M_x-M_{x+n})}{D_{x+n}} \quad . \quad . \quad . \quad (2) $$

5. It is easy to show that the expressions for ${}_nV_x$ given in equations 1 and 2 respectively are identical. By Art. 27 of Chap. xvi, $P_xN_{x-1}=M_x$; and therefore these terms may be omitted from equation 2, which then becomes ${}_nV_x=\frac{M_{x+n}-P_xN_{x+n-1}}{D_{x+n}}$; or $=A_{x+n}-P_x(1+a_{x+n})$. The policy-value in equation 1 was found by the *Prospective Method;* and in equation 2 by the *Retrospective Method.* These two methods are of great and equal importance.

6. It may be useful to the student to give him a numerical illustration of the fact that the Prospective and Retrospective methods of obtaining policy-values produce identical results. Assuming that each of 89685 ($=l_{30}$) persons, all aged exactly 30, effects on a given day a policy of 1 on his life, at an annual premium of ·01873208, the following table, A, shows the operation of the fund for 10 years at 3 per-cent interest. The premiums (Col. 3) and the interest (Col. 5) are passed to the credit of the fund, and the claims (Col. 7) are withdrawn; and in Col. 8 the amount of the fund at the end of each year is shown. The number of surviving members ($=l_{30+n}$) is entered in Col. 9, and the share of each in the fund is shown in Col. 10. It will be found that ${}_nV_{30}$ calculated at 3 per-cent by either of the formulas 1 or 2, is exactly equal to the corresponding number in Col. 10.

7. In Table B, Col. 2 shows the number of survivors at the end of each year, which is also the total amount remaining assured at the age then attained by the lives; and in Col. 4 the present value of that amount is displayed; the value of the reversion being given for convenience in Col. 3. The aggregate of the annual premiums just due appears in Col. 5, and the value of the future premiums in Col. 7; the annuity-due used as multiplier being entered in Col. 6. The difference between the value of the sums assured and the value of the future premiums is entered in Col. 8; and this represents the liability under the contracts. It will be seen that, with insignificant differences due to the last place of decimals in the annuity-values being only approximately correct, the figures in

TABLE A.

Showing the progress of a Fund formed by the Contributions of 89,685 *Members, each assuring for the Whole of Life at Age* 30 *for* 1, *at an Annual Premium of* ·01873208.—*Interest* 3 *per-cent.*

Retrospective Method.

(1) Year n	(2) Fund at Commencement of Year.	(3) Premiums paid at Commencement of Year.	(4) Sum of (2) and (3).	(5) Interest earned in Year.	(6) Sum of (4) and (5).	(7) Claims in Year.	(8) Fund at End of Year.	(9) Survivors at End of Year.	(10) (8) ÷ (9) $= {}_nV_{30}$	(11) Year n
1	0000·000	1679·987	1679·987	50·400	1730·387	691	1039·387	88994	·01168	1
2	1039·387	1667·043	2706·430	81·193	2787·623	700	2087·623	88294	·02364	2
3	2087·623	1653·930	3741·553	112·247	3853·800	709	3144·800	87585	·03590	3
4	3144·800	1640·649	4785·449	143·564	4929·013	719	4210·013	86866	·04847	4
5	4210·013	1627·181	5837·194	175·116	6012·310	729	5283·310	86137	·06133	5
6	5283·310	1613·525	6896·835	206·905	7103·740	742	6361·740	85395	·07450	6
7	6361·740	1599·626	7961·366	238·841	8200·207	756	7444·207	84639	·08795	7
8	7444·207	1585·465	9029·672	270·890	9300·562	770	8530·562	83869	·10171	8
9	8530·562	1571·041	10101·603	303·048	10404·651	786	9618·651	83083	·11577	9
10	9618·651	1556·317	11174·968	335·249	11510·217	806	10704·217	82277	·13010	10

Table B.

Showing the Liability for each Year under the Policies of 89,685 *Persons, each assuring at Age* 30 *for* 1, *at an Annual Premium of* ·01873208.—*Interest* 3 *per-cent.*

Prospective Method.

(1) Year n	(2) Survivors at End of Year.	(3) A_{30+n}	(4) Value of Total Sums Assured.	(5) Amount of Premiums just due.	(6) $(1+a_{30+n})$	(7) Value of Future Premiums.	(8) (4)—(7), being Total Liability.	(9) Liability under each Contract $={}_nV_{30}$	(10) Year n
0	89685	·3914073	35103·363	1679·987	20·89502	35103·363	0·000	·00000	0
1	88994	·3985152	35465·462	1667·043	20·65098	34426·072	1039·390	·01168	1
2	88294	·4057968	35829·423	1653·930	20·40098	33741·793	2087·630	·02364	2
3	87585	·4132592	36195·307	1640·649	20·14477	33050·497	3144·810	·03590	3
4	86866	·4209031	36562·169	1627·181	19·88233	32352·150	4210·019	·04847	4
5	86137	·4287360	36930·033	1613·525	19·61340	31646·711	5283·322	·06133	5
6	85395	·4367460	37295·925	1599·626	19·33839	30934·191	6361·734	·07450	6
7	84639	·4449344	37658·803	1585·465	19·05725	30214·603	7444·200	·08795	7
8	83869	·4533089	38018·564	1571·041	18·76973	29488·015	8530·549	·10171	8
9	83083	·4618647	38373·105	1556·317	18·47598	28754·482	9618·623	·11577	9
10	82277	·4705851	38718·330	1541·219	18·17658	28014·090	10704·240	·13010	10

Col. 8 of Table B are the same as those in Col. 8 of Table A, thus showing that the liability under the contracts is identical with the balance of the accumulations of the premiums and claims. The liability under each individual contract appears in Col. 9 of Table B; and it is again seen to be equal to ${}_nV_x$.

8. In Table B, year 0 means the moment before the policies are effected; and it is inserted in order to make the table complete. It shows that just before the first premium is paid the policy has no value,—that the sum assured and the future premiums are of equal present worth.

9. The value of the policy is formed of the accumulations of the balance of past premiums not absorbed by the risk already incurred; and from this we may learn that the reason why policies have any value is, that the progressively increasing risk from age to age is paid for by an average premium, uniform throughout life. The average premium must always be greater in the early years of assurance than is required for the risk in those years; and the surplus part of the premium which remains in hand after providing for the risk must be husbanded, and accumulated at the fundamental rate of interest, in order that it may provide for the years in which the value of the risk is greater than the premium. If the policyholder paid merely the value of each year's risk, the policy would possess no value. If, on the other hand, he paid a premium without the office incurring any present risk, as in the case of endowments, the value of the policy would be the accumulated amount of the premiums paid by all the assured, apportioned amongst those who survived.

10. These principles will be more easily grasped if further numerical examples be given; and therefore three tables, C, D and E, are added, showing the progress of a fund formed by persons effecting Term Assurances, Endowment Policies, and Endowment Assurances, respectively.

11. In the case of Term Assurances, (Table C), the average premium is only slightly in excess of the value of the risk at the commencement, and consequently the policies have very small values; and it will be seen from the example that the value, after increasing for a time, diminishes again, until at the end of the term, it entirely disappears. In this particular example, the average premium is greater for five years than the value of the risk; and until the end of the fifth year the funds increase. After the fifth year the increasing rate of mortality causes the value of the risk to become greater than the average premium; and the excess of premium of former years begins to be drawn upon, and the funds diminish. It is evident that the excess in hand is not profit, but only a necessary reserve for liabilities which will certainly mature; therefore the excess must be scrupulously husbanded and improved at interest, because otherwise the fund would inevitably become insolvent.

12. In the case of Endowments, there is no risk incurred until the end of the term, and then the whole liability matures at once. Table D shows that, therefore, the whole of the premiums paid by all the members must be kept in hand, and accumulated at compound interest; and that if this is done, at the end of the term there will be invested a sum exactly sufficient to provide the endowment for each of the survivors. The policy-values are large, and increase rapidly throughout the whole term. They are larger than would be produced by the accumulation of an annuity-certain of the premium, because the premiums are accumulated with benefit of survivorship.

13. Endowment Assurances are a combination of Term Assurances and Endowments. The risk incurred in each year is small compared with the average premium, and the policy-values are large, increasing rapidly until the end of the term, when the major portion of the liabilities matures. Table E could be formed by combining Tables C and D, or it could be calculated independently.

TABLE C.

Showing the progress of a Fund formed by the Contributions of 89,685 *Members, each assuring for* 1 *at age* 30 *for* 10 *Years only, at an Annual Premium of* ·00826797. *Interest* 3 *per-cent.*

(1) Year n	(2) Funds at Commencement of Year.	(3) Premiums paid at Commencement of Year.	(4) Sum of (2) and (3).	(5) Interest earned in Year.	(6) Sum of (4) and (5).	(7) Claims in Year.	(8) Funds at End of Year.	(9) Policy Value $_nV^1_{30:\overline{10}\vert}$
1	00·000	741·511	741·511	22·245	763·756	691	72·756	·00082
2	72·756	735·800	808·556	24·256	832·812	700	132·812	·00150
3	132·812	730·010	862·822	25·885	888·707	709	179·707	·00205
4	179·707	724·150	903·857	27·116	930·973	719	211·973	·00244
5	211·973	718·204	930·177	27·905	958·082	729	229·082	·00266
6	229·082	712·176	941·258	28·238	969·496	742	227·496	·00266
7	227·496	706·042	933·538	28·006	961·544	756	205·544	·00243
8	205·544	699·791	905·335	27·160	932·495	770	162·495	·00194
9	162·495	693·425	855·920	25·678	881·598	786	95·598	·00115
10	95·598	686·927	782·525	23·475	806·000	806	00·000	·00000

(10)
Survivors

TABLE D.

Showing the progress of a Fund formed by the Contributions of 89,685 *Members, each taking out an Endowment of* 1, *at age* 30, *payable at the end of* 10 *Years, at an Annual Premium of* ·08043157. *Interest* 3 *per-cent.*

741·511
7213·505
7955·016

(1) Year n	(2) Funds at Commencement of Year.	(3) Premiums paid at Commencement of Year.	(4) Sum of (2) and (3).	(5) Interest earned in Year.	(6) Funds at End of Year.	(7) Policy Value ${}_nV^{\,1}_{30:\overline{10}|}$
1	000·000	7213·505	7213·505	216·405	7429·910	·08349
2	7429·910	7157·927	14587·837	437·635	15025·472	·17017
3	15025·472	7101·625	22127·097	663·813	22790·910	·26021
4	22790·910	7044·599	29835·509	895·065	30730·574	·35378
5	30730·574	6986·769	37717·343	1131·520	38848·863	·45101
6	38848·863	6928·134	45776·997	1373·310	47150·307	·55214
7	47150·307	6868·454	54018·761	1620·563	55639·324	·65737
8	55639·324	6807·649	62446·973	1873·411	64320·384	·76692
9	64320·384	6745·716	71066·100	2131·984	73198·084	·88103
10	73198·084	6682·497	79880·581	2396·419	82277·000	1·00000

TABLE E.

Showing the progress of a Fund formed by the Contributions of 89,685 *Members, each effecting an Endowment Assurance Policy for* 1, *at age* 30, *payable at the end of* 10 *Years or on previous death, at an Annual Premium of* ·08869954. *Interest* 3 *per-cent.*

(1) Year n	(2) Funds at Commencement of Year.	(3) Premiums paid at Commencement of Year.	(4) Interest earned in Year.	(5) Sum of (2), (3), and (4).	(6) Claims in Year.	(7) Funds at End of Year.	(8) Policy Value ${}_nV_{30:\overline{10}|}$
1	000·000	7955·016	238·650	8193·666	691	7502·666	·08431
2	7502·666	7893·727	461·891	15858·284	700	15158·284	·17167
3	15158·284	7831·635	689·698	23679·617	709	22970·617	·26226
4	22970·617	7768·749	922·181	31661·547	719	30942·547	·35622
5	30942·547	7704·973	1159·425	39806·945	729	39077·945	·45367
6	39077·945	7640·310	1401·548	48119·803	742	47377·803	·55480
7	47377·803	7574·496	1648·569	56600·868	756	55844·868	·65980
8	55844·868	7507·440	1900·571	65252·879	770	64482·879	·76886
9	64482·879	7439·141	2157·662	74079·682	786	73293·682	·88218
10	73293·682	7369·424	2419·894	83083·000	806	82277·000	1·00000

14. In the preceding tables it has been shown numerically, that according to the mortality table at the end of this volume, for age at entry 30, at 3 per-cent interest, and n being not greater than 10, the following relation holds:—

$$(l_{x+n}\cdot{}_nV_x+l_{x+n}\cdot P_x)(1+i)=d_{x+n}+l_{x+n+1}\cdot{}_{n+1}V_x\,;$$

which may be written

$${}_nV_x+P_x=v(q_{x+n}+p_{x+n}\cdot{}_{n+1}V_x)\,.$$

This equation is of some importance, because it shows that the reserve held against a policy at the beginning of a year, together with the premium then paid, is equal to the present value of the sum assured payable at the end of the year should the life fail, together with the present value of the reserve which will have to be set aside at the end of the year should the life survive. It will be well therefore to remove the restrictions of the numerical example, and prove that the relation holds universally for every table of mortality, rate of interest, age at entry, and duration of policy.

15. Writing ${}_nV_x$ and P_x in terms of the annuities, we have, since by equation 1, ${}_nV_x=A_{x+n}-P_x(1+a_{x+n})=1-(P_x+d)(1+a_{x+n})$ $=1-\dfrac{1+a_{x+n}}{1+a_x}=\dfrac{a_x-a_{x+n}}{1+a_x}$,

$$\begin{aligned}{}_nV_x+P_x&=\frac{a_x-a_{x+n}}{1+a_x}+\frac{1}{1+a_x}-d\\&=\frac{a_x-a_{x+n}}{1+a_x}-\frac{a_x}{1+a_x}+v\\&=v-\frac{a_{x+n}}{1+a_x}\\&=v-vp_{x+n}\frac{1+a_{x+n+1}}{1+a_x}\\&=v-vp_{x+n}(1-{}_{n+1}V_x)\\&=v(q_{x+n}+p_{x+n}\cdot{}_{n+1}V_x)\,.\quad .\quad .\quad .\quad .\quad (3)\end{aligned}$$

16. By following different lines of reasoning we may arrive at expressions for the value of a policy other than those of formulas 1 and 2.

17. If the unit assured were payable at once, its value would be 1; but it is not payable until the end of the year of the death of the life assured aged $x+n$, and we must therefore deduct $d(1+a_{x+n})$, the value of the interest as shown in Chap. vii, Art. 43. The receipt of the sum assured is also burdened with the condition that a premium of P_x must be paid at the beginning of each year until the failure of the life assured;

and therefore $P_x(1+a_{x+n})$, the value of the premiums, must also be deducted. The remainder is the policy-value. That is,

$$_nV_x = 1-(P_x+d)(1+a_{x+n}) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The analogy between the policy-value and the value of a reversion (*see* Chap. vii, formula 25) is apparent. The reversion is merely an assurance under which there are no more premiums payable.

18. Again:—A person aged $x+n$ is entitled to a policy on his life, under which a premium of only P_x is payable. If he were to apply afresh to the assurance office he would be called upon to pay a premium of P_{x+n}. The value of the existing contract is therefore the present value of the difference between the two premiums. That is,

$$_nV_x = (P_{x+n}-P_x)(1+a_{x+n}) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

19. Again:—A person aged $x+n$ applying afresh to an assurance office, and offering a premium of P_x, would receive a policy for $\dfrac{P_x}{P_{x+n}}$; because, as P_{x+n} is the premium at age $x+n$ to secure 1, $\dfrac{1}{P_{x+n}}$ is the sum which a premium of 1 will secure, and $\dfrac{P_x}{P_{x+n}}$ is the sum which the premium P_x will secure. Therefore, if there is an existing policy for 1, under which a premium of P_x is payable, only $\dfrac{P_x}{P_{x+n}}$ of the amount is secured by the future premiums, and the difference, $1-\dfrac{P_x}{P_{x+n}}$, must be provided for by the funds in hand—that is, by the value of the policy. The present value of this difference is therefore the value of the policy. That is,

$$_nV_x = A_{x+n}\left(1-\frac{P_x}{P_{x+n}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

20. Lastly:—If a person aged x invest the sum of 1 in an annuity-due on his life, the amount he will receive at the beginning of each year will be $\dfrac{1}{1+a_x}$; and, by Chap. vii, Art. 61, this annual amount will consist of d, the interest on the unit during the life of (x), and P_x, the premium required to secure a return of the unit at his death. If, therefore, at the moment of purchasing the annuity of P_x+d, (x) also effects a policy of 1 on his life, the value of the two contracts together will always be 1, because the annuity will provide the interest on the investment and also

the premium payable annually to the assurance office, and the policy will secure the return of the sum invested when the annuity ceases. After n years, therefore, the value of the policy alone will be 1, less the value of the remainder of the annuity. But the value of the remainder of the annuity is $\frac{1+a_{x+n}}{1+a_x}$; and therefore

$$ {}_nV_x = 1 - \frac{1+a_{x+n}}{1+a_x} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

21. Equation 7 may be put in the form

$$ {}_nV_x = \frac{a_x - a_{x+n}}{1+a_x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

which shows that the value of the policy is proportional to the *fall* in the value of the annuity.

22. When the same mortality table and the same rate of interest are used, all the foregoing expressions for ${}_nV_x$ yield identical results, and the student will have no difficulty in proving that, by mere algebraical transformation, each of the expressions may be derived from any one of the others. Also, he will be able easily to show that

$$ {}_nV_x = 1 - \frac{1-A_{x+n}}{1-A_x} \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

$$ = \frac{A_{x+n} - A_x}{1-A_x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

$$ = 1 - \frac{P_x + d}{P_{x+n} + d} \quad . \; . \; . \; . \; . \; . \; . \quad (11)$$

$$ = \frac{P_{x+n} - P_x}{P_{x+n} + d} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

23. The value of a policy taken out at age x, and which has been $m+n$ years in force, may be conveniently expressed in terms of the values of two policies, one effected at age x, and which has been m years in force, and the other effected at age $x+m$, and which has been n years in force. Thus:—

$$ 1 - {}_{m+n}V_x = \frac{1+a_{x+m+n}}{1+a_x}$$

$$ = \frac{1+a_{x+m}}{1+a_x} \cdot \frac{1+a_{x+m+n}}{1+a_{x+m}}$$

$$ = (1 - {}_mV_x)(1 - {}_nV_{x+m}).$$

Whence $$ {}_{m+n}V_x = 1 - (1 - {}_mV_x)(1 - {}_nV_{x+m}) \quad . \; . \; . \; . \quad (13)$$

24. A similar relation holds if there be more terms of years than two. If, therefore, there be n terms of one year each, we shall have

$$_{n}V_{x}=1-(1-{}_{1}V_{x})(1-{}_{1}V_{x+1})\ldots(1-{}_{1}V_{x+n-1}) \quad . \quad . \quad (14)$$

25. It has been proved, both algebraically and by numerical example, that the prospective and retrospective methods of obtaining the policy-value give identical results; but in the process it has been tacitly assumed that the valuation has been made on the *net premium* method—that is, that all the annuities, reversions, and premiums, involved in the expressions are based upon the same mortality table, and calculated at the same rate of interest as the premium payable under the policy; and that the premium payable under the policy does not carry any loading. If these conditions do not hold, then the prospective and retrospective methods produce very different policy-values. Thus, for instance, if the premium, P_x, payable under the policy, be loaded to the extent of ϕ_x, by the prospective method we should have

$$_{n}V'_{x}=A_{x+n}-(P_{x}+\phi_{x})(1+a_{x+n})={}_{n}V_{x}-\phi_{x}(1+a_{x+n});$$

and by the retrospective method we should have

$$_{n}V''_{x}=\frac{(P_{x}+\phi_{x})(N_{x-1}-N_{x+n-1})-(M_{x}-M_{x+n})}{D_{x+n}}={}_{n}V_{x}+\phi_{x}\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}.$$

It thus appears that in the case of a loaded premium, the value of a policy by the prospective method is smaller than the net premium value, while by the retrospective method it is larger. In the case of the prospective method the whole of the future loading is discounted, and treated as an asset, and deducted from the liability; while in the case of the retrospective method, the loading paid in the past is accumulated at interest with benefit of survivorship, and treated as an addition to the liability.

26. In making the valuations of an Assurance Company it is usual to employ the prospective method, and, in general, to discard the loading and deal only with net premiums. Sometimes, however, the net premiums are not those valued. Perhaps from the office premium a fixed percentage is thrown off, which may be either greater or less than the loading included in the office premium; or perhaps the net premium calculated by one table of mortality is valued by the annuity calculated by another. These two cases are identical in principle, because in each there is a difference—positive or negative—between the premium valued and the net premium according to the mortality table and rate of interest

used in the valuation; and therefore the two cases, although they arise in different circumstances, may be treated as one.

27. To find the relation between the resulting policy-value where net premiums are not valued, and the policy-value by the net premium method. Let P_x be the ordinary net premium, and ${}_nV_x$ the ordinary net policy-value, P'_x the altered premium, and ${}_nV'_x$ the corresponding altered policy-value; and let $P'_x = P_x + \phi$, where ϕ may be either positive or negative. Then,

$$\begin{aligned} {}_nV_x &= (P_{x+n} - P_x)(1 + a_{x+n}) \\ {}_nV'_x &= (P_{x+n} - P'_x)(1 + a_{x+n}) \\ &= (P_{x+n} - P_x - \phi)(1 + a_{x+n}) \\ &= {}_nV_x - \phi(1 + a_{x+n}). \end{aligned}$$

But from the ordinary equation for the value of a policy we have

$$(1 + a_{x+n}) = (1 - {}_nV_x)(1 + a_x).$$

Therefore,
$${}_nV'_x = {}_nV_x - (1 - {}_nV_x)(1 + a_x)\phi;$$

and if we put $(1 + a_x)\phi = \psi$, then

$$\begin{aligned} {}_nV'_x &= {}_nV_x - (1 - {}_nV_x)\psi \\ &= {}_nV_x(1 + \psi) - \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15) \end{aligned}$$

28. The quantity ψ is positive or negative according to the sign of ϕ, but it is always fractional. Therefore, when the premium valued is increased, the policy-value is increased by a constant percentage, and diminished by the same percentage of unity; but as the policy-value is always less than unity, it is under these circumstances actually diminished. On the other hand, when the premium valued is decreased, the policy-value is actually increased, being diminished by a constant percentage, and increased by the same percentage of unity.

29. The difference between the premium valued and the net premium may be any quantity we please. If, therefore, it be taken negative, and equal to the net premium, we shall have the case where no premium at all is valued; that is, the policy-value becomes simply the value of the reversion, and ${}_nV'_x = A_{x+n}$. Also, $\psi = -P_x(1 + a_x) = -A_x$; and from formula 15 we have

$$A_{x+n} = {}_nV_x(1 - A_x) + A_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

30. Formulas 15 and 16 are independent of the value of n; and therefore, if we know the value of a policy and the age at which it was

effected, we can, without knowing its duration, find the value of the sum assured, and as a consequence, the value of the future premiums also. Thus, to take a numerical example, let the value of a policy at 3 per-cent interest, effected at age 30, and which has been an unknown period in force, be ·13010. To find the value of the sum assured, and the value of the future premiums. By Table VI, $A_{30}=\cdot39141$, and therefore $1-A_{30}=\cdot60859$. Performing the calculations indicated by formula 16, we have $A_{30+n}=\cdot47059$. Also, since ${}_nV_x=A_{x+n}-P_x(1+a_{x+n})$, therefore $P_x(1+a_{x+n})=A_{x+n}-{}_nV_x$: whence $P_{30}(1+a_{30+n})=\cdot34049$, which is the value of the future premiums.

The example is that of a policy effected at age 30, and ten years in force; and on reference to Table VI, it will be seen that $A_{30+n}=A_{40}$, which is as it should be.

31. Formulas 15 and 16 being independent of the value of n, if there be any number, r, of policies of uniform or of various durations, but all effected at age x; and if ΣV_x be the sum of their values by the net premium method, and $\Sigma V'_x$ the sum of their values when the altered premium P'_x, $(=P_x+\phi)$, is valued; then by formula 15,

$$\Sigma V'_x=\Sigma V_x(1+\psi)-r\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and by formula 16,

$$\Sigma A_{x+n}=\Sigma V_x(1-A_x)+rA_x \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where ΣA_{x+n} is the sum of the values of the reversions at the present ages of the lives, these present ages being unknown.

32. As a numerical example of formula 18, let ·69105 be the sum of the values at 3 per-cent interest of ten policies of unknown durations, but all effected at age 30. To find the value of the sums assured, and the value of the future premiums separately. As before, we have $A_{30}=\cdot39141$, and $1-A_{30}=\cdot60859$; and making $r=10$, and performing the calculations, we have $\Sigma A_{30+n}=4\cdot3347$, which is the value of the sums assured. Also the value of the future premiums is $4\cdot3347-\cdot6911$, or $3\cdot6436$. In this example ΣV_x has been so selected as to consist of the values of ten policies effected at age 30, and which have been 1, 2, 3, &c., 10 years respectively in force. The sum of their values, according to Table B, page 317, is ·69105, as above stated; and ΣA_{30+n} should be $A_{31}+A_{32}+\&c.+A_{40}$. On adding together the values of these reversions as given in Table VI, we have 4·33463, which agrees with the result of formula 18.

33. Hitherto, in the investigations in this chapter, it has been taken for granted that the valuation of the premiums and of the sums assured is to be made by a table supposed to represent actual rates of mortality; and this assumption has been made even in paragraphs 26 to 32, in which the valuation of premiums other than net premiums has been discussed. There is, however, a method of deducing policy-values, based upon a different principle, which at one time was much in vogue with assurance companies. It consists in forming from the table of office premiums a table of annuity-values, and in then using these hypothetical annuity-values in valuing the policies. On this account the method has been called the *Hypothetical Method* of valuation. When it was used, interest at 3 per-cent was generally assumed, without reference to what might have been the rate on which the table of office premiums was based. The office premiums being greater than the net premiums, it follows that the hypothetical annuity-values deduced therefrom are smaller than the annuity-values derived directly from the mortality table, the effect upon them being the same as if a mortality table showing a higher rate of mortality were employed.

34. If P' represent the office premium, and a' the corresponding hypothetical annuity-value, so that $a'=\frac{1}{P'+d}-1$; and if V' represent the policy-value by the hypothetical method, then we have the following formulas:—

$$ {}_nV'_x=A'_{x+n}-P'_x(1+a'_{x+n}) \quad \ldots\ldots \quad (19)$$

$$=(P'_{x+n}-P'_x)(1+a'_{x+n}) \quad \ldots\ldots \quad (20)$$

$$=A'_{x+n}\left(1-\frac{P'_x}{P'_{x+n}}\right) \quad \ldots\ldots \quad (21)$$

$$=1-\frac{1+a'_{x+n}}{1+a'_x} \quad \ldots\ldots \quad (22)$$

$$=1-\frac{P'_x+d}{P'_{x+n}+d} \quad \ldots\ldots \quad (23)$$

$$=\frac{P'_{x+n}-P'_x}{P'_{x+n}+d} \quad \ldots\ldots \quad (24)$$

In fact, seeing that in the hypothetical method of valuation the annuities and premiums correspond to each other exactly as if they were based upon an actual mortality table, the forms of expressions for the policy-value are precisely the same as if the net-premium method were used.

35. The formulas for the hypothetical value of a policy may be arrived at in another way, as follows.

If there be a policy of 1, which was effected n years ago, on a life then aged x, at the office premium of P'_x; and if P'_{x+n} be the office premium to assure 1 at the present age of the life; then the sum which can now be assured for P'_x, the premium payable under the policy, is $\frac{P'_x}{P'_{x+n}}$. The balance of the sum assured, namely, $1-\frac{P'_x}{P'_{x+n}}$ or $\frac{P'_{x+n}-P'_x}{P'_{x+n}}$, must be provided for by the value of the policy, and therefore the value of the policy is the single premium for which that balance can be assured. A single premium of 1 will assure a sum of $1+\frac{d}{P'_{x+n}}$ or $\frac{P'_{x+n}+d}{P'_{x+n}}$, because d, the annual interest in advance, will assure $\frac{d}{P'_{x+n}}$, and the unit will remain intact to be returned on the death of the life assured. Therefore the single premium to assure 1 is $\frac{P'_{x+n}}{P'_{x+n}+d}$, and the single premium to assure $\frac{P'_{x+n}-P'_x}{P'_{x+n}}$ is $\frac{P'_{x+n}-P'_x}{P'_{x+n}+d}$: whence ${}_nV'_x=\frac{P'_{x+n}-P'_x}{P'_{x+n}+d}$, which is the same as formula 24 above.

36. The argument in the last preceding paragraph may be put in a somewhat modified form. The policy-value, ${}_nV'_x$, together with the interest upon it, $d_nV'_x$, and together with P'_x, the premium payable under the policy, must assure the whole unit. But the amount which ${}_nV'_x$ and its interest will assure is ${}_nV'_x+\frac{d_nV'_x}{P'_{x+n}}$, or ${}_nV'_x\cdot\frac{P'_{x+n}+d}{P'_{x+n}}$; and the amount which P'_x will assure is $\frac{P'_x}{P'_{x+n}}$; and these two together must be equal to 1. That is, ${}_nV'_x\cdot\frac{P'_{x+n}+d}{P'_{x+n}}+\frac{P'_x}{P'_{x+n}}=1$. Whence ${}_nV'_x=\frac{P'_{x+n}-P'_x}{P'_{x+n}+d}$ as above.

37. From these arguments it is seen that the value of a policy under the hypothetical method is the sum which the company that granted the policy should pay to another company charging the same scale of premiums and making the assumed rate of interest on its funds, in order to transfer the liability. The hypothetical method has therefore been also called the *Reinsurance Method.*

38. It was remarked, in Art. 33, that to assume the hypothetical annuity-values is equivalent to assuming a higher rate of mortality; but it does not therefore follow that the hypothetical value of a policy is greater than the net value. Taking the formula ${}_nV'_x=1-\dfrac{P'_x+d}{P'_{x+n}+d}$, we see that, by including office premiums instead of net premium, both the numerator and denominator of the fraction are increased; but it is not possible to say from this fact alone whether the fraction itself is increased or diminished, or whether it remains unchanged in value; and therefore it is impossible to say, without further investigation, whether the hypothetical value of the policy is greater than, equal to, or less than, the net value.

39. In order to determine the relative magnitudes, let it be assumed that $P'_x=P_x+\phi_x$, and $P'_{x+n}=P_{x+n}+\phi_{x+n}$. Then

$${}_nV'_x >=< {}_nV_x,$$

according as $$1-\frac{P'_x+d}{P'_{x+n}+d} >=< 1-\frac{P_x+d}{P_{x+n}+d},$$

according as $$\frac{P_x+d}{P_{x+n}+d} >=< \frac{P'_x+d}{P'_{x+n}+d},$$

$$>=< \frac{P_x+\phi_x+d}{P_{x+n}+\phi_{x+n}+d},$$

according as $$\frac{P_{x+n}+\phi_{x+n}+d}{P_{x+n}+d} >=< \frac{P_x+\phi_x+d}{P_x+d},$$

according as $$1+\frac{\phi_{x+n}}{P_{x+n}+d} >=< 1+\frac{\phi_x}{P_x+d},$$

according as $$\frac{\phi_{x+n}}{P_{x+n}+d} >=< \frac{\phi_x}{P_x+d} \quad . \; . \; . \; . \; . \; . \; . \quad (25)$$

40. It therefore appears that the magnitude of the policy-value by the hypothetical method, as compared with that by the net method, depends on the way in which the premiums are loaded. If the loading be such that at all ages $\phi=\kappa(P+d)$, then

$$\begin{aligned}{}_nV'_x&=1-\frac{P_x+\kappa(P_x+d)+d}{P_{x+n}+\kappa(P_{x+n}+d)+d}\\&=1-\frac{(1+\kappa)(P_x+d)}{(1+\kappa)(P_{x+n}+d)}\\&={}_nV_x.\end{aligned}$$

Under these circumstances, therefore, the hypothetical value of the policy is equal to the net value.

41. If, to obtain the office premium, the net premium be loaded with a percentage and a constant—that is, if $P'=(1+\kappa)P+c$ at all ages, so that $\phi=\kappa P+c$, then

$${}_nV'_x-{}_nV_x=\frac{P_x+d}{P_{x+n}+d}-\frac{P_x(1+\kappa)+c+d}{P_{x+n}(1+\kappa)+c+d}.$$

But $P_{x+n}+d=\dfrac{P_x+d}{1-{}_nV_x}$. Therefore

$${}_nV'_x-{}_nV_x=(1-{}_nV_x)-\frac{P_x(1+\kappa)+c+d}{\left\{\dfrac{P_x+d}{1-{}_nV_x}-d\right\}(1+\kappa)+c+d},$$

and

$${}_nV'_x=1-\frac{P_x(1+\kappa)+c+d}{\dfrac{P_x+d}{1-{}_nV_x}(1+\kappa)+c-\kappa d}$$

$$=\frac{{}_nV_x(P_x+d)(1+\kappa)}{(P_x+d)(1+\kappa)+(1-{}_nV_x)(c-\kappa d)}$$

$$=\frac{{}_nV_x}{1+(1-{}_nV_x)\dfrac{c-\kappa d}{(P_x+d)(1+\kappa)}} \quad . \quad . \quad . \quad (26)$$

42. Examining formula 26 we see that the only term involving the loading is in the denominator; and by watching the consequences in that term of any change in the method of loading the premiums, we can perceive how the hypothetical policy-value is influenced. If the premiums are loaded with a percentage only, then $c=0$, and the equation becomes

$${}_nV'_x=\frac{{}_nV_x}{1-(1-{}_nV_x)\dfrac{\kappa d}{(P_x+d)(1+\kappa)}};$$

thus showing, that under these circumstances, the hypothetical policy-value is greater than the net value. If, on the other hand, the amount of loading is equal at all ages, then κ vanishes, and we have

$${}_nV'_x=\frac{{}_nV_x}{1+(1-{}_nV_x)\dfrac{c}{P_x+d}};$$

and we see that in this case the hypothetical value is less than the net value.

43. When the constant addition to the premium is the same proportion of d that the constant percentage is of P, that is to say, when $c=\kappa d$, and $P'=(1+\kappa)P+\kappa d$, then the algebraical term in the denominator of the right-hand member of equation 26 disappears, and ${}_nV'_x={}_nV_x$, a relation proved otherwise in Art. 40.

In general, according as $c > = < \kappa d$, the expression $\dfrac{c-\kappa d}{(P_x+d)(1+\kappa)}$ is positive, equal to zero, or negative, and ${}_nV'_x < = > {}_nV_x$.

44. Returning to equation 26, it will be observed that the expression $\dfrac{c-\kappa d}{(P_x+d)(1+\kappa)}$ is constant, not varying with the duration of the policy. Writing for it θ, we have ${}_nV'_x=\dfrac{{}_nV_x}{1+\theta(1-{}_nV_x)}$. Taking now the reciprocal of each side of the last equation, we have the remarkable relation

$$\frac{1}{{}_nV'_x}=\frac{1}{{}_nV_x}(1+\theta)-\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

45. The consideration of the hypothetical method of valuation has shown that a mortality table exhibiting a higher rate of mortality than another does not necessarily give greater policy-values. It therefore becomes of interest and importance to investigate in what way the rate of mortality affects policy-values. We have

$$\begin{aligned}{}_nV_x&=1-\frac{1+a_{x+n}}{1+a_x}\\ &=1-\frac{1+a_{x+n}}{1+a_{x:\overline{n-1}|}+v^n.{}_np_x(1+a_{x+n})}\\ &=1-\frac{1}{\dfrac{1+a_{x:\overline{n-1}|}}{1+a_{x+n}}+v^n.{}_np_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)\end{aligned}$$

46. In the expression for ${}_nV_x$ found in equation 28, there are three quantities involving the rate of mortality, namely, $1+a_{x:\overline{n-1}|}$, $v^n.{}_np_x$, and $1+a_{x+n}$; and it will be noticed that the first two involve the rate of mortality between the ages x and $x+n$ only, while the third involves the rate of mortality only for ages above $x+n$. Now let the rate of mortality below age $x+n$ be increased, while above that age let it remain unchanged. In this case the quantities $1+a_{x:\overline{n-1}|}$ and $v^n.{}_np_x$ will be diminished, the other quantities remaining unaffected; and therefore the fraction $\dfrac{1}{\dfrac{1+a_{x:\overline{n-1}|}}{1+a_{x+n}}+v^n.{}_np_x}$ will be increased, and as a consequence, ${}_nV$-

will be diminished. In this case an increase in the rate of mortality diminishes the policy-value.

47. Let now the rate of mortality between the ages x and $x+n$ remain unchanged, while above age $x+n$ it is increased. In this case $1+a_{x+n}$ will be diminished, and it will be the only quantity to vary. The fraction $\dfrac{1+a_{x:\overline{n-1}|}}{1+a_{x+n}}$ will be increased; and therefore the fraction $\dfrac{1}{\dfrac{1+a_{x:\overline{n-1}|}}{1+a_{x+n}}+v^n.{}_np_x}$ will be diminished, and ${}_nV_x$ will be increased. In this case an increase in the rate of mortality increases the policy-value.

48. The same principles apply if the rate of mortality be increased throughout the whole mortality table. The increases, respectively below and above age $x+n$, will have opposite effects on the policy-value, and will tend to neutralize each other; and according to some law of progression in the rate, the nature of which the expression does not disclose, there is a certain increase in the rate of mortality which will not change the policy-value. If, however, the increase be proportionally greater at the younger ages, the policy-value will be diminished, and if the increase be proportionally greater at the older ages, the policy-value will be augmented. In Arts. 56 and 57 the law of equal policy-values, above indicated, will be defined.

49. Let there be two mortality tables, T and T′; and let the letters a, V, &c., refer to the table T, and the accentuated letters a', V′, &c., to the table T′. To ascertain in what circumstances the value of a policy as given by the table T′ will be greater or less than that given by another table T. We have

$$ {}_nV'_x >=< {}_nV_x $$

according as
$$ \frac{1+a_{x+n}}{1+a_x} >=< \frac{1+a'_{x+n}}{1+a'_x} $$

according as
$$ \frac{1+a'_x}{1+a_x} >=< \frac{1+a'_{x+n}}{1+a_{x+n}}. $$

50. If, therefore, we take any two tables of annuities, and calculate throughout the value of the ratio $\dfrac{1+a'_x}{1+a_x}$ for all values of x, a simple inspection of the results will show for what ages at entry and what durations the policy-values given by the one table will be greater or less than those given by the other. If the value of the ratio becomes continually less in passing to higher ages, then T′, the annuity table

which has its values put in the numerator, gives throughout larger policy-values than the other.

51. If it should happen that the value of the ratio is constant, then the two annuity tables would give equal policy-values for all ages at entry and all durations. In this case we may write $1+a_x=(1+\kappa)(1+a'_x)$; whence $P'_x+d=(1+\kappa)(P_x+d)$; and $P'_x=(1+\kappa)P_x+\kappa d$, a relation already found in Arts. 40 and 43.

52. The foregoing demonstration is perfectly general. It holds for all mortality tables, and it is not necessary that the rate of interest should be the same in the two annuity-values compared. When the rates of interest are different, we shall have, in the case of equal policy-values,

$$P'_x+d'=(1+\kappa)(P_x+d)$$

and

$$P'_x=(1+\kappa)P_x+\kappa d+d-d'.$$

If here we make $\kappa=0$, so that $1+\kappa=1$; that is, if we take two mortality tables which give equal annuity-values, and consequently equal policy-values, at different rates of interest; we shall have $P'_x=P_x+d-d'$. In other words the premiums differ by a constant. This was proved in a different way in Chap. viii, Art. 24.

53. When we compare the policy-values obtained from two mortality tables, we are not generally to expect that the values obtained from the one are either throughout greater, or throughout less, than those obtained from the other; but rather that, for certain intervals, the policy-values given by the first table will be greater, while for certain other intervals they will be less, than those given by the other. In fact we must compare the two tables age by age.

54. If ${}_mV'_x>{}_mV_x$, and ${}_nV'_{x+m}>{}_nV_{x+m}$, it follows from equation 13 that ${}_{m+n}V'_x>{}_{m+n}V_x$. Similarly, from equation 14, it follows that if, ${}_1V'_x>{}_1V_x$, ${}_1V'_{x+1}>{}_1V_{x+1}$, and so on up to ${}_1V'_{x+n-1}>{}_1V_{x+n-1}$; then ${}_nV'_x>{}_nV_x$. In order, therefore, to compare the values of policies of n years' duration it is convenient to compare those of one year's duration. Now,

$${}_1V'_x>=<{}_1V_x,\text{ as }\frac{1+a_{x+1}}{1+a_x}>=<\frac{1+a'_{x+1}}{1+a'_x},$$

$$\text{as }\frac{1+a'_x}{1+a'_{x+1}}>=<\frac{1+a_x}{1+a_{x+1}}.$$

We have therefore only to compute the ratio $\dfrac{1+a_x}{1+a_{x+1}}$ for two annuity tables; and a comparison of the values will show through what intervals the one table gives greater or less policy-values than the other. This demonstration holds equally well whether the annuity-values are taken from the same or different tables; so that the table of the ratio in question enables us to compare any part of one annuity table with any part of the same or of any other annuity table, as regards the effect upon policy-values. It will not, however, enable us to compare policy-values when the ages at entry and valuation lie in parts of the table exhibiting different progressions in the ratio.

55. The relative magnitudes of ${}_1V'_x$ and ${}_1V_x$ may be displayed in terms of the rates of mortality and the annual premiums. We have $a_x = vp_x(1+a_{x+1})$; therefore $\dfrac{1+a_x}{1+a_{x+1}} = \dfrac{1}{1+a_{x+1}} + vp_x$

$$= P_{x+1} + 1 - v + v(1-q_x) = P_{x+1} + 1 - vq_x.$$

Hence $${}_1V'_x >=< {}_1V_x,$$

according as $$P'_{x+1} - vq'_x >=< P_{x+1} - vq_x;$$

according as $$P'_{x+1} - P_{x+1} >=< vq'_x - vq_x;$$

according as $$(1+i)(P'_{x+1} - P_{x+1}) >=< (q'_x - q_x).$$

56. Returning now to the case where two mortality tables give equal policy-values. Let
$$1+a_x = (1+\kappa)(1+a'_x),$$

so that $$1+a'_x = \frac{1}{1+\kappa}(1+a_x).$$

Then, since $$1+a'_x = 1+vp'_x(1+a'_{x+1})$$
$$= 1+vp'_x \cdot \frac{1}{1+\kappa}(1+a_{x+1})$$
$$= 1+p'_x \cdot \frac{1}{1+\kappa} \cdot \frac{1}{p_x} \cdot a_x;$$

therefore, $$1+\kappa+\frac{p'_x}{p_x}a_x = 1+a_x,$$

and $$p'_x = p_x\left(1-\frac{\kappa}{a_x}\right) \quad \ldots\ldots\ldots \quad (29)$$

57. Formula 29 gives the definition of the law which was indicated in Art. 48. If κ be positive, then $p'_x < p_x$, and the rate of mortality is increased; whereas, if κ be negative, $p'_x > p_x$, and the rate of mortality is diminished. In neither case are policy-values affected.

58. Since a_x diminishes with the age, $\frac{\kappa}{a_x}$ increases, and if κ be positive, the factor $\left(1-\frac{\kappa}{a_x}\right)$, by which p_x is multiplied, diminishes. Therefore, under these circumstances, p'_x diminishes with the age in a more rapidly increasing ratio than does p_x. If κ be negative, and equal say, to $-\lambda$, then the factor becomes $\left(1+\frac{\lambda}{a_x}\right)$, and it increases with the age, and under these circumstances p'_x diminishes in a less rapidly increasing ratio than p_x.

59. If, instead of multiplying p_x by the varying factor $\left(1-\frac{\kappa}{a_x}\right)$, we multiply by the constant factor $(1-r)$; that is, if p_x be diminished at each age by a constant percentage, then it is evident that p'_x will not be reduced from year to year in so great a ratio as when the factor $\left(1-\frac{\kappa}{a_x}\right)$ is employed; and therefore q'_x, the rate of mortality, will not increase so rapidly as is required to produce equal policy-values. In accordance, therefore, with the principles of Art. 48, the effect of diminishing p_x by a constant percentage is to diminish policy-values.

60. Similarly, if we multiply p_x at each age by the factor $(1+r)$, instead of by the increasing factor $\left(1+\frac{\lambda}{a_x}\right)$; that is, if we increase p_x by a constant percentage, the result will be that p'_x will diminish with the age in a more rapidly increasing ratio than is provided for in formula 29; and in accordance with the principles of Art. 48 the policy-values will be increased.

61. By means of formula 29 a mortality table may very conveniently be constructed to show either a greater or a less rate of mortality than that of a given table, but which at the same time will produce equal policy-values.

62. Let there be three mortality tables, giving at a specified rate of interest equal policy-values, and so connected that

$$1+a_x=(1+\kappa)(1+a'_x)=(1+\lambda)(1+a''_x);$$

then by formula 29, $p'_x=p_x\left(1-\frac{\kappa}{a_x}\right)$, and $p''_x=p_x\left(1-\frac{\lambda}{a_x}\right)$. Hence $p_x-p'_x=p_x\cdot\frac{\kappa}{a_x}$, and $p_x-p''_x=p_x\cdot\frac{\lambda}{a_x}$. Therefore

$$p_x-p''_x=(p_x-p'_x)\frac{\lambda}{\kappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

63. The coefficient $\frac{\lambda}{\kappa}$ is constant for the mortality tables in question, and does not depend in any way on the age. From this it follows that if we suppose curves to be drawn representing the chance of living a year by a series of hypothetical mortality tables which all give the same policy-values for an assumed rate of interest, the abscissa representing the age, and the ordinate the chance of living a year; then the distances between the various curves measured along the ordinates will be proportional at all ages.

64. Because $p_x - p'_x = q'_x - q_x$, the same is true when the ordinate represents the chance of dying in a year instead of the chance of living a year.

65. Tables F, G, and H, afford a numerical illustration of the principles discussed. Table F shows the annuity-values and annual premiums, calculated by three mortality tables, T, T′, and T″, producing equal policy-values at 3 per-cent interest. The table T is that at the end of this volume, and in which the mortality may be styled "normal." The table T′ is calculated from the relation $(1+a'_x) = 1{\cdot}05(1+a_x)$, and the table T″ from the relation $(1+a''_x) = \frac{1}{1{\cdot}05}(1+a_x)$. Therefore T′ exhibits "sub-normal", and T″ "super-normal", mortality. On account of the relation between the annuity tables, we also have $P'_x = \frac{1}{1{\cdot}05}P_x - \frac{{\cdot}05}{1{\cdot}05}d = \frac{1}{1{\cdot}05}P_x - {\cdot}001386$; and $P''_x = 1{\cdot}05P_x + {\cdot}05d = 1{\cdot}05P_x + {\cdot}001456$. By means of these numerical relations, the columns of P may be checked by summation.

66. Table G shows the probability of living a year, and the probability of dying in a year, respectively, by the three mortality tables; these probabilities having been calculated by formula 29. We have $(p_x - p''_x) = -1{\cdot}05(p_x - p'_x)$, and $(q''_x - q_x) = 1{\cdot}05(q_x - q'_x)$.

67. In Table H the differences $q_x - q'_x$, and $q''_x - q_x$, are displayed; as also the ratios $\frac{q'_x}{q_x}$ and $\frac{q''_x}{q_x}$.

The diagram on page 339 gives graphically the same facts as are shown numerically in Table H. The abscissa represents the age, and (in the case of the *dotted curves*) the thick horizontal line represents the abscissa axis, the positive direction being taken downwards, and the negative direction upwards; and the distance between each pair of horizontal lines being ·005. In the case of the curves marked by *continuous* lines, the abscissa as before represents the age; but here the abscissa axis falls below the lower edge of the diagram, and the thick horizontal line represents unity, or the ratio $\frac{q_x}{q_x}$, and the distance between each pair of horizontal lines is ·1. In the left margin of the diagrams the italic numerals, such as *·70*, refer to the unbroken lines, while the numerals slanting to the left, such as ·015, refer to the dotted lines.

68. It will be seen from the diagram that the ratios $\frac{q''_x}{q_x}$ and $\frac{q'_x}{q_x}$, are very wide apart at the younger ages, but that they approach each other towards the older ages, and evidently both tend towards the limit unity. On the other hand the differences $q_x - q'_x$ and $q_x - q''_x$ are small at the younger ages, and in advancing towards the older ages rapidly increase. The curves on the two sides of the thick line are similar, on account of the mode of construction of the three mortality tables.

By altering the value of κ in formula 29, an infinite number of curves might be drawn, representing mortality tables all giving equal policy-values at an assumed rate of interest.

Table F.

Showing the Values of Annuities, and the Annual Premiums, by three Mortality Tables producing Equal Policy-Values, at 3 per-cent Interest.

Age.	a_x			P_x			Age.
	T Normal Mortality.	T′ Sub-Normal Mortality.	T″ Super-Normal Mortality.	T Normal Mortality.	T′ Sub-Normal Mortality.	T″ Super-Normal Mortality.	
20	22·064	23·217	20·965	·01423	·01217	·01640	20
25	21·025	22·127	19·976	·01628	·01412	·01855	25
30	19·895	20·940	18·900	·01873	·01645	·02113	30
35	18·613	19·594	17·679	·02186	·01943	·02441	35
40	17·177	18·086	16·311	·02589	·02327	·02864	40
45	15·591	16·420	14·801	·03115	·02828	·03417	45
50	13·878	14·622	13·170	·03809	·03488	·04145	50
55	12·072	12·726	11·449	·04737	·04373	·05121	55
60	10·223	10·784	9·689	·05997	·05573	·06443	60
65	8·395	8·865	7·948	·07732	·07224	·08263	65
70	6·656	7·039	6·291	·10149	·09527	·10803	70
75	5·074	5·378	4·785	·13551	·12766	·14373	75
80	3·702	3·937	3·478	·18354	·17342	·19419	80
85	2·571	2·750	2·401	·25087	·23754	·26491	85
90	1·686	1·820	1·558	·34319	·32548	·36181	90
95	1·035	1·137	·938	·46234	·43881	·48687	95

TABLE G.

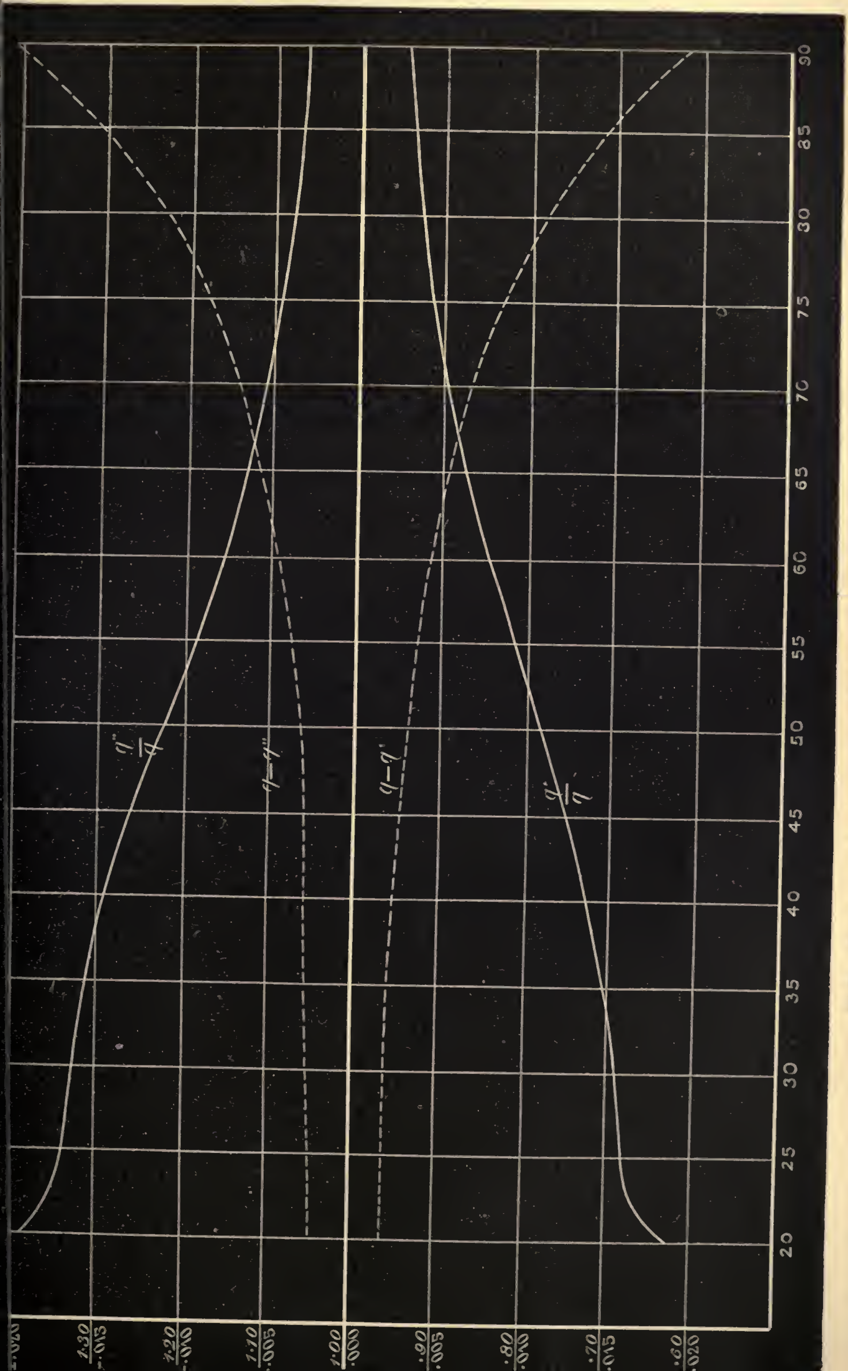
Showing the Probability of Living a Year, and the Probability of Dying within a Year, by three Mortality Tables producing Equal Policy-Values, at 3 per-cent Interest.

Age.	p_x Normal Mortality.	p'_x Sub-Normal Mortality.	p''_x Super-Normal Mortality.	q_x Normal Mortality.	q'_x Sub-Normal Mortality.	q''_x Super-Normal Mortality.	Age.
20	·99428	·99643	·99203	·00572	·00357	·00797	20
25	·99293	·99518	·99057	·00707	·00482	·00943	25
30	·99229	·99467	·98980	·00771	·00533	·01020	30
35	·99138	·99392	·98872	·00862	·00608	·01128	35
40	·98999	·99273	·98711	·01001	·00727	·01289	40
45	·98776	·99079	·98458	·01224	·00921	·01542	45
50	·98428	·98766	·98073	·01572	·01234	·01927	50
55	·97877	·98263	·97472	·02123	·01737	·02528	55
60	·97017	·97469	·96542	·02983	·02531	·03458	60
65	·95673	·96216	·95103	·04327	·03784	·04897	65
70	·93590	·94260	·92887	·06410	·05740	·07113	70
75	·90396	·91244	·89505	·09604	·08756	·10495	75
80	·85574	·86675	·84418	·14426	·13325	·15582	80
85	·78478	·79931	·76952	·21522	·20069	·23048	85
90	·68421	·70353	·66392	·31579	·29647	·33608	90
95	·55039	·57572	·52379	·44961	·42428	·47621	95

TABLE H.

Showing the Ratios, and the Differences, between the Rates of Mortality by three Mortality Tables producing Equal Policy-Values, at 3 per-cent Interest.

Age.	$\frac{q'_x}{q_x}$	$\frac{q''_x}{q_x}$	$q_x - q'_x$	$q''_x - q_x$
20	·624	1·393	·00215	·00225
25	·682	1·334	·00225	·00236
30	·691	1·323	·00238	·00249
35	·705	1·309	·00254	·00266
40	·726	1·288	·00274	·00288
45	·752	1·260	·00303	·00318
50	·785	1·226	·00338	·00355
55	·818	1·191	·00386	·00405
60	·848	1·159	·00452	·00475
65	·875	1·132	·00543	·00570
70	·895	1·110	·00670	·00703
75	·912	1·093	·00848	·00891
80	·924	1·080	·01101	·01156
85	·932	1·071	·01453	·01526
90	·939	1·064	·01932	·02029
95	·944	1·059	·02533	·02660



N.B.—For an explanation of this Diagram, see Art. 67, page 336.

69. An increase in the rate of interest has the effect of reducing policy-values. In the expression for the value of an annuity, the vitality of the life and the discount of the money may be interchanged, without affecting the result. Let $w=v(1-r)$, where $w=\dfrac{1}{1+j}$, and $v=\dfrac{1}{1+i}$; so that $v>w$, and therefore $j>i$; and let a'_x be calculated at rate j, and a_x at rate i. Then

$$\begin{aligned} a'_x &= wp_x+w^2.p_x.p_{x+1} \\ &\qquad +w^3.p_x.p_{x+1}.p_{x+2}+\&c. \\ &= v(1-r)p_x+v^2(1-r)p_x(1-r)p_{x+1} \\ &\qquad +v^3(1-r)p_x(1-r)p_{x+1}(1-r)p_{x+2}+\&c. \\ &= vp'_x+v^2.p'_x.p'_{x+1} \\ &\qquad +v^3.p'_x.p'_{x+1}.p'_{x+2}+\&c. \end{aligned}$$

when we write p'_x for $(1-r)p_x$ for all values of x. Therefore to increase the rate of interest from i to j is exactly equivalent to reducing p_x in the ratio $(1-r)$ throughout the mortality table; and by Art. 59 the result of this step is to reduce policy-values.

[70] By means of the differential calculus the theorem may be proved in a manner that to some minds may be more conclusive.

Considering first the policy-value at the end of one year, we have ${}_1\mathrm{V}_x=1-\dfrac{1+a_{x+1}}{1+a_x}=1-\dfrac{a_x}{vp_x(1+a_x)}$; or, omitting the suffix, $\mathrm{V}=1-\dfrac{a}{vp(1+a)}$. Differentiating with respect to v;

$$\frac{d\mathrm{V}}{dv}=-\frac{1}{p}\cdot\frac{v(1+a)\dfrac{da}{dv}-a(1+a)-av\dfrac{da}{dv}}{v^2(1+a)^2}$$

$$=-\frac{1}{p}\cdot\frac{v\dfrac{da}{dv}-a(1+a)}{v^2(1+a)^2}$$

Now, $$a=vp+v^2.{}_2p+v^3.{}_3p+\&c.$$

therefore $$v\frac{da}{dv}=vp+2v^2.{}_2p+3v^3.{}_3p+\&c$$

$$=a_x+vpa_{x+1}+v^2.{}_2pa_{x+2}+\&c.$$

Also $$a(1+a)=a_x+vpa_x+v^2.{}_2pa_x+\&c.$$

Hence if a_x is greater than a_{x+1}, a_{x+2}, &c., that is, if the value of an annuity on the life of (x) is greater than the value of an annuity on any life older than (x), then $a(1+a)>v\frac{da}{dv}$; that is, $\frac{d\text{V}}{dv}$ is a positive quantity. In other words, as v increases, so also does V. But as v increases the rate of interest diminishes; therefore the lower the rate of interest, the greater is the value of a policy of one year's standing on the life of (x). The same argument applies to policies on the lives of $(x+1)$, $(x+2)$, $(x+3)$, &c. Therefore, when the annuity-value decreases with the age, then policies of one year's standing decrease with an increase in the rate of interest.

But by formula 14, if ${}_1\text{V}_x$, ${}_1\text{V}_{x+1}$, ${}_1\text{V}_{x+2}$, &c., ${}_1\text{V}_{x+n-1}$, are all diminished, so also is ${}_n\text{V}_x$ diminished. Therefore, generally, if a_x decrease with the age, ${}_n\text{V}_x$ decreases with an increase in the rate of interest.

71. The limitation that a_x must decrease with the age, applies also to the argument in Art. 69, because otherwise $\left(1-\frac{\kappa}{a_x}\right)$ would not be a diminishing ratio. We must therefore enquire under what circumstances a_{x+1} is greater than a_x.

Since $a_x=vp_x(1+a_{x+1})$, therefore, if $a_x<a_{x+1}$, $vp_x(1+a_{x+1})<a_{x+1}$; and $p_x<\frac{(1+i)a_{x+1}}{1+a_{x+1}}$. Therefore

$$q_x>1-\frac{(1+i)a_{x+1}}{1+a_{x+1}}$$

$$>\frac{1-ia_{x+1}}{1+a_{x+1}}$$

$$>\frac{(1+i)\text{A}_{x+1}}{1+a_{x+1}}$$

Or $$vq_x>\text{P}_{x+1}$$

That is $$\text{P}^{1}_{x:\overline{1|}}>\text{P}_{x+1}$$

Also, since $a_x=vp_x(1+a_{x+1})$ and, by hypothesis, $a_{x+1}>a_x$; therefore $a_x>vp_x(1+a_x)$, and $p_x<\frac{(1+i)a_x}{1+a_x}$; whence as above $\text{P}^{1}_{x:\overline{1|}}>\text{P}_x$.

We have therefore proved that if the premium for an assurance for a single year at age x is greater than the whole life premium at age $x+1$, it will also be greater than the whole life premium at age x; and a_x will be less than a_{x+1}.

72. Under these circumstances also, the value of the policy will be *negative.* If a_x be less than a_{x+1}, then P_x must be greater than P_{x+1}, because $P_x = \dfrac{1}{1+a_x} - d$, and $P_{x+1} = \dfrac{1}{1+a_{x+1}} - d$; and by formula 5 ${}_1V_x = (P_{x+1} - P_x)(1+a_{x+1})$, a negative quantity.

73. It is only under exceptional circumstances that the annuity increases with the age. This condition obtains at the infantile ages of all mortality tables, and at very advanced ages of some tables which have been badly graduated. For practical purposes these special cases may be ignored, and it may be said generally that policy-values decrease with an increase in the rate of interest.

74. In order to find the policy-value by the formula ${}_nV_x = 1 - \dfrac{1+a_{x+n}}{1+a_x}$, there is no need to know by what table of mortality or at what rate of interest the annuity-values are calculated. If the values of a_x, the initial annuity, and a_{x+n}, the terminal annuity, be given, the value of the policy can be at once ascertained.

75.—Chisholm (James) took advantage of this principle, and prepared* "Tables for finding the Values of Policies of all durations, according to any Table of Mortality or any Rate of Interest," these tables resembling the conversion tables described in Chap. viii. On each page there are ten columns, corresponding to ten values of $1+a_x$, the initial annuity-due; and down the left of the page are placed the values of $1+a_{x+n}$, the terminal annuity-due. The values of $1+a_x$ are given from 23·95 to 2·00 for differences of ·05, and the values of $1+a_{x+n}$ are given from 23·9 to 1·0 for differences of ·1. Having therefore the values of the two annuities, the policy-value is found in the column headed with $1+a_x$ and on the same line as $1+a_{x+n}$. The tables are so planned that the *initial* annuity must be taken to the nearest value tabulated, but tables of proportional parts are given for the *terminal* annuity-values. These tables may be used for other purposes than to find policy-values, as they give, by inspection, the value of any function of the form $1 - \dfrac{A}{B}$, when A and B are known.

76. In the case of a policy at yearly premiums, which has been in force $n + \dfrac{t}{m}$ years, the problem of finding its value may be looked at in either of two ways.

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* Published in 1885, by C. & E. Layton.

77. The age of the life being $x+n+\frac{t}{m}$, the value of the sum assured is $\mathrm{A}_{x+n+\frac{t}{m}}$. The next premium will fall due $\frac{m-t}{m}$ of a year hence, and therefore the value of the future premiums is $\mathrm{P}_x.{}_{\frac{m-t}{m}|}\mathrm{a}_{x+n+\frac{t}{m}}$. To find ${}_{\frac{m-t}{m}|}\mathrm{a}$, we must write $\frac{m-t}{m}$ for $\frac{1}{t}$ in equation 27 of Chap. ix. This gives (omitting the terms involving Δ_x, &c.) ${}_{\frac{m-t}{m}|}\mathrm{a}=\frac{t}{m}+a$. Hence the value of the future premiums is $\mathrm{P}_x\left(\frac{t}{m}+a_{x+n+\frac{t}{m}}\right)$; and

$$ {}_{n+\frac{t}{m}}\mathrm{V}_x=\mathrm{A}_{x+n+\frac{t}{m}}-\mathrm{P}_x\left(\frac{t}{m}+a_{x+n+\frac{t}{m}}\right). \quad . \quad . \quad . \quad (31) $$

78. Taking the other aspect of the problem:—

Just after the $(n+1)$th premium has been paid, the value of the policy, which at that moment may be represented by the symbol ${}_{n+0}\mathrm{V}_x$, is ${}_n\mathrm{V}_x+\mathrm{P}_x$: just before the $(n+2)$th premium falls due the value will be ${}_{n+1}\mathrm{V}_x$. Interpolating between these, we have

$$ \begin{aligned} {}_{n+\frac{t}{m}}\mathrm{V}_x &= {}_{n+0}\mathrm{V}_x-\frac{t}{m}\left({}_{n+0}\mathrm{V}_x-{}_{n+1}\mathrm{V}_x\right) \\ &= {}_n\mathrm{V}_x+\mathrm{P}_x-\frac{t}{m}\,{}_n\mathrm{V}_x-\frac{t}{m}\mathrm{P}_x+\frac{t}{m}\,{}_{n+1}\mathrm{V}_x \\ &= {}_n\mathrm{V}_x+\frac{t}{m}\left({}_{n+1}\mathrm{V}_x-{}_n\mathrm{V}_x\right)+\mathrm{P}_x\left(1-\frac{t}{m}\right) \quad . \quad . \quad (32) \end{aligned} $$

That is, we must interpolate between ${}_n\mathrm{V}_x$ and ${}_{n+1}\mathrm{V}_x$; and further, of the premium paid at the beginning of the $(n+1)$th year, we must reserve a proportionate part for the portion of that year unexpired.

79. The identity of formulas 31 and 32 may be proved as follows:—

$$ \begin{aligned} &{}_n\mathrm{V}_x+\frac{t}{m}\left({}_{n+1}\mathrm{V}_x-{}_n\mathrm{V}_x\right)+\mathrm{P}_x\left(1-\frac{t}{m}\right) \\ &\quad=\mathrm{A}_{x+n}-\mathrm{P}_x(1+a_{x+n})+\frac{t}{m}\mathrm{A}_{x+n+1}-\frac{t}{m}\mathrm{P}_x(1+a_{x+n+1}) \\ &\qquad-\frac{t}{m}\mathrm{A}_{x+n}+\frac{t}{m}\mathrm{P}_x(1+a_{x+n})+\mathrm{P}_x\left(1-\frac{t}{m}\right) \\ &\quad=\mathrm{A}_{x+n}+\frac{t}{m}\left(\mathrm{A}_{x+n+1}-\mathrm{A}_{x+n}\right)-\mathrm{P}_x\left\{\frac{t}{m}+a_{x+n}-\frac{t}{m}\left(a_{x+n}-a_{x+n+1}\right)\right\} \\ &\quad=\mathrm{A}_{x+n+\frac{t}{m}}-\mathrm{P}_x\left(\frac{t}{m}+a_{x+n+\frac{t}{m}}\right). \end{aligned} $$

If there be a prepared table of policy-values, formula 32 will be found the more convenient.

80. Taking the great mass of contracts of an assurance office, in the case of policies subject to yearly premiums, the next premium may fall due at any time from one day to twelve months hence; and, taken one with another, and assuming the premium income to be uniformly distributed over the year, they may be considered to be due, on the average, six months hence.

81. In valuing the liabilities of an assurance office it is therefore a common custom to collect together all the policies on lives of the same present age, say $x+n$, and to value the aggregate of the sums assured by the reversion A_{x+n}, and the aggregate of the premiums payable by the annuity $\frac{1}{2}+a_{x+n}$, which, according to Chap. ix, Art. 38, is an approximate value for ${}_{\frac{1}{2}|}\mathrm{a}_{x+n}$. Experience shows that when this brief process is adopted, the final result differs only to an insignificant extent from the result of valuing each policy separately, and making allowance for the exact due date of the premium. Care must, however, be taken at the outset to see whether the premium income of the office is uniformly distributed over the financial year. If it be found that, on the average, the premiums are due eight months hence instead of six months hence, then the annuity-value to employ is ${}_{\frac{8}{12}|}\mathrm{a}_{x+n}$, or $\frac{1}{3}+a_{x+n}$, and not $\frac{1}{2}+a_{x+n}$. If $\frac{1}{2}+a_{x+n}$ were used under these circumstances, the liability of the company would be considerably understated.

82. In valuing the policies thus in groups there is one danger that should be guarded against. If the net premium be not valued, but the office premium less an arbitrary percentage, it may easily happen that the premium valued is greater than the net premium. In such case, if the policy be of recent date, the value may be negative. If the policy were valued by itself this would be noticed, and the negative value would be discarded; because it is not legitimate to treat a policy as an asset instead of a liability, the assured being under no binding obligation to the company to keep the policy in force. But if the policy be included in a group for valuation, the negative value may escape attention, because the value of the group may be positive, the negative value of the recent policy simply going to reduce the liability under others of older date. When a valuation of this nature is made in groups, an investigation should therefore take place to discover and eliminate all negative values.

83. In the case of a policy secured by premiums payable m times a year, let $V^{(m)}$ represent the value. The annuities and premiums to correspond are $a_x^{(m)}$ and $P_x^{(m)}$; and for all practical purposes it will be sufficient to write, according to formulas 1 and 30 respectively of Chap. ix, $a_x^{(m)}=a_x+\frac{m-1}{2m}$, and $P_x^{(m)}=\frac{A_x}{\frac{m+1}{2m}+a_x}$.

84. The value of a policy which is subject to premiums payable m times a year, and which has been exactly n years in force, is

$$_nV_x^{(m)}=A_{x+n}-P_x^{(m)}\left(\frac{m+1}{2m}+a_{x+n}\right) \quad . \quad . \quad . \quad . \quad (33)$$

85. To find the difference between $_nV_x^{(m)}$ and $_nV_x$; we have

$$\begin{aligned}
_nV_x^{(m)}-{}_nV_x&=A_{x+n}-P_x^{(m)}\left(\frac{m+1}{2m}+a_{x+n}\right)-A_{x+n}+P_x(1+a_{x+n})\\
&=P_x(1+a_{x+n})-P_x^{(m)}\left(\frac{m+1}{2m}+a_{x+n}\right)\\
&=P_x^{(m)}(1+a_{x+n})\frac{\frac{m+1}{2m}+a_x}{1+a_x}-P_x^{(m)}\left(\frac{m+1}{2m}+a_{x+n}\right)\\
&=\frac{m-1}{2m}\cdot P_x^{(m)}\left(1-\frac{1+a_{x+n}}{1+a_x}\right)\\
&=\frac{m-1}{2m}\cdot P_x^{(m)}.{}_nV_x.
\end{aligned}$$

Hence

$$_nV_x^{(m)}={}_nV_x\left(1+\frac{m-1}{2m}\cdot P_x^{(m)}\right) \quad . \quad . \quad . \quad . \quad . \quad (34)$$

86. This result may be explained as follows:—By accepting premiums payable m times a year instead of yearly, the office each year stands the chance of losing part of the premium of that year, by the life failing in the early part of the year, before the premium for the whole of that year has been paid. If death occur in the first mth part of the year, the amount lost will be $\frac{m-1}{m}\cdot P_x^{(m)}$; and the chance of this occurring is $\frac{1}{m}$: hence the amount lost multiplied by the probability of the loss occurring is $\frac{m-1}{m^2}\cdot P_x^{(m)}$. Similarly, for the second mth part, we have $\frac{m-2}{m^2}\cdot P_x^{(m)}$; for the third, $\frac{m-3}{m^2}\cdot P_x^{(m)}$; and so on. Therefore, taking the sum of all these amounts, we shall have $\frac{m-1}{2m}\cdot P_x^{(m)}$, for the

value of the risk which the office runs in accepting the premiums m times instead of once a year. Hence, when premiums are payable m times a year, the office must reserve ${}_n V_x$ in respect of the sum assured, just as when premiums are payable yearly; and it must also reserve the further sum of ${}_n V_x \cdot \frac{m-1}{2m} \cdot P_x^{(m)}$ in respect of the premiums for the year of death which may be lost.

87. The difference between ${}_n V_x^{(m)}$ and ${}_n V_x$ is small. It is greatest when m is infinite, and the premiums are payable momently. For example, at age 30 and 3 per-cent interest $\overline{P}_{30} = {\cdot}01919$, and at age 60, $\overline{P}_{60} = {\cdot}06277$. Therefore, at age 30, ${}_n\overline{V}_{30} = {}_n V_{30} \times 1{\cdot}0096$, and at age 60, ${}_n\overline{V}_{60} = {}_n V_{60} \times 1{\cdot}0314$. For smaller values of m the differences are less. In half-yearly cases ${}_n V_{30}^{(2)} = {}_n V_{30} \times 1{\cdot}0047$, and ${}_n V_{60}^{(2)} = {}_n V_{60} \times 1{\cdot}0153$; and in quarterly cases ${}_n V_{30}^{(4)} = {}_n V_{30} \times 1{\cdot}0072$, and ${}_n V_{60}^{(4)} = {}_n V_{60} \times 1{\cdot}0257$. For practical purposes it is usual to discard the difference, and write in all cases ${}_n V_x^{(m)} = {}_n V_x$.

88. If a policy subject to premiums payable m times a year have been in force $n + \frac{t}{m}$ years, so that one of the instalments of premium is just due, we have

$${}_{n+\frac{t}{m}} V_x^{(m)} = A_{x+n+\frac{t}{m}} - P_x^{(m)} \left(\frac{m+1}{2m} + a_{x+n+\frac{t}{m}} \right) \quad . \quad . \quad . \quad (35)$$

By a process similar to that followed in Art. 85, it may be shown that the last equation is equivalent to

$${}_{n+\frac{t}{m}} V_x^{(m)} = \left\{ {}_n V_x + \frac{t}{m} ({}_{n+1}V_x - {}_n V_x) \right\} \left\{ 1 + \frac{m-1}{2m} P_x^{(m)} \right\} \quad . \quad (36)$$

But the quantity $\frac{m-1}{2m} P_x^{(m)}$, as already remarked, is small, and may generally be disregarded, and we may write approximately

$${}_{n+\frac{t}{m}} V_x^{(m)} = {}_n V_x + \frac{t}{m} ({}_{n+1}V_x - {}_n V_x) \quad . \quad . \quad . \quad . \quad (37)$$

89. If the policy have not been in force for the full term of $n + \frac{t}{m}$ years, but only for $n + \frac{t - \frac{1}{s}}{m}$ years, so that the next premium will not fall due for $\frac{1}{sm}$ of a year, a reserve of premium for that period must be

made in accordance with the principles on which formula 32 was based, and we must write,

$$ {}_{n+\frac{t-\frac{1}{s}}{m}}V_x^{(m)} = {}_nV_x + \frac{t-\frac{1}{s}}{m}({}_{n+1}V_x - {}_nV_x) + \frac{1}{sm}P_x \quad . \quad . \quad (38) $$

90. To take an example:—if a policy by quarterly premiums have been in force n years and 4 months, we shall have $m=4$, $t=2$, $\frac{1}{s}=\frac{2}{3}$, $\frac{1}{sm}=\frac{1}{6}$, and $\frac{t-\frac{1}{s}}{m}=\frac{1}{3}$; and ${}_{n+\frac{1}{3}}V_x^{(4)} = {}_nV_x + \frac{1}{3}({}_{n+1}V_x - {}_nV_x) + \frac{1}{6}P_x$. Formula 38 may at first sight appear complicated, but that is only because it is made perfectly general. It becomes quite simple the moment it is translated into numbers.

91. The foregoing investigations show that if the policies of an office be valued each separately, and if in each case the proper allowance be made for the portion of time until the next premium falls due, then a table of the values of policies at yearly premiums may conveniently and safely be used, whether the premiums are payable only once a year, or at more frequent intervals; and throughout the calculations the yearly premium P_x may be substituted for $P_x^{(m)}$.

92. If, however, the policies are to be valued in groups as explained in Art. 81, the case is different. Assuming as before that the premium income is equally distributed over the year, then in the case of policies subject to premiums payable m times a year, on the average the next premium will be due $\frac{1}{2m}$ of a year hence; and if the present age of the life be $x+n$, and if the annual premium, P_x, be employed in accordance with formula 38, then the annuity by which to value the premiums will be ${}_{\frac{1}{2m}|}a_{x+n}$, or by Chap. ix, formula 27, $\frac{2m-1}{2m}+a_{x+n}$, and not $\frac{1}{2}+a_{x+n}$ as in the case of yearly premiums. Thus, in half-yearly cases the annuity-value would be $\frac{3}{4}+a_{x+n}$, and in quarterly cases, $\frac{7}{8}+a_{x+n}$. If the yearly, half-yearly, quarterly, &c., cases were treated in separate groups, then the annual premium P_x could be adopted in all cases, the annuity $\frac{1}{2}+a_{x+n}$ being used in valuing the premiums of the annual group, $\frac{3}{4}+a_{x+n}$ those of the half-yearly group, $\frac{7}{8}+a_{x+n}$ those of the quarterly group, &c. But if in all cases the annuity $\frac{1}{2}+a_{x+n}$ were employed, the

value of the premiums in half-yearly, quarterly, &c., cases would be understated, and an unnecessarily large reserve would be made for these policies.

The reason of this is not far to seek, because it can be proved easily that, by adopting the annual premium, P_x, the assumption is tacitly made that the premiums are really yearly premiums payable by instalments; and that the periodical premiums are not premiums in themselves, but that those instalments for the current year, unpaid at the time of death, should be deducted from the sum assured on settlement of the claim. First, to ascertain what should be the premium to be charged under the instalment principle. The premium not being payable in full at the beginning of the year, but by m equal instalments throughout the year, there will be loss of interest on every instalment except the first; but, because the unpaid instalments for the year of death are to be deducted on settlement of the claim, there will, in the end, be as many full years' premiums paid as if premiums were payable in full annually in advance. The first instalment will not be deferred at all; the second will be deferred $\frac{1}{m}$ of a year; the third, $\frac{2}{m}$ of a year; and so on; and the last $\frac{m-1}{m}$ of a year; and it will come to the same thing, on the average, if the whole premium be deferred, as regards interest but not as regards mortality, by $\frac{m-1}{2m}$ of a year. Therefore, if P'_x be the total premiums per annum under these conditions, we must have

$$v^{\frac{m-1}{2m}} \times P'_x \times (1+a_x) = A_x\,;$$

whence $P'_x = (1+i)^{\frac{m-1}{2m}} \times P_x$.

Passing now to the Policy-value: At any time the annuity for finding the value of the future premiums (leaving out of account for the moment the unpaid instalments for the current insurance year) will be the same as if the premiums were payable in full annually in advance, except that, because of the conditions of the case, the premiums must be discounted for $\frac{m-1}{2m}$ of a year more. That is, the value of the premiums payable in full annually in advance being $P_x(\frac{1}{2}+a_{x+n})$, the value of the premiums payable by instalments must be $P'_x \times v^{\frac{m-1}{2m}} \times (\frac{1}{2}+a_{x+n})$; and this becomes, when we write P'_x in terms of P_x, $P_x(\frac{1}{2}+a_{x+n})$, as before. To this, however, we must now add the actual amount of unpaid instalments for the current insurance year, as these are certain to be received, and in their case the operation of discount is insignificant. On the average, the actual amount of outstanding instalments is $\frac{m-1}{2m}P'_x$, or, what is practically the same thing, $\frac{m-1}{2m}\mathrm{P}_x$. Therefore, the total value of the future premiums is $\mathrm{P}_x(\frac{1}{2}+a_{x+n})+\frac{m-1}{2m}\mathrm{P}_x$, that is, $\mathrm{P}_x\left(\frac{2m-1}{2m}+a_{x+n}\right)$, as before.

It therefore appears that, if an office conduct its business on the system of instalment premiums, that is, if it adopt the practice of deducting from the sum assured, on settlement of the claim, the instalments of premium for the current year unpaid at the time of death, then all the policies, whether by annual, half-yearly, or quarterly premiums, may properly be valued in the same groups and by the same annuities, provided that to the value of the premiums be added the amount of the instalments for the current insurance year actually outstanding at the date of valuation.

93. If, however, for policies subject to premiums payable m times a year, the premium $\mathrm{P}_x^{(m)}$ be employed, so that the periodical premiums are really premiums in themselves, and not merely instalments of yearly premiums, as in Art. 92, then the annuity to be used in the valuation is $_{\frac{1}{2m}|}\mathrm{a}_{x+n}^{(m)}$, which, by Chap. ix, Art. 38, is approximately equal to $\frac{1}{2}+a_{x+n}$, no matter what may be the value of m. Therefore, by inserting the proper premium, $\mathrm{P}_x^{(m)}$, in each case, the yearly, half-yearly, quarterly, &c., cases may all be valued in one group, the annuity adopted being $\frac{1}{2}+a_{x+n}$.

94. In all the preceding formulas it has been assumed that the sum assured is payable at the *end* of the year of death; that is, on the average, *six months after death*. What is the effect on the policy-value if the sum assured be made payable at any other period, say, at the moment of death?

95. Seeing that the date of payment of the sum assured is altered, its value is altered; and, as a consequence, the net premium by which it is to be secured must be altered also. At the outset, the present value of the sum assured must be exactly equal to the present value of the premiums; whence, if the sum assured be payable at the moment of death, its value being $\overline{\mathrm{A}}_x$, the annual premium will be $\frac{\overline{\mathrm{A}}_x}{1+a_x}$.

96. It will be convenient to have symbols for the premiums and policy-values in cases where the sum assured is payable otherwise than at the end of the year of death. In Chap. x, Art. 3, the symbol $\mathrm{A}_x^{(m)}$ was appropriated for the value of an assurance payable at the end of the interval $\frac{1}{m}$ in which (x) shall die; but we cannot write $\mathrm{P}_x^{(m)}$ and ${}_n\mathrm{V}_x^{(m)}$ for the corresponding premium and policy-value, because these symbols have

already been employed to denote that the premium is payable m times a year. The notation will, however, be sufficiently distinctive if the letter (m) in brackets be placed on the other side of the principal letter, and if we write ${}^{(m)}P_x$ for the annual premium for an assurance payable at the end of the interval in which (x) shall die, and ${}^{(m)}_{\ n}V_x$ for the corresponding policy-value. We shall then have ${}^{(m)}P_x = \dfrac{A_x^{(m)}}{1+a_x}$. If the assurance be payable at the moment of death, the symbols will be ${}^{(\infty)}P_x$ and ${}^{(\infty)}_{\ n}V_x$.

97. The policy-value after n years being the difference between the value of the sum assured and the value of the premiums, we have

$$ {}^{(\infty)}_{\ n}V_x = \overline{A}_{x+n} - {}^{(\infty)}P_x(1+a_{x+n}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39) $$

98. Writing $\overline{A}_x = A_x(1+i)^{\frac{1}{2}}$, in accordance with formula 2 of Chap. x, we have

$$ {}^{(\infty)}P_x = \frac{A_x(1+i)^{\frac{1}{2}}}{1+a_x} = P_x(1+i)^{\frac{1}{2}}; $$

whence
$$ \begin{aligned} {}^{(\infty)}_{\ n}V_x &= A_{x+n}(1+i)^{\frac{1}{2}} - P_x(1+i)^{\frac{1}{2}}(1+a_{x+n}) \\ &= {}_nV_x(1+i)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40) \end{aligned} $$

99. A precisely similar line of reasoning applies to the values of policies which have been $n+\dfrac{1}{t}$ years in force, and to policies subject to premiums payable m times a year. It thus appears that, whether policies be valued singly or in groups, there are two ways in which allowance may be made for the immediate payment of claims. Either the proper reversions and premiums, $\overline{A}_x$ and ${}^{(\infty)}P_x$, may be used in the valuation, as in formula 39, in which case the required result is obtained without further process; or, the policy may be valued as if the sum assured were payable at the end of the year of death, and the result so produced multiplied by $(1+i)^{\frac{1}{2}}$ as in formula 40.

100. A corresponding argument will hold good if the sum assured be payable three months after death, in which case the multiplier would be $(1+i)^{\frac{3}{4}}$; or at any other interval. Either we may use in the valuation the reversions and premiums applicable to the benefit, or we may use the more ordinary reversions and premiums, and multiply the resulting value by the proper power of $(1+i)$.

101. Up to this point policy-values have been considered solely in connection with assurances for the whole of life, subject to uniform premiums; but the same principles are applicable to policies of every description. In finding the value of a policy, whatever its kind, either the prospective or the retrospective method may be used, and which of

these methods is the more convenient will depend upon the circumstances of the particular case in hand. For policies of a simple description, the prospective method will usually be the more easy to apply; while for those of a complex character, such as assurances with return of premium, the retrospective method will generally be preferable.

102. In formulas 33 and 44 of Chap. vii, and in Art. 33 of Chap. viii, it was shown that the equations between the annuity, the assurance, and the annual premium are the same for endowment assurances as for whole-term policies, the temporary annuity $a_{x:\overline{t-1}|}$ taking the place of the whole-term annuity, a_x. This is the case also as regards the policy-value. We have

$$_nV_{x:\overline{t}|}=A_{x+n:\overline{t-n}|}-P_{x:\overline{t}|}(1+a_{x+n:\overline{t-n-1}|} \quad . \quad . \quad . \quad (41)$$

and by writing $A_{x+n:\overline{t-n}|}$ and $P_{x:\overline{t}|}$ in terms of the annuities, this becomes

$$_nV_{x:\overline{t}|}=1-\frac{1+a_{x+n:\overline{t-n-1}|}}{1+a_{x:\overline{t-1}|}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

103. It must, however, be noted that endowment assurances cannot be valued in groups by precisely the same methods as are applicable to whole-term policies. Assuming, as in Art. 80, that the premium income is uniformly distributed over the year, and that consequently, taken one with another, the premiums in yearly cases are due on the average six months hence; then, in the case of an endowment assurance policy under which there are $t-n$ premiums still remaining to be paid, it will not be correct to use the annuity-value $\frac{1}{2}+a_{x+n:\overline{t-n-1}|}$; but we must rather interpolate between the annuity-due of $t-n$ payments, and the ordinary annuity of $t-n$ payments. That is, the annuity by which to value the premiums is $\frac{1}{2}(1+a_{x+n:\overline{t-n-1}|}+a_{x+n:\overline{t-n}|})$, or $\left(\frac{1}{2}+\frac{a_{x+n:\overline{t-n-1}|}+a_{x+n:\overline{t-n}|}}{2}\right)$.

104. Another method which will have the same effect as the foregoing, is to take the valuation age as the age at the nearest birthday, and when there are $t-n$ premiums still remaining to be paid, to multiply the premium by $1+a_{x+n:\overline{t-n-1}|}$ if the next renewal falls due within six months of the date of valuation; or by $a_{x+n:\overline{t-n}|}$, if the next renewal falls due beyond the six months.

105. As regards endowment assurance policies under which the sum assured is payable at the moment of death, we shall have, by Chap. x, Art. 21.

$$^{(\infty)}P_{x:\overline{t}|}=\frac{(M_x-M_{x+t})(1+i)^{\frac{1}{2}}+D_{x+t}}{N_{x-1}-N_{x+t-1}};$$

and

$$^{(\infty)}_{\ n}V_{x:\overline{t}|}=\bar{A}_{x+n:\overline{t-n}|}-{}^{(\infty)}P_{x:\overline{t}|}(1+a_{x+n:\overline{t-n-1}|}) \quad . \quad . \quad . \quad (43)$$

106. Manifestly it would be incorrect to write ${}^{(\infty)}_{\ n}V_{x:\overline{t}|} = {}_nV_{x:\overline{t}|}(1+i)^{\frac{1}{2}}$, because that portion of the sum assured consisting of the endowment is not affected by the change in the date of payment of the death claim.

107. Assurances with return of premium may next be briefly considered. In such cases the retrospective method of valuation will be the most convenient.

108. The premiums for Endowment Assurances with return were discussed in Chap. xvi., Arts. 59 to 61, and 82. Writing π for the net premium, whether the return is to take place only at death, or with the sum assured whenever that may be payable, and using the single letter V for the value of the policy, we have

$$V = \frac{\pi(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n}) - \pi(R_x - R_{x+n} - nM_{x+n})}{D_{x+n}} \quad (44)$$

This value is found merely by accumulating the premiums and accumulating the claims, and then striking the balance.

109. Should the contract be that the premiums are to be returned only in the event of the life assured surviving until the endowment becomes payable, then in respect of the return of premium there will not be any claims to accumulate, and we shall have simply

$$V = \frac{\pi(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} \quad . \ . \ . \ . \quad (45)$$

110. If the *office premiums* are to be returned, then in formula 44 the *office premium*, π', which can be found by Chap. xvi, Art. 82, must be used in that part of the expression which relates to the return of premium; but the *net premium* must be retained in that portion of the expression which relates to the accumulation of the premiums. That is

$$V = \frac{\pi(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n}) - \pi'(R_x - R_{x+n} - nM_{x+n})}{D_{x+n}} \quad (46)$$

111. If Endowments be in question, instead of Endowment Assurances, the principles are exactly the same, the only difference in detail being that in respect of the sum assured there are no claims to accumulate. The premiums for Endowments were discussed in Chap. vii, Art. 76, and in Chap. xvi, Arts. 50 and 77.

In the case of a simple Endowment without return of premium, we have

$$V = \frac{P^{\ 1}_{x\overline{t}|}(N_{x-1} - N_{x+n-1})}{D_{x+n}} \quad . \ . \ . \ . \ . \ . \quad (47)$$

But by formula 46 of Chap. vii, $\dfrac{D_{x+n}}{N_{x-1}-N_{x+n-1}}=P_{x:\overset{1}{\overline{n}|}}$; whence $\dfrac{N_{x-1}-N_{x+n-1}}{D_{x+n}}=\dfrac{1}{P_{x:\overset{1}{\overline{n}|}}}$. Therefore

$$V=\frac{P_{x:\overset{1}{\overline{t}|}}}{P_{x:\overset{1}{\overline{n}|}}}\quad\ldots\ldots\ldots\quad(48)$$

112. In the case of an Endowment subject to premiums to be returned in the event of the nominee dying within the term, writing $\pi_{x:\overset{1}{\overline{t}|}}$ for the premium, we have

$$V=\frac{\pi_{x:\overset{1}{\overline{t}|}}\{N_{x-1}-N_{x+n-1}-(R_x-R_{x+n}-nM_{x+n})\}}{D_{x+n}}\quad\ldots\quad(49)$$

But by Chap. xvi, formula 31,

$$\frac{D_{x+n}}{N_{x-1}-N_{x+n-1}-(R_x-R_{x+n}-nM_{x+n})}=\pi_{x:\overset{1}{\overline{n}|}};$$

Whence

$$\frac{N_{x-1}-N_{x+n-1}-(R_x-R_{x+n}-nM_{x+n})}{D_{x+n}}=\frac{1}{\pi_{x:\overset{1}{\overline{n}|}}}.$$

Therefore

$$V=\frac{\pi_{x:\overset{1}{\overline{t}|}}}{\pi_{x:\overset{1}{\overline{n}|}}}\quad\ldots\ldots\ldots\quad(50)$$

113. Formulas 48 and 50 are more curious than useful, because to employ them in practice extensive tables of premiums would be required, and these are not forthcoming. The meaning of the formulas can be easily explained. Taking formula 48, $P_{x:\overset{1}{\overline{n}|}}$ is the premium which will secure an endowment of 1 at the end of n years, and $\dfrac{1}{P_{x:\overset{1}{\overline{n}|}}}$ is therefore the endowment which an annual premium of 1 will secure; or, in other words, $\dfrac{1}{P_{x:\overset{1}{\overline{n}|}}}$ is the share belonging to (x) at the end of n years in a fund formed of the accumulations of premiums of 1 per annum. Hence $\dfrac{P_{x:\overset{1}{\overline{t}|}}}{P_{x:\overset{1}{\overline{n}|}}}$ is the share belonging to (x) at the end of n years in a fund formed of the accumulations of the premiums actually paid. This is as it should be, because, there being no claims, the value of the endowment policy consists merely of the premiums accumulated with benefit of survivorship.

By a very similar line of argument, formula 50 may be deduced.

114. Formulas 49 and 50 are applicable only if *net* premiums are to be returned. If the return is to consist of the *office* premiums, then the *office* premium, $\pi'_{x:\overset{1}{t}|}$, must be inserted in that part of the expression in formula 49 which relates to the return of premium, while the *net* premium is retained in that part of the expression which relates to the accumulation of the premiums. We shall then have

$$V=\frac{\pi_{x:\overset{1}{t}|}(N_{x-1}-N_{x+n-1})-\pi'_{x:\overset{1}{t}|}(R_x-R_{x+n}-nM_{x+n})}{D_{x+n}} \quad . \quad (51)$$

115. If a whole-term assurance be granted in consideration of a single premium, or of premiums payable for a limited number of years only, then the loading (which, as explained in Chap. xvi, Art. 72, is the fund provided for necessary expenses, and for profits), is not spread uniformly over the duration of the policy, but is wholly paid up, with the net premium, in one sum, or within a term of years. But a certain portion of the expenses will continue after the premiums have ceased, and it is desirable that the profits should not be entirely realized and distributed within a limited period, but that they should extend over the whole duration of the policy. It is therefore well, in the case of policies subject to a limited number of premiums, to reserve a part of the loading received in the early years, over and above the reserve made by the net premium method in respect of the sum assured.

116. If P_x be the ordinary whole-term premium, and ϕ_x the loading thereon; and if P'_x be the office premium, so that $P'_x=P_x+\phi_x$; and if ${}_tP_x$, ${}_t\phi_x$, and ${}_tP'_x$ be respectively the net premium, the loading, and the office premium, limited to t payments, then the net-premium value of an ordinary whole-term policy may be written,

$${}_nV_x=A_{x+n}-P'_x(1+a_{x+n})+\phi_x(1+a_{x+n}) \quad . \quad . \quad . \quad (52)$$

We see here that the value of the reserve of loading for future expenses and profits is $\phi_x(1+a_{x+n})$.

If the value of the policy by limited payments be denoted by ${}_nU_x$, we have, on the same principles, when $n<t$,

$${}_nU_x=A_{x+n}-{}_tP'_x(1+a_{x+n:\overline{t-n-1}|})+{}_t\phi_x(1+a_{x+n:\overline{t-n-1}|}) \quad . \quad (53)$$

and the reserve of loading for future expenses and profits, under these circumstances, is ${}_t\phi_x(1+a_{x+n:\overline{t-n-1}|})$.

117. To place the two policies on an equality as regards reserve for future expenses and profits, we must add to the expression in equation 53 the difference between the reserves of loading in equations 52 and 53, namely $\phi_x(1+a_{x+n})-{}_t\phi_x(1+a_{x+n:\overline{t-n-1}|})$, and the value of the policy by limited payments then becomes,

$${}_nU_x=A_{x+n}-{}_tP'_x(1+a_{x+n:\overline{t-n-1}|})+\phi_x(1+a_{x+n}) \quad . \quad . \quad (54)$$

That is, in the case of a policy subject to premiums payable for a limited number of years, the value of the *office premium* must be deducted from the value of the sum assured, or, as it is called, a *gross premium valuation* of the policy must be made, and to the result must be added the value at the date of valuation of the loading on the whole term premium at the original age at entry.

118. If the course explained in Art. 117 be followed, the entire loading on the whole term premium is reserved; but possibly such a full reserve may be deemed unnecessary, because part of the expenses, such as commission to the agent, may cease when all the premiums are paid up. In such case, assuming these special expenses to be at the rate of r per unit of the premium, we may write, instead of formula 54,

$${}_nU_x=A_{x+n}-{}_tP'_x(1-r)(1+a_{x+n:\overline{t-n-1}|})+\{P'_x(1-r)-P_x\}(1+a_{x+n}) \quad . \quad . \quad . \quad . \quad (55)$$

119. If n be not less than t, that is, if the valuation be made after all the premiums have been paid up, then, in equations 54 and 55, the term involving $a_{x+n:\overline{t-n-1}|}$ will disappear.

120. The *Surrender-Value* of a policy is that sum which the assuring office will pay in consideration of the cancelment of the contract. It is usual in calculating the surrender-value to deduct some portion of the theoretical value as found by the formulas investigated in this chapter. The customs of the different companies are very various in this respect, and it is not intended to discuss the subject here.

121. Analogous to the question of policy-values is that of *Free*, or *Paid-up, Policies.* A *Free Policy* is a policy free from all future premiums, and in this connection the name is applied to policies granted, instead of cash surrender-values, on cancelment of other policies.

122. For instance, if there be a whole-term policy which was effected n years ago on a life then aged x, at the annual premium of P_x:—How much by way of paid-up policy may be given in exchange for it?

The value of the existing policy being ${}_nV_x$, this is the sum available for securing the paid-up policy. But A_{x+n} will secure a paid-up policy of 1; and therefore, by simple proportion, $\frac{{}_nV_x}{A_{x+n}}$ is the sum which ${}_nV_x$ will secure. That is, using the symbol W for *Free Policy*,*

$$W=\frac{{}_nV_x}{A_{x+n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

123. The question may be looked at in another way. As shown in Art. 19, $\frac{P_x}{P_{x+n}}$ is the sum which the future premiums will secure; and as these premiums are to be waived, $\frac{P_x}{P_{x+n}}$ is the sum to be deducted from the original assurance when it is exchanged for a free policy. That is

$$W=1-\frac{P_x}{P_{x+n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

124. The expressions in equations 56 and 57 are identical. In equation 56, ${}_nV_x=A_{x+n}-P_x(1+a_{x+n})=A_{x+n}-P_x\frac{A_{x+n}}{P_{x+n}}$. Whence, dividing by A_{x+n}, $W=1-\frac{P_x}{P_{x+n}}$.

125. Equation 57 has been deduced for ordinary whole-term assurances, but it is general in its character. Let P be the premium at the outset for a benefit of any kind, and let ${}^nP$ be the premium for *the same benefit* after n years. Then, because after n years the premium to secure 1 is ${}^nP$, therefore the benefit which a premium of 1 will secure is $\frac{1}{{}^nP}$, and the benefit which a Premium of P will secure is $\frac{P}{{}^nP}$. But, no matter what may be the nature of the benefit, it may be looked upon as consisting of two parts; first that portion which the reserve will secure, and which is the amount for which a free policy can be granted, because it will remain even if the payment of premium cease; and second, that portion which the future premiums will secure. The second portion we have found to be $\frac{P}{{}^nP}$: whence

$$W=1-\frac{P}{{}^nP} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

126. In order that the foregoing argument may be valid, the benefit after n years must be precisely the same as that assured at the commencement, and the premium ${}^nP$ must cease at the same time as the premium P. Thus, in the case of a policy subject to a limited number of premiums, $\mathrm{W}=1-\dfrac{{}_{t}\mathrm{P}_{x}}{{}_{t-n}\mathrm{P}_{x+n}}$; and in the case of a short-term policy, $\mathrm{W}=1-\dfrac{\mathrm{P}^{1}_{x:\overline{t}|}}{\mathrm{P}_{\overset{1}{x+n}:\overline{t-n}|}}$; and in the case of an endowment, $\mathrm{W}=1-\dfrac{\mathrm{P}_{x:\overset{1}{\overline{t}|}}}{\mathrm{P}_{x+n:\overset{1}{\overline{t-n}|}}}$; and in the case of an endowment assurance, $\mathrm{W}=1-\dfrac{\mathrm{P}_{\overline{x:t}|}}{\mathrm{P}_{x+n:\overline{t-n}|}}$. For a policy on joint lives, $\mathrm{W}=1-\dfrac{\mathrm{P}_{xy}}{\mathrm{P}_{x+n:y+n}}$; and for a contingent assurance on (x) against (y), $\mathrm{W}=1-\dfrac{\mathrm{P}^{1}_{xy}}{\mathrm{P}_{\overset{1}{x+n}:y+n}}$. If the assurance be payable only should (x) die after (y), two cases present themselves. After n years, if (x) survive, (y) may be either alive or dead. In the former case $\mathrm{W}=1-\dfrac{\mathrm{P}^{2}_{xy}}{\mathrm{P}_{\overset{2}{x+n}\ y+n}}$: while in the latter case, $\mathrm{W}=1-\dfrac{\mathrm{P}^{2}_{xy}}{\mathrm{P}_{x+n}}$. Similarly, if the assurance be on the survivor of (x) and (y), there are two cases to be considered, namely, that wherein both the lives survive, and that wherein one of the lives, say (x), has died. In the former case, $\mathrm{W}=1-\dfrac{\mathrm{P}_{\overline{xy}}}{\mathrm{P}_{\overline{x+n:y+n}}}$; and in the latter case, $\mathrm{W}=1-\dfrac{\mathrm{P}_{\overline{xy}}}{\mathrm{P}_{y+n}}$.

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* This symbol, W, was assigned to the Free Policy in the Universal Notation adopted by the Actuarial Congress held in London in 1898. In the first edition of the *Text-Book* the symbol (FP) was employed.

127. Where either the benefit or the premium alters with the lapse of time, formula 58 is not applicable. Thus, it cannot be used for varying assurances, or for assurances by increasing or decreasing premiums. Also, benefits with return of premium are beyond its range, because the benefit varies with each premium paid.

128. In deducing formula 57, *net* premiums were considered, but in theory the argument applies equally to *office* premiums. In practice, however, care must be exercised in dealing with office premiums, because the results will differ widely according to the manner in which the premiums are loaded, and anomalies may very possibly arise.

129. There is a problem which is frequently met with in the valuations of small friendly societies, and which therefore may conveniently be dealt with here, although it does not belong strictly to the subject of policy-values. In a society of m members of various ages, a levy of 1 is made, at the death of every member, upon all the surviving members. What is the present value of all these prospective payments?

130. Let it first be supposed that all the members are of the same age, x, and that the levies are paid at the end of each year by all the members who survive the year, in respect of each who died in the year. Let also the radix of the mortality table be so taken that $l_x=m$. Then, if V represent the value of all the levies,

$$\begin{aligned}
V &= v\,.\,l_{x+1}\,.\,d_x + v^2.\,l_{x+2}\,.\,d_{x+1} + \&c.\\
&= vl_{x+1}(l_x - l_{x+1}) + v^2 l_{x+2}(l_{x+1} - l_{x+2}) + \&c.\\
&= v\,.\,l_x\,.\,l_{x+1} + v^2.\,l_{x+1}\,.\,l_{x+2} + \&c.\\
&\quad -(vl_{x+1}l_{x+1} + v^2 l_{x+2}l_{x+2} + \&c.)\\
&= l_{x-1}\,.\,l_x\,.\,a_{x-1:x} - l_x l_x a_{xx}\\
&= \frac{l_x l_x}{p_{x-1}}\,.\,a_{x-1:x} - l_x l_x a_{xx}\\
&= m^2\left(\frac{a_{x-1:x}}{p_{x-1}} - a_{xx}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (59)
\end{aligned}$$

131. Formula 59 understates the value of the levies, because it assumes that only those members who survive the year pay the levies of the year, whereas, according to the terms of the question, even those members who die in the year will have to pay the levies in respect of the members who die before them in that year.

132. A more accurate solution may be obtained as follows. Still assuming that all the members are aged x, let each member effect by a single premium a contingent assurance on the life of each other member against his own life, by means of which to pay his share of the levies as they arise. The value of the levies will then be exactly equal to the value of all the contingent assurances. The value of each contingent assurance is $\overline{\mathrm{A}}^{1}_{xx}$, which by formula 13 of Chap. xiii, is equal to $\frac{1}{2}\overline{\mathrm{A}}_{xx}$. There are m members, and each of them effects $m-1$ contingent assurances, so that there are $m(m-1)$ such assurances at the outset, and therefore their value is $\frac{m(m-1)}{2}\overline{\mathrm{A}}_{xx}$. That is

$$V = \frac{m(m-1)}{2}\overline{\mathrm{A}}_{xx} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (60)$$

133. The ages of the actual members of the society being various, an average age, x, must be found in order to apply either formula 59 or 60. Of the m members let a be aged y, b aged z, &c. Then applying Makeham's formula, by Chap. vi, Art. 28, and Chap. xii, Art. 23, taking x as the average age, $m\mu_x = (a\mu_y + b\mu_z + \&c.)$; whence, by referring to the table of the force of mortality, x may be found.

134. Theoretically, formula 60 is the correct solution of the problem, but in practice it is not entirely satisfactory. If members secede from the society, or if new members join, the value of the levies, even as regards the original members who remain, is changed; and there is no practical method by which account can be taken of these causes of disturbance.

CHAPTER XIX.

Life Interests and Reversions.

1. It is not intended to discuss in this chapter the many limited interests in property which may be created by deed or will, but only to investigate in an elementary way some of the mathematical formulas which have been proposed for the practical valuation of the four most important of such interests, namely, *Life Interests in Possession, Reversionary Life Interests, Absolute Reversions,* and *Contingent Reversions.*

2. When a person is entitled to the income of settled property for life, he is said to possess a *Life Interest* in the property, and is called the *Life Tenant in Possession*, or often simply the *Life Tenant.* The Life Interest therefore is a benefit of the same kind as an immediate annuity, the difference being that in the one case the income may possibly vary from time to time, while in the other case it is fixed.

3. What is the value of a Life Interest to (x) of 1 per annum, to pay the purchaser a given rate of interest on his investment, and to secure by means of a life policy a return of the principal on the death of the life tenant?

Assuming that the rate of interest intended to be realized is i, and that P'_x is the *office premium* for which a whole-world policy on (x) may be obtained, and writing $d_{(i)}$ and $v_{(i)}$ for the values of these functions calculated at rate i, then for an annuity of $P'_x+d_{(i)}$ payable at the end of each year a purchaser can afford to lay out $v_{(i)}$, because at the end of each year survived by the life tenant the annuity will supply P'_x, the premium then falling due, and $d_{(i)}$, the interest for that year on $v_{(i)}$, and at the end of the year of death the sum assured of 1 will repay the investment of $v_{(i)}$, and provide $d_{(i)}$, the interest upon it for that year.

The total sum which can be laid out for an annuity of $P'_x+d_{(i)}$ being thus $v_{(i)}$, the amount which can be paid to the vendor is $v_{(i)}-P'_x$, because P'_x, the first premium, must be paid to the assurance office at the commencement of the transaction.

Writing a'_x for the value of the life interest of 1 per annum, we therefore have, since $v_{(i)}-P'_x$, or $1-(P'_x+d_{(i)})$, is the value of a life interest of $P'_x+d_{(i)}$ per annum,

$$a'_x=\frac{1}{P'_x+d_{(i)}}-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

4. Seeing that for a life interest of $P'_x+d_{(i)}$ per annum the sum to be assured is 1, and that the amount invested is $v_{(i)}$, therefore, for a life interest of 1 per annum the sum to be assured is $\frac{1}{P'_x+d_{(i)}}$, and the amount invested is $\frac{v_{(i)}}{P'_x+d_{(i)}}$.

5. The line of reasoning adopted here to arrive at the practical value of a life interest, is similar to that which was followed in explaining formula 39 in Art. 64 of Chap. vii. In fact, formula 1 of this chapter is the same as No. 39 of Chap. vii, except that for P_x, the net premium formerly used, we have substituted the office premium at which the life can be actually assured, and that now d is to be taken at the rate of interest which it is desired to realize, that rate not being necessarily or even probably the same as was used in calculating the premium.

6. In Art. 28 of Chap. viii, it was shown how to employ a Conversion Table for the purpose of finding the values of life interests. Taking the premium at which the life may be assured, and entering inversely the table under the rate of interest to be realized, the result is the value of the life interest.

7. So also a table of reciprocals may be employed. Entering the table with $P'_x+d_{(i)}$, and deducting unity from the result, we have the value of the life interest.

8. It must be noted that P'_x is the annual premium for a *whole-world* policy: otherwise, if the life assured were to proceed abroad, an extra premium might become payable, for which provision is not made in the calculations.

9. Formula 1 gives the value of a curtate annuity payable yearly, but nevertheless it is very generally employed in practical transactions, although the income from life interests is almost always apportionable, and is usually receivable at shorter intervals than once a year. The effect of the formula therefore is, to give the purchaser a slightly better return on his outlay than the calculations seem to provide.

10. Supposing a policy to be effected for the full amount of $\frac{1}{P'_x+d_{(i)}}$, it is seen that after the lapse of any number of complete years the value of the life interest and policy together will always be the same, namely, $\frac{1}{P'_x+d_{(i)}}-1$; the annuity providing for the premium of assurance and the interest on the outlay during the continuance of the life, and the policy returning the capital at the end of the year of death.

11. If in the contract of sale there be a stipulation that the vendor may redeem his life interest, then the amount of redemption-money should not be less than $\frac{v_{(i)}}{P'_x+d_{(i)}}$, no matter what time may have elapsed since the contract was entered into, because that is the amount which was actually laid out by the purchaser and which is still standing to the debit of the transaction in his books. In addition, however, to a re-transfer of the life interest, it has been contended that the life tenant should have an assignment of the policy, or else that its value at the date of redemption should be deducted from the redemption-money.

12. As a numerical example, let it be required to find the value of a life interest of 150 per annum, to pay 4½ per-cent interest, the life being assurable at an annual premium of 4·6125 per-cent.

Here $P'_x=\cdot046125$, $d_{(i)}=\cdot043062$, and $v_{(i)}=\cdot956938$. Whence, for a life interest of 1 per annum, the amount to be assured, $\frac{1}{P'_x+d_{(i)}}$, is 11·2124, on which the annual premium is ·51717. The value of the life interest is 10·2124; and adding to this the annual premium, we have 10·7296, the amount invested, the annual interest on which at 4½ per-cent is ·48283. The premium, ·51717, and the interest, ·48283, added together, are equal to unity, and thus the accuracy of the work is proved. It will also be noticed that the amount of the policy, 11·2124, multiplied by v, ·956938, is exactly 10·7296, the amount invested, as it should be.

From the foregoing figures we at once have, for a life interest of 150 per annum,

Sum Assured	1681·860
Annual Premium	77·576
Value paid Vendor	1531·860
Amount Invested	1609·436
Annual Interest	72·424
Annual Premium, as above	77·576
Annual Amount of Life Interest	150·000
Redemption-money, being Amount Invested .	1609·436

13. In practice the question is often put in another way. A person possessed of a life interest of ample value requires a sum of 5,000. What annual charge must he give to cover interest and premium, and for what amount must his life be assured, assuming interest at 4½ per-cent, and an annual premium of 4·6125 per-cent?

We have seen that for a life interest of 1 per annum a vendor will receive $\frac{1}{P'_x+d_{(i)}}-1$: therefore for a sum down of 1 he must give an annual charge of $\frac{P'_x+d_{(i)}}{1-(P'_x+d_{(i)})}$. Also, for a sum down of $v_{(i)}-P'_x$, or $1-(P'_x+d_{(i)})$, the amount to be assured is 1: therefore, for a sum down of 1 the amount to be assured is $\frac{1}{1-(P'_x+d_{(i)})}$.

Working out by means of these expressions the figures for a sum down of 5,000, we have

Sum Assured	5489·602
Purchase-money	5000·000
Annual Premium	253·208
Amount Invested	5253·208
Annual Interest	236·394
Annual Premium, as above	253·208
Annual Charge on Life Interest	489·602
Redemption-money, being Amount Invested .	5253·208

14. Sometimes a life interest is offered for sale coupled with a policy of assurance which is not of sufficient amount to cover the whole. What is the value of the life interest under such circumstances?

Let the life interest be R per annum, and let the existing policy be for S, at an annual premium of P'_1 per unit, the premium actually payable being therefore $P'_1 \times S$. Also, let a'_1 represent the value of a life interest of 1 per annum when the life was assurable at the premium P'_1, and let a'_2 be the value now when the life may be assured at the premium P'_2.

By Art. 3, a policy of 1 covers a life interest of $P'_1 + d_{(i)}$, and therefore a policy of S will cover a life interest of $S(P'_1 + d_{(i)})$. Moreover, by formula 1 and by Art. 10 the value of the policy of S and the life interest of $S(P'_1 + d_{(i)})$ is still the same as when the policy was effected, that is $S(P'_1 + d_{(i)})\left\{\frac{1}{P'_1 + d_{(i)}} - 1\right\}$, or $S\{1 - (P'_1 + d_{(i)})\}$.

The balance of the life interest not covered by the policy S is $R - S(P'_1 + d_{(i)})$, and its value now is $\{R - S(P'_1 + d_{(i)})\}\left\{\frac{1}{P'_2 + d_{(i)}} - 1\right\}$, and the amount of additional policy required to cover it is $\frac{R - S(P'_1 + d_{(i)})}{P'_2 + d_{(i)}}$, the annual premium being $P'_2 \frac{R - S(P'_1 + d_{(i)})}{P'_2 + d_{(i)}}$.

Adding together the values of the two portions, we have the value of the whole life interest of R per annum, namely,

$$S\{1 - (P'_1 + d_{(i)})\} + \{R - S(P'_1 + d_{(i)})\}\left\{\frac{1}{P'_2 + d_{(i)}} - 1\right\}$$

or

$$S - S\frac{P'_1 + d_{(i)}}{P'_2 + d_{(i)}} + R\left\{\frac{1}{P'_2 + d_{(i)}} - 1\right\}$$

or

$$Ra'_2 + S(P'_2 - P'_1)(1 + a'_2).$$

15. The question may be looked at from a somewhat different standpoint. The benefit offered for sale is composed of a life interest of R, of which the value is Ra'_2, and of a policy of S effected at rate of premium P'_1, of which the value has to be found. By Art. 10, the value of a life interest of 1 coupled with a policy of $\frac{1}{P'_1 + d_{(i)}}$ effected at rate of premium P'_1 is always a'_1, while the present value of the life interest alone is a'_2. Therefore the present value of the policy alone is $a'_1 - a'_2$, or $(P'_2 - P'_1)\frac{1}{(P'_2 + d_{(i)})(P'_1 + d_{(i)})}$. This is the value of a policy of

$\frac{1}{P'_1+d_{(i)}}$, and therefore the value of a policy of 1 is $\frac{P'_2-P'_1}{P'_2+d_{(i)}}$ or $(P'_2-P'_1)(1+a'_2)$, and the value of the policy of S is $S(P'_2-P'_1)(1+a'_2)$; adding which to the value of the life interest alone, we have, as before, $Ra'_2+S(P'_2-P'_1)(1+a'_2)$ for the value of a life interest of R coupled with the policy of S.

16. To take a numerical example:—If interest at the rate of $4\frac{1}{2}$ per-cent is to be realized, what is the value of an annuity of 100 during the continuance of a life which can now be assured at 3·5 per-cent, and which is covered to the extent of 500 by a policy effected some years ago at a premium of 2·5 per-cent?

Here $a'_1=13{\cdot}6925$ and $a'_2=11{\cdot}8103$; $R=100$, $S=500$, $P'_1={\cdot}025$ and $P'_2={\cdot}035$. Therefore the value of the life interest alone is 1181·030, and that of the policy alone is 64·052; and the value of the two together is 1245·082. The amount of new policy to be effected is 845·083, on which the annual premium will be 29·578, while the premium on the old policy for 500 is 12·500.

The transaction may be set out thus:—

Purchase-money of Annuity and Policy . .	1245·082
First Premium on New Policy . . .	29·578
Premium falling due on Old Policy . .	12·500
Total Amount Invested	1287·160

To verify the work, we have

Interest on Investment	57·922
Premium on Old Policy	12·500
„ New „	29·578
Total . .	100·000

which is exactly the annual amount of the life interest, as it should be.

17. The value of a *Reversionary Life Interest* differs from that of an immediate life interest, in that provision must be made for the assurance premium and the interest on the investment until the life interest falls into possession. Assuming that the annual income in reversion is 1, the purchaser must provide for an annuity of 1 during the joint lives, and the cost of that annuity must be borne by the vendor. The value paid to the vendor will be diminished by the sum required to render the life interest immediate instead of reversionary, but the total sum invested will not be changed.

18. If the life of the tenant in reversion can be assured at the rate of premium P'_x, and if an annuity on the joint lives can be obtained for $a_{xy(j)}$, while the purchaser desires to realize interest at rate i on his investment; then, adding suffixes as before to indicate the rates at which the several values are to be calculated, and writing $a'_{y|x}$ for the practical value of the reversionary life interest, we have

$$a'_{y|x} = \frac{1}{P'_x + d_{(i)}} - (1 + a_{xy(j)}) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

19. Formula 2 assumes that a full year's payment of the life interest will be made at the end of the year of the death of (y), whereas in reality payment will be made only, on the average, for half a year. This defect in the formula may be sufficiently corrected by taking for a_{xy} the practical value of a complete annuity. In actual transactions it at one time was not unusual to employ the Carlisle Table at 3½ per-cent interest (or perhaps 3¼ per-cent) for the purpose, using the curtate value, such being approximately the price at which the complete annuity could be purchased from a good company, but conditions change, and an annuity-value must be employed, suited to the circumstances at the date of the transaction.

20. The amount to be assured is the same as if the life interest were immediate, namely, $\frac{1}{P'_x + d_{(i)}}$, and the assurance must be for the whole of the life of (x), and not only on his life against that of (y), because when the life interest falls into possession the policy will still be required to protect it.

21. The redemption-money will also be the same as in the case of a life interest in possession, namely, $\frac{v_{(i)}}{P'_x + d_{(i)}}$, that being the total outlay.

22. The following statement shows the items of which the investment is composed:—

Value paid Vendor . .	$\frac{1}{P'_x + d_{(i)}} - (1 + a_{xy(j)})$
First Year's Premium .	$\frac{P'_x}{P'_x + d_{(i)}}$
Cost of Annuity . . .	$a_{xy(j)}$
Total Outlay . . .	$\frac{v_{(i)}}{P'_x + d_{(i)}}$

23. The amount assured being $\frac{1}{P'_x + d_{(i)}}$, the annual premium is $\frac{P'_x}{P'_x + d_{(i)}}$; and the amount invested being $\frac{v_{(i)}}{P'_x + d_{(i)}}$, the annual interest on it is $\frac{d_{(i)}}{P'_x + d_{(i)}}$; and the annual premium and interest added together are unity. The premium and interest during the joint lives will therefore be met by the annuity provided for the purpose, and afterwards they will be met out of the life interest itself when it has fallen into possession. Moreover, at the end of the year of death the capital will be returned, and interest upon it for that year will be provided, by means of the life policy.

24. To take a numerical example:—Let it be required to find the value of a reversionary life interest of 150 per annum coming to a person now aged 30, on the death of a person now aged 60, the purchaser desiring to realize 4½ per-cent interest on his investment, and by means of a policy and an annuity to protect himself against loss through the premature death of the life tenant in reversion or the unusual prolongation of the joint lives.

Let it be assumed that the life aged 30 can be assured at an annual premium of 2·1 per-cent; and in order to work the example from the tables in this volume, let the joint-life annuity be calculated from these tables at 3½ per-cent, so that $a_{30:60} = 9{\cdot}130$. Then, performing the calculations, we have

Sum to be Assured	2341·482
Annual Premium	49·170
Cost of Annuity	1369·500
Value paid to Vendor	822·000
Amount Invested	2240·670
Annual Interest	100·830
Annual Premium, as above	49·170
Annual Amount of Life Interest	150·000
Redemption-money, being Amount Invested .	2240·670

25. It may sometimes happen that the transaction can be conveniently carried out by commuting the annual premiums during the joint lives. Assuming that the premiums may be commuted by a 4 per-cent annuity table, the formula becomes

$$a'_{y|x} = \frac{1}{\mathrm{P}'_x + d_{(i)}} - \frac{\mathrm{P}'_x(1 + a_{xy(4\,\%)}) + d_{(i)}(1 + a_{xy(j)})}{\mathrm{P}'_x + d_{(i)}} \quad . \quad . \quad (3)$$

where the annuities, &c., are to be taken at the rates named in the suffixes. Using the same example as before, the value of the joint-life annuity at 4 per-cent is 8·797; and we have

Sum to be Assured	2341·482
Sum paid in Commutation of Premiums during Joint Lives	481·718
Cost of Annuity for Interest during Joint Lives	920·578
Value paid to Vendor	838·374
Amount Invested	2240·670

26. Property in reversion generally yields a better return than property in possession, there being less competition for it among investors. Formula 2 may be easily transformed so as to give the value of a reversionary life interest to pay the purchaser interest at rate i after it falls into possession, and in the meantime interest at rate i', where from the nature of the case i' is greater than i. Let $d_{(i)}$ be the rate of discount calculated at rate i, and $d_{(i')}$ that calculated at rate i'; and let, as before, P'_x be the office annual premium for a whole-world policy on the life of the tenant in reversion, and $a_{xy(j)}$ the practicable joint-life annuity-value. In formula 2 the calculations are planned to pay interest at rate i throughout the whole duration of the transaction, and provision has therefore been made for premium and interest by deducting $(1 + a_{xy(j)})$, being the value during the joint lives of the premium P'_x and of the interest at rate i, from the value of a life interest in possession. Under the new arrangement the deduction must, however, be made to correspond to interest at rate i', while the premium remains unchanged. Therefore the amount deducted must be altered in the ratio $\frac{\mathrm{P}'_x + d_{(i')}}{\mathrm{P}'_x + d_{(i)}}$; that is, instead of deducting simply $(1 + a_{xy(j)})$, we must deduct $\frac{\mathrm{P}'_x + d_{(i')}}{\mathrm{P}'_x + d_{(i)}}(1 + a_{xy(j)})$, and we have

$$a'_{y|x} = \frac{1}{P'_x + d_{(i)}} - \frac{P'_x + d_{(i')}}{P'_x + d_{(i)}}(1 + a_{xy(j)}) \quad . \quad . \quad . \quad . \qquad (4)$$

27. Applying formula 4 to the same numerical example as before, the value of the reversionary life interest of 150 per annum to pay 4½ per-cent interest after it falls into possession, and in the meantime to pay 5 per-cent, comes out at 713·9. This is a smaller value than that produced by formula 2, because provision has been made for a higher rate of interest while the annuity remains in reversion. The sum to be assured, the annual premium, and the redemption-money, remain unchanged, but a larger sum is absorbed in providing a larger joint-life annuity.

28. To find the annual reversionary charge which must be given in consideration of an advance, the reciprocals must be taken of the expressions in equations 2, 3, or 4, as the case may be. Thus, in consideration of an advance of 1, the lender, if he is to realize interest at the rate i throughout the whole transaction, must, by equation 2, receive an annual reversionary charge of

$$\frac{P'_x + d_{(i)}}{1 - (P'_x + d_{(i)})(1 + a_{xy(j)})}.$$

If the premiums are to be commuted at 4 per-cent interest during the joint lives, then, by equation 3, the annual reversionary charge must be

$$\frac{P'_x + d_{(i)}}{1 - P'_x(1 + a_{xy(4°/_o)}) - d_{(i)}(1 + a_{xy(j)})}.$$

And if interest at rate i is to be realized when the annuity falls into possession, while in the meantime the rate is to be i', then, by formula 4, the annual reversionary charge must be

$$\frac{P'_x + d_{(i)}}{1 - (P'_x + d_{(i')})(1 + a_{xy(j)})}.$$

29. It is assumed in all the foregoing formulas that the purchaser of a reversionary life interest will actually buy an annuity to provide the premium and interest during the joint lives, but in this respect there are great divergencies of practice, even when these formulas are used. Supposing the reversionary life interest to be sold to an assurance company, the company could, it is true, without buying the annuity, set one up *pro formâ* in its books. By doing so it would throw on the annuity branch of its business the chances of profit and of loss, and all risks would be removed from the investment department. The amount invested in the security, which would include the sum set apart for the *pro formâ* annuity, would always remain the same: the interest and premium would be regularly provided by means of the *pro formâ* annuity during the joint lives, and afterwards by means of the life interest itself; and eventually the capital would be returned by means of the life policy.

30. If, however, an office enter on transactions of this kind in numbers large enough to form an average, then the setting up of the joint-life annuity is not really necessary. The value of a joint-life annuity has still to be deducted from the amount to be paid the vendor, and if the deduction be calculated at the rate of interest intended to be realized on the investment, the a_{xy} in formula 2 will be taken at the same rate of interest as is used in d, and thus there will be no profit made in the annuity branch. The correction discussed in Art. 19 should be attended to by deducting $\frac{1}{2}\mathrm{A}_{x\overset{1}{y}}$ from the value of the reversionary life interest. We then have

$$a'_{y|x}=\frac{1}{\mathrm{P}'_x+d_{(i)}}-(1+a_{xy(i)})-\tfrac{1}{2}\mathrm{A}_{x\overset{1}{y}(i)} \quad . \quad . \quad . \quad . \quad (5)$$

31. The contingent assurance $\mathrm{A}_{x\overset{1}{y}}$ is a little troublesome to calculate; but in practice it will be sufficiently accurate to substitute for it A_{xy}, which is nearly equal to it in the common case of (y) being much older than (x). With this slight modification, remembering that $\mathrm{A}_{xy}=1-d(1+a_{xy})$, formula 5 becomes

$$a'_{y|x}=\frac{1}{\mathrm{P}'_x+d_{(i)}}-\tfrac{1}{2}-(1-\tfrac{1}{2}d_{(i)})(1+a_{xy(i)}) \quad . \quad . \quad . \quad (6)$$

32. If formula 6 be used the whole profit of the transaction, except such as may be derived from the life policy, will appear in the rate of interest realized; and the reversionary life interests, taken one with another, will so fall in that the rate assumed in the formula will on the average be secured.

33. Formula 5 is a modification of formula 2; and so now, from formula 4, we may write

$$a'_{y|x}=\frac{1}{\mathrm{P}'_x+d_{(i)}}-\frac{\mathrm{P}'_x+d_{(i')}}{\mathrm{P}'_x+d_{(i)}}(1+a_{xy(i')})-\tfrac{1}{2}\mathrm{A}_{xy(i')} \quad . \quad . \quad (7)$$

for the value of a reversionary life interest to realize rate i' whilst it remains in reversion, and rate i after it falls into possession.

34. In the case of an *Absolute Reversion*, if interest on the investment is to be provided for until the benefit falls in, for a reversion of 1 receivable on the death of a person now aged x we have the formula

$$A'_x = 1 - d_{(i)}(1 + a_{x(j)}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where d is to be taken at the rate to be realized by the investor, and where $a_{x(j)}$ is at the rate at which an annuity can be secured from a good company. The formula is seen by inspection to be correct. It is identical in principle with formula 25 of Chap. vii, and the verbal interpretation given of that formula (Chap. vii, Art. 43) applies here also.

35. The amount invested is v, of which da_x is spent in purchasing the annuity for interest, and $v - da_x$, or $1 - d(1 + a_x)$, is paid to the vendor.

36. To take a numerical example:—What sum can be given for a reversion to 5,000, if the purchaser is to realize interest at 4½ per-cent on the outlay, assuming that an annuity on the life of the life tenant can be purchased for 10·223. Applying the formula, we have

Value paid Vendor	2583·600
Cost of Annuity of 215·312 for Interest .	2201·100
Total Sum Invested	4784·700

37. Conversion Tables may be used to find the practical value of a reversion by means of formula 8. If the Single-Premium Conversion Table, under the rate of interest to be realized, be entered with the practicable annuity, the result will be the value sought.

38. If, instead of the reversion being sold, a reversionary charge is to be given in consideration of a fixed sum to be paid down, what should be the amount of reversionary charge?

By formula 8, the value of a reversion of 1 is $1 - d_{(i)}(1 + a_{x(j)})$; and therefore the reversionary charge for a sum of 1 should be $\frac{1}{1 - d_{(i)}(1 + a_{x(j)})}$. For a reversion of 1, we have seen that an annuity o d must be purchased; therefore $\frac{d_{(i)}}{1 - d_{(i)}(1 + a_{x(j)})}$ is the annuity to b purchased for a reversion of $\frac{1}{1 - d_{(i)}(1 + a_{x(j)})}$, and the cost thereo will be $\dfrac{d_{(i)}a_{x(j)}}{1-d_{(i)}(1+a_{x(j)})}$. The total investment will therefore be $1+\dfrac{d_{(i)}a_{x(j)}}{1-d_{(i)}(1+a_{x(j)})}$ or $\dfrac{v_{(i)}}{1-d_{(i)}(1+a_{x(j)})}$.

39. To return to the example of Art. 36:—What reversionary charge should be given in consideration of a sum down of 2,000—if the investor is to realize 4½ per-cent interest on his investment, when an annuity may be obtained on the life of the life tenant for 10·223?

Working out the formula, we have 3870·568 for the amount of reversionary charge. Also we have

Amount paid Reversioner	2000·000
Cost of Annuity of 166·676 for Interest .	1703·914
Total Amount Invested	3703·914

40. As explained in Art. 19, it at one time was not unusual in practical transactions to employ the Carlisle 3½ per-cent (or 3¼ per-cent) annuity for $a_{x(j)}$, but, seeing that an annuity is seldom or never actually bought there is no need to assume one at a practicable rate. If a suitable mortality table be used, and if the calculations be made throughout at the rate of interest to be realized, then, on the average of a large number of transactions, the reversions will so fall in as to produce the rate of interest calculated on. By this method we therefore have

$$A'_x = A_{x(i)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

41. In the case of a *Contingent Reversion*, if the principles of Art. 34 be adopted, in addition to the provision for interest a policy must be effected for the amount in reversion, to guard against the chance of (x) dying before (y). The effect of the policy is to transform the benefit into an absolute reversion receivable on the failure of the joint lives, and it must be valued accordingly. Writing P'^{1}_{xy} for the office premium for the necessary contingent assurance, and A'^{1}_{xy} for the practical value of a reversion coming to (x) on the death of (y), we have

$$A'^{1}_{xy} = 1-(P'^{1}_{xy}+d_{(i)})(1+a_{xy(j)}) \quad . \quad . \quad . \quad . \qquad (10)$$

42. As in the case of the absolute reversion, the amount invested is $v_{(i)}$, of which $(P'^{1}_{xy}+d_{(i)})a_{xy(j)}$ is spent in the purchase of the annuity, and P'^{1}_{xy} is paid to the assurance office for the first premium; and the

balance, namely, $v_{(i)} - \mathrm{P}'^{1}_{xy} - (\mathrm{P}'^{1}_{xy} + d_{(i)})a_{xy(j)}$, or $1 - (\mathrm{P}'^{1}_{xy} + d_{(i)})(1 + a_{xy(j)})$, as above, is paid to the vendor.

43. To take a numerical example:—What is the value of a contingent reversion to 5,000, receivable on the death of a person aged 60, provided a person aged 30 survive him, to pay interest at 4½ per-cent to the purchaser, assuming that the life aged 30 may be assured against the life aged 60 at an annual premium of 1·45 per-cent, and that, as in Art. 24, a joint-life annuity may be purchased for 9·130. Performing the calculations, we have

Amount paid Vendor	2084·485
First Premium on 5,000 paid to the Assurance Company	72·500
Cost of Annuity of 287·810 for Premium and Interest	2627·705
Total Amount Invested	4784·690

44. Instead of effecting a contingent policy at annual premiums, sometimes it is found convenient to pay a single premium. Using the italic capital A'^{1}_{xy} to represent the office single premium for the contingent assurance, formula 10 becomes

$$\mathrm{A}'^{\,1}_{xy} = 1 - d_{(i)}(1 + a_{xy(j)}) - \mathit{A}'^{1}_{xy} \quad . \; . \; . \; . \; . \quad (11)$$

45. In both formulas 10 and 11 it is assumed that an annuity is to be purchased, but the transaction may be carried out in accordance with the principles already discussed, by using throughout the whole of the formulas the rate of interest to be realized by the purchaser. Under this arrangement formula 10 becomes

$$\mathrm{A}'^{\,1}_{xy} = 1 - (\mathrm{P}'^{1}_{xy} + d_{(i)})(1 + a_{xy(i)}) \quad . \; . \; . \; . \; . \quad (12)$$

and formula 11 becomes

$$\mathrm{A}'^{\,1}_{xy} = \mathrm{A}_{xy(i)} - \mathit{A}'^{1}_{xy} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

46. If the question be to find the reversionary charge which should be granted in consideration of an advance, it is only necessary to take the reciprocals of the expressions in equations 10 to 13.

47. The following is an example of a transaction that is met with occasionally in practice:—A person aged x desires an advance on a contingent reversion which he will receive should he survive a person aged y. He wishes to be entitled to redeem at any time within the first

n years, whether the life tenant be alive or dead, by simply repaying the sum advanced with compound interest, together with the assurance premiums actually paid by the lender, these to be accumulated at the same rate. If he do not exercise his right to redeem within the n years, then the charge is to remain permanent. What amount of reversionary charge should he give.

Looking first at a contingent reversion of 1, what sum can be paid for it under the conditions named? The full amount of policy must be effected at once, and the premium will therefore be ${P'}^{1}_{xy}$. Writing ${}^{n}{A'}^{1}_{xy}$ for the value of the contingent reversion after n years, and S for the sum to be paid now by the investor, clearly we must have

$$S(1+i)^n + {P'}^{1}_{xy}\,\frac{(1+i)^{n+1}-(1+i)}{i} = {}^{n}{A'}^{1}_{xy}$$

whence

$$S = {}^{n}{A'}^{1}_{xy} \times v^n - {P'}^{1}_{xy}(1+a_{\overline{n-1}|}) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Equation 14 gives the value under the conditions named of a reversionary charge of 1; and therefore the charge which must be given for an advance of 1 is $\dfrac{1}{{}^{n}{A'}^{1}_{xy} \times v^n - {P'}^{1}_{xy}(1+a_{\overline{n-1}|})}$.

48. An arrangement may be made in the case of an absolute reversion, or of a reversionary life interest, similar to that described in Art. 47 in respect of a contingent reversion.

49. On the principles discussed in this chapter, what is the value to a purchaser of a life policy as an investment?

The assurance is simply an absolute reversion, burdened, however, with the premium which must be paid during the continuance of the life. Therefore, besides deducting the value of the interest as in formula 8, we must also deduct the value of the premium. Assuming that the present age of the life is x, and that an annual premium of π is actually payable under the policy, then

$$V' = 1 - (\pi + d_{(i)})(1 + a_{x(j)}) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

50. In formula 15 it is assumed that the policy is without profits. If the policy is with profits, then allowance must be made for the bonuses already declared, and also for future bonuses. The bonuses already declared are merely an absolute reversion, not subject to any premium. As regards future bonuses, of course only a rough estimate can be made.

51. Formulas 1, 2, 8, and 10, are called Jellicoe's formulas, because the late Charles Jellicoe was the first to publish them (*J.I.A.*, ii, 159). They were, however, in general use before the date of his paper.

52. The transactions are almost infinite in variety which may be entered into in connection with reversionary securities, and each change in the conditions demands a corresponding change in the formula to be applied. Also the same question may generally be looked at from various points of view, and one actuary will not always follow the method of solution adopted by another. In this chapter only fundamental principles have been discussed, not details. Many important papers have appeared in the pages of the *Journal of the Institute of Actuaries* on the subject, to the following of which the reader is more particularly referred:—C. Jellicoe, ii, 159; vi, 61; viii, 310: R. Tucker, v, 162: A. Day, viii, 326: T. B. Sprague, viii, 12; xiv, 145; xiv, 417; xvii, 229; xviii, 77; xix, 372: A. Baden, xvi, 269: C. J. Bunyon, xviii, 1.

CHAPTER XX.

Sickness Benefits.

1. If there be m members of a society all aged x, and if among them they experience r weeks of sickness in the year of age from x to $x+1$, then the average number of weeks of sickness in the year to each member is $r \div m$, and this ratio is called the *Rate of Sickness* at age x. Writing z_x for this rate of sickness we therefore have the equation $z_x = \frac{r}{m}$.

2. If each sick member is to receive compensation at the rate of 1 for each week during which he may be sick, then the value per member, at the beginning of the year, of the sick pay for that year is $v^{\frac{1}{2}} z_x$, because the amount which will be actually payable per member is z_x, as above; and, taking one member with another, it may be assumed that the payments will so fall as to be equivalent to an aggregate payment made in the middle of the year.

3. What is the present value of an allowance to be made to a person now aged x, of 1 per week in sickness during the year of age $x+n$ to $x+n+1$?

At age $x+n$ the value of the benefit will be $v^{\frac{1}{2}} z_{x+n}$. Therefore, discounting this value for the n years, and multiplying by the probability of its being received, we have the value sought, namely, $v^n {}_np_x v^{\frac{1}{2}} z_{x+n}$, which may be written $\frac{D_{x+n} v^{\frac{1}{2}} z_{x+n}}{D_x}$.

4. To find the value of a sick allowance of 1 per week in sickness throughout life, we must in the last expression give to n every integral value from 0 upwards. Using the symbol s_x for the value of a sick allowance of 1 per week in sickness throughout life, we therefore have the equation

$$s_x = \frac{D_x v^{\frac{1}{2}} z_x + D_{x+1} v^{\frac{1}{2}} z_{x+1} + D_{x+2} v^{\frac{1}{2}} z_{x+2} + \&c.}{D_x} \quad . \quad . \quad . \quad (1)$$

5. If we write $D_x v^{\frac{1}{2}} z_x + D_{x+1} v^{\frac{1}{2}} z_{x+1} + D_{x+2} v^{\frac{1}{2}} z_{x+2} + \&c. = K_x$, we have

$$s_x = \frac{K_x}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It thus appears that commutation columns may be formed, by means of which to calculate the values of sickness benefits. These columns are similar to those discussed in Chap. xvi, and may be employed to find the values of temporary and deferred sickness benefits. Thus we may have, using analogous notation to that adopted for annuities,

$$_{n|}s_x = \frac{K_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$_{|n}s_x = \frac{K_x - K_{x+n}}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

6. Sickness benefits are very frequently paid for by weekly or monthly contributions. The intervals being usually so short, the contributions form approximately a continuous annuity. Using the symbol c_x for the total contribution per annum for an allowance of 1 per week during sickness, we therefore have approximately

$$c_x = \frac{s_x}{\bar{a}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$= \frac{K_x}{\frac{1}{2}D_x + N_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The contribution payable at the beginning of each week will therefore be $\frac{c_x}{52}$; or at the beginning of each calendar month $\frac{c_x}{12}$; &c.

7. Instead of the continuous annuity being used for the divisor, the ordinary annuity payable at the end of each year has sometimes been employed. Under these circumstances formula 6 becomes

$$c_x = \frac{K_x}{N_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The results of formulas 6 and 7 differ only very slightly in practical cases, the lives being generally comparatively young to whom sickness benefits are granted.

8. The rate of sickness, like the rate of mortality, increases with the age, until in advanced life it becomes very heavy. Moreover, in the case of old people it is difficult to distinguish sickness properly so called from infirmity of age, and therefore it is not uncommon with friendly societies to limit sick pay to a certain period of life, say up to age 65, 70, or 75, the contributions being likewise limited; and some societies substitute a pension for sick pay after the stipulated age has been attained. In this last case the benefit consists of a sickness allowance limited to, say, n years, and an annuity to be entered on in n years. If the sickness allowance and also the annuity be 1 per week, the value of this benefit is ${}_{|n}s_x + 52{}_{n|}\bar{a}_x$, or $\dfrac{\mathrm{K}_x - \mathrm{K}_{x+n} + 52(\frac{1}{2}\mathrm{D}_{x+n} + \mathrm{N}_{x+n})}{\mathrm{D}_x}$, and the contribution per annum for the same is

$$\frac{\mathrm{K}_x - \mathrm{K}_{x+n} + 52(\frac{1}{2}\mathrm{D}_{x+n} + \mathrm{N}_{x+n})}{\mathrm{N}_x - \mathrm{N}_{x+n} + \frac{1}{2}(\mathrm{D}_x - \mathrm{D}_{x+n})}.$$

In practice the pension after age $x+n$ would probably not be for the full amount of the sick pay. It might be for half the amount, or for some other proportion.

9. Among friendly societies it is a very common custom to give only reduced sick pay in prolonged illness. Thus, with some, the allowance is, full pay for the first six months of illness, half pay for the second six months, and quarter pay for the remainder of the illness. The practice is, however, very various both as regards the durations of full and reduced pay respectively, and the amount of reduction in prolonged sickness. Where reductions in sick pay take place, for the purpose of calculating the values of the benefits the sickness experience must be separated into periods, first six months, second six months, &c., or such other periods as may be required; and tables must be prepared for each period. The complete benefit will then consist of several partial benefits. Thus, for example, if $\mathrm{K}_x^{\mathrm{I}}$ represent the column for the first six months' sickness, $\mathrm{K}_x^{\mathrm{II}}$ that for the second six months, and $\mathrm{K}_x^{\mathrm{III}}$ that for the remainder of the illness, and if there is to be full pay for the first six months, half pay for the second six months, and quarter pay thereafter, then the value of the benefit will be $\dfrac{\mathrm{K}_x^{\mathrm{I}} + \frac{1}{2}\mathrm{K}_x^{\mathrm{II}} + \frac{1}{4}\mathrm{K}_x^{\mathrm{III}}}{\mathrm{D}_x}$.

10. Because the rate of sickness, like the rate of mortality, increases progressively with the age, therefore the sickness benefits granted by a society constitute an increasing pecuniary liability for which suitable

reserves should be made. In estimating the value of a sickness benefit the balance must be struck between the value of the benefit contracted for, and that of the contributions. In the practical valuation of friendly societies another element must be taken into consideration, namely, the expenses of conducting the business. Sometimes the rules of the society provide a management fund to which every member is bound to contribute; and, when that is the case, credit may generally be taken in the valuation for the whole of the contributions payable for the sickness benefits. In other circumstances a sufficient portion of the contributions must be set aside for future expenses.

11. In Chap. xviii the effect on policy-values of varying rates of mortality was discussed, and now a similar enquiry might be made with regard to the values of sickness benefits; but it would be much more intricate. Here the rates of mortality and sickness are both involved, and they have contrary effects on the values. This is seen clearly from equation 1. If only the rate of sickness be increased, then z_x, z_{x+1}, &c., will be increased, and s_x will also be increased. If, on the other hand, only the rate of mortality be increased, then D_{x+1}, D_{x+2}, &c., will be diminished, and s_x will also be diminished. If the rates of both sickness and mortality be increased, the increase in the one will tend to neutralize the increase in the other; and, without actual trial, it is impossible to say what the ultimate result will be.

12. The secessions of members have a very material influence on the financial position of friendly societies, because if a member retires after having paid his contributions for some time, he relieves the society, as we have seen, of a distinct liability, and the society is thereby financially strengthened. Tables have been constructed involving the rate of secession, by which to calculate the contributions and the reserves of friendly societies, making allowance for this element. There is, however, not so much constancy in the rate of secession as in the rate of mortality or even as in that of sickness, and special caution must be exercised in bringing it into account. Not only does it vary with the age at entry, and with the duration of membership, but the experience of one society is very different from that of another, and in the same society the rate will vary considerably from time to time according to the condition of the society, whether prosperous or otherwise, and according to many other circumstances which it is needless to particularize here.

CHAPTER XXI.

Construction of Tables.

1. An intimate acquaintance with the nature of the tables he may find in his hands is important to the actuary, for without that knowledge he cannot turn them to the best account. It is therefore very desirable for the student to practise the construction of tables for himself, although those he requires may already be in print. Tables may be said to be the tools of the actuary; and by actual experience in their manufacture he will more readily obtain a clear knowledge of their properties, than by only theoretical study.

2. For constructing a table there are generally two courses open. Each individual value may be calculated independently; or each succeeding value may be formed from the value immediately preceding it. If the first course be chosen, each value must usually be also verified independently; but if the second, verification at stated intervals will be sufficient, because if one value be correct, so also must be those that go before it.

3. Facility and certainty of verification are not the only advantages of the *Continued* Process*. On account of regular sequence of operations, the calculations may generally be performed more rapidly, and with less mental effort, than when each value is computed by itself; and it also frequently happens that the actual amount of arithmetical work is less, the differences of functions, instead of the functions themselves, being made use of.

4. In order that the continued process may be successfully applied there are three things essential. There must be a *Continued Formula*, an *Initial Value*, and a *Verification Formula.* Frequently the most obvious formulas are not those most convenient; and the formula chosen should be such as best to suit the aids to calculation that are available.

5. To the actuary, the most powerful aids to calculation in the construction of tables are Ordinary Logarithms, the Logarithms of Gauss, and the Arithmometer. Other aids might be mentioned, such as the Spiral Slide Rule, of Fuller; but they are not much used in the construction of tables.

It is assumed that the reader has already attained a competent knowledge, both theoretical and practical, of ordinary logarithms; and it is proposed, before proceeding further with the special subject of this chapter, to explain briefly the logarithms of Gauss, and the arithmometer.

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* The word *Continuous* has sometimes been used in this connection; but it is suggested that *Continued* is preferable, in order that possible confusion with the word *Continuous*, in its sense as used in the Differential Calculus, may be avoided.

6. In a table of logarithms arranged in the form first given by Gauss, the number with which the table is entered, called the "argument", is $\log x$, and the result is $\log(1+x)$. The late Peter Gray was the first to apply, in his book "*Tables and Formulæ for the Computation of Life Contingencies*", Gauss's logarithmic tables to the calculation of annuities and assurances; and for that purpose he re-computed and extended Gauss's table. When Gray's work was first published in 1849, the extent of the table was too restricted to be practically convenient; and, as a consequence, for many calculations there had to be two distinct formulas, to be used at different parts of the operations. In 1870, however, there was a second issue of the volume, comprising a large extension of the principal table; and now in no case is more than one formula required. This fact should be borne in mind in reading Gray's book, because, by the extension of the table, all the formulas and illustrations relating to *negative* arguments are rendered obsolete, and Gray's methods are thereby very much shortened and simplified.

7. In Gray's table as extended, $\log(1+x)$ is given to six places of decimals for all values of $\log x$, from $x = \cdot 001$ to $x=100$. The argument consists of four figures, and therefore two figures must be proportioned for; but only one entry for the proportional parts is required, these being arranged in a special and convenient manner.

8. Since Gray's Tables were issued, Wittstein's more extensive ones of the same kind have been published. These give to seven places $\log(1+x)$ for all values of $\log x$, from $x = \cdot 0000001$ to $x=1{,}000{,}000$. The argument of Wittstein's tables consists, however, of only four figures, like Gray's; so that in using them three figures have to be proportioned for. This is tedious, and attended with considerable risk of error. Moreover, the proportional parts are arranged like those in the majority of tables of ordinary logarithms; and consequently, as pointed out by Gray, from the nature of the table, in certain parts the error is accumulative in continued operations. It was with a view to obviate this radical defect of tables in the same shape as Wittstein's that Gray adopted the special arrangement of proportional parts already referred to.

9. Besides Gray's six-figure, and Wittstein's seven-figure tables, a small five-figure table has been published by Galbraith and Haughton, giving to five places $\log(1+x)$ for all values of $\log x$, from $x=1$ to $x=10{,}000$. Seeing, however, that it does not contain any negative arguments, it is not of much use to the actuary. In the calculation of

life contingencies, Gray's Tables are the most convenient, and, on account of the arrangement of the proportional parts, the most trustworthy.

10. An important application of the logarithms of Gauss is to find the logarithm of the sum, or the difference, of two numbers, the logarithms of which are given. Let a and b be the two numbers. Log a and log b being given, it is required thence to determine $\log(a+b)$. For convenience, let $[t]\log x$ represent the result of entering Gauss's table with $\log x$, so that $[t]\log x = \log(1+x)$. Converting into logarithms the identical equation $(a+b) = b\left(\frac{a}{b}+1\right)$, we have

$$\begin{aligned}\log(a+b) &= \log b + \log\left(\frac{a}{b}+1\right)\\ &= \log b + [t](\log a - \log b).\end{aligned}$$

Therefore, entering Gauss's table with the difference between the logarithms of two numbers, and adding the result to the logarithm of the second number, we have the logarithm of the sum of the numbers.

11. Seeing that by entering Gauss's tables directly with $\log x$, the result is $\log(1+x)$, it follows that by entering the tables inversely with $\log x$, we obtain $\log(x-1)$. The operation of entering the table inversely may be conveniently represented by the symbol $[t^{-1}]$, and in this notation we therefore have $[t^{-1}]\log x = \log(x-1)$.

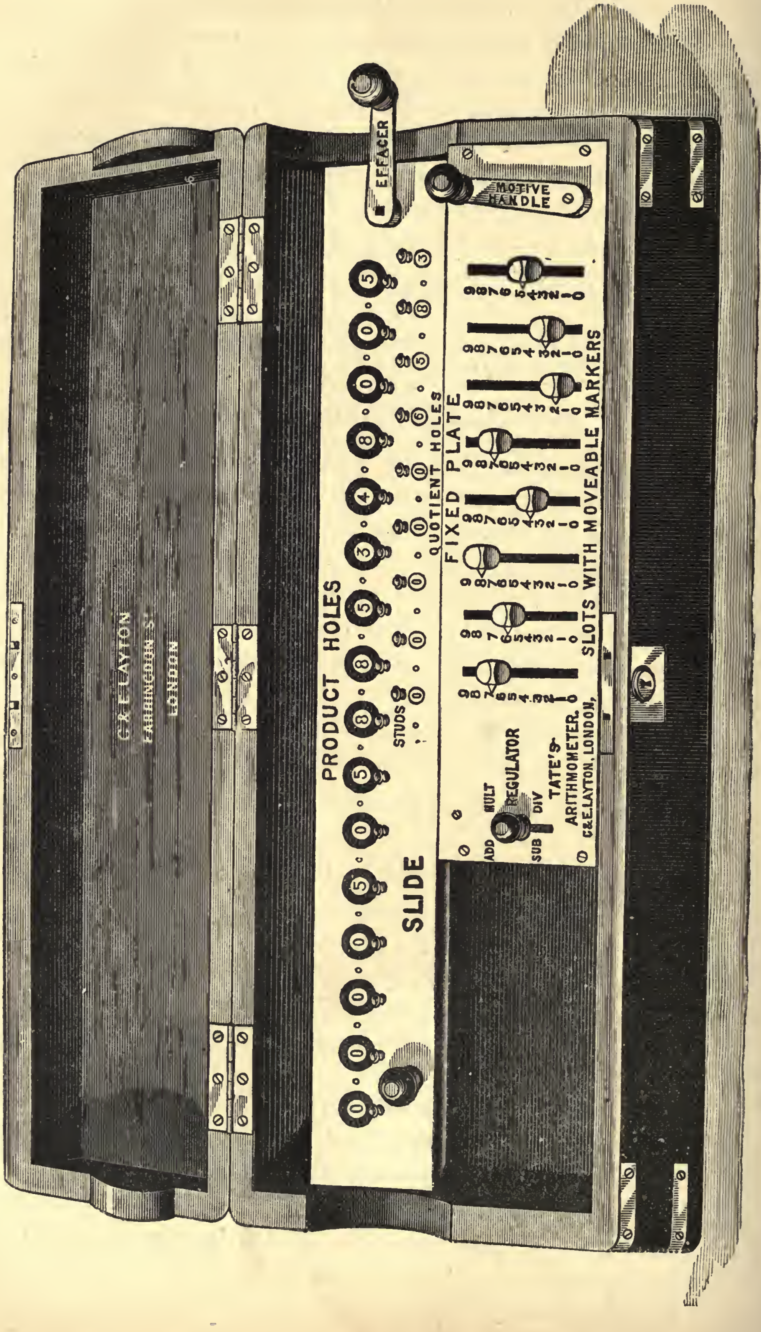
12. The logarithms of two numbers, a and b, being given, it is required to find the logarithm of their difference.

Converting into logarithms the identical equation $(a-b) = b\left(\frac{a}{b}-1\right)$ we have

$$\begin{aligned}\log(a-b) &= \log b + \log\left(\frac{a}{b}-1\right)\\ &= \log b + [t^{-1}](\log a - \log b).\end{aligned}$$

Therefore, entering Gauss's table *inversely*, with the difference between the logarithms of two numbers, and adding the result to the logarithm of the smaller number, we have the logarithm of the difference between the numbers.

13. Calculating machines of various kinds have been constructed; but the only one which has been found practically useful in the daily work of the actuary is the arithmometer. This was invented by a French mechanician, and patented by Mons. Thomas de Colmar, formerly manager of the Soleil Assurance Office in Paris. The accompanying engraving represents one of these machines as modified and improved by an Englishman named Tate.


C & E LAYTON
FARRINGDON St
LONDON
EFFACER
MOTIVE HANDLE
PRODUCT HOLES
STUDS
QUOTIENT HOLES
FIXED PLATE
SLOTS WITH MOVEABLE MARKERS
SLIDE
REGULATOR
ADD
MULT
DIV
SUB
TATE'S ARITHMOMETER.
C&E. LAYTON, LONDON,

14. The arithmometer is constructed to add and to subtract; and by means of continued addition or subtraction it can perform multiplication and division with great rapidity. The top of the machine consists of two plates, which may be called the "fixed plate" and the "slide" respectively. On the fixed plate are slots with moveable "markers", each of which can be placed against any number desired, from 0 to 9, these numbers appearing in order along the left of each slot. On the left of the fixed plate is the "regulator", by means of which the machine may be set for addition or subtraction, as may be required; and on the right is the "motive handle", which can be turned only in the direction of the hands of a clock, and by means of which the machine is set in motion. In the slide is a row of "product holes", and the slide, on being lifted, can be passed from left to right or the contrary, so that the position of the product holes with respect to the slots in the fixed plate may be varied at pleasure. A number is "placed on" the machine by marking it on the fixed plate by means of the markers in the slots. The number so placed on the machine can be "thrown up" on to the slide, by placing the regulator at addition, and giving one turn to the motive handle. If a second turn be given to the handle, the number is again thrown up, and added to the number already on the slide, and so on; by each turn of the motive handle the number on the fixed plate being thrown up once, and added to whatever number may already be on the slide. In this way multiplication is performed. The number on the fixed plate will be thrown up in the units place, or the tens place, &c., according to the position of the slide with regard to the fixed plate.

15. The method of procedure will best be illustrated by a numerical example. Let it be required to multiply together the numbers 76,847,235 and 6,583. Set the regulator at addition, and place the larger number on the machine by means of the moveable markers in the slots, as shown in the engraving. Let the slide be raised and drawn as far as it will go to the left, and then let it be dropped into position. Now turn the motive handle three times: this multiplies the number 76,847,235 by 3, the unit figure of the multiplier, and throws the result up on to the slide, where it will appear in the product holes. Next move the slide one station to the right, and turn the motive handle eight times for the second figure of the multiplier. This multiplies 76,847,235 by 80, and adds the result to what had already appeared in the product holes. By successively moving the slide and turning the motive handle, in the same way complete the multiplication by the remaining figures, 5 and 6, of the multiplier; and finally bring the slide into its original position. The machine is now as it appears in the engraving. The multiplicand, 76,847,235, is seen in the slots in the fixed plate where it had been placed; and the product, 505,885,348,005, is in the product holes. On the slide of the machine is a second row of holes called "quotient holes." In these are registered the number of turns given to the handle. Therefore the multiplier appears in the quotient holes, as shown in the engraving. On the slide is also "the effacer", by means of which the machine is set at zero before commencing an operation.

16. In order to perform division, the dividend is set as far as possible to the left upon the slide of the machine, and the slide is carried as far as it will go to the right. The divisor is placed on the fixed plate, also as far as possible to the left, and the regulator is placed at subtraction. By turning the handle and moving the slide station by station to the left, the division is effected, and the quotient appears in the quotient holes; and when the work has been completed, the remainder will be found in the product holes.

17. For the purposes of the actuary the principal advantage of the arithmometer lies in its power to multiply two numbers together, and add the product to a third number, without recording the intermediate steps. Thus, if it be desired to find the value of $P+QR$, the number P must be placed on the slide of the machine, and the number Q, which may be called the *in-factor*, upon the fixed plate. The in-factor, Q, must then be multiplied by the *out-factor*, R, in the manner sketched above. The effect will be that the product $Q \times R$ will be formed, and added to P; and as a final result we shall have, as desired, $P+QR$.

18. This property of the arithmometer is specially useful in continued operations when Q, the in-factor, is constant, and R, the out-factor, is a series of comparatively small numbers, such as the differences of some function involved in the operation. This will abundantly appear in the illustrations to follow in this chapter.

19. By means of the arithmometer, a continued product, QR, may equally readily be subtracted from a number, P. It is only necessary for this purpose to place the regulator at subtraction, and proceed as in Art. 17.

20. In proceeding now to treat more directly of the construction of actuarial tables, it will be convenient and sufficient to explain fully the methods adopted in forming those at the end of the volume, and to give such examples of the formation of other tables as will illustrate the general principles to be observed. When the student has made himself thoroughly acquainted with the principles, it will be easy for him to apply them to other kinds of tables when occasion requires.

21. The fundamental column in the tables at the end of this volume is the column of l_x in Table No. I. It was constructed in the way described in Chap. vi, and from it all the other tables have been derived, for the most part by means of logarithms.

22. In the case of the logarithmic functions, only five places of figures have been tabulated. In taking out log l_x the values were made accurate in the fifth place; that is, where at the beginning of the table l_x contains six figures, the sixth figure was proportioned for in Babbage's seven-figure table, and the results were then cut down to five figures. In all other cases, except in constructing the values of a_x at 3 per-cent interest, five-figure logarithms only were used; and numbers containing more than five figures, as for instance N_x at the majority of the ages, were cut down to five figures before the logarithms were taken out. In some cases this method of procedure results in an error of 1 in the last place of the logarithm. Thus, for example, at 3 per-cent interest, $N_3 = 2{,}378{,}133$; and $\log N_3 = 6{\cdot}3762361$, or, cut down to five figures, 6·37624; but the value tabulated is 6·37623, being the logarithm of 2,378,100.

23. In the same way, when numbers have been derived from logarithms, the numbers have been taken to the nearest figure in the fifth place, and no attempt has been made to obtain more figures. These rules have been followed rigidly throughout the whole of the tables, with the single exception already mentioned, and they have great advantages in the saving of labour. A five-figure table such as Scott's can be used; or if a seven-figure table like Babbage's be employed, it can be worked to only five figures with considerable economy of trouble. It is the interpolations for the sixth and subsequent figures which occupy most of the time in using logarithms, and cause the greatest liability to error.

24. The loss of accuracy in using logarithms to only five figures for calculations in life contingencies is not often of any moment. As showing the degree of accuracy attained it may be mentioned that, the values of annuities at 3 per-cent interest having been calculated accurately by seven-figure logarithms, and the results tabulated to five places of decimals in Table No. XI, these were compared with the annuity-values calculated as above described by five-figure logarithms; and it was found that, there being 101 values in all, 84 were identical by the two processes; in nine cases the five-figure logarithms gave 1 in excess in the third place of decimals of the annuity-value, and in seven cases 1 in defect; while in only one instance was there a larger error,

namely, 2 in excess in the third place of decimals: the annuity-value at age 16 being 22·926 by five-figure logarithms, and 22·92431 by seven-figure logarithms.

25. Although for some purposes it may be useful to have annuity-values tabulated to more than three places of decimals, yet it must be observed that the degree of accuracy thereby attained is more apparent than real. The facts at our disposal are not capable of producing so much minuteness of result. The values of l_x, on which all the other tables depend, are only approximately true in the last figure tabulated; and if they were carried out, say to one other place, the fourth and succeeding decimal places in the annuity-values would be affected. Moreover, a very slight change in the method of constructing, or even of graduating, the mortality table alters the decimal places of the annuity-values. Such being the case, it has been thought that every important end has been served by using only five-figure logarithms in constructing the tables given in this volume.

26. An exception, as already mentioned, has been made in respect of the annuity-values at 3 per-cent interest, for which seven-figure logarithms were used. The principal reason for this departure from the otherwise universal rule was, that the examples in Chaps. ix, x, and xi might be carried out in greater detail. The adjustments there discussed in some cases do not affect the third place of decimals in the annuity-value; and the effects of the several formulas are not apparent except with annuity-values carried out to at least five places of decimals.

27. The column $\log p_x$ was formed simply by differencing $\log l_x$. Verification was obtained by addition at periodical intervals; because $\log l_{x+n} = \log l_x + \log p_x + \log p_{x+1} + \&c. + \log p_{x+n-1}$. In the same way the column $\Delta \log p_x$, the use of which will appear later on in this chapter, was constructed and verified from the column $\log p_x$. The numbers in the column $\text{colog}\, l_x$, or $\log \dfrac{1}{l_x}$, are simply the arithmetical complements of those in column $\log l_x$. The column $\text{colog}\, l_x$ was checked by addition, the sum of any section, when added to the sum of the corresponding section of the column $\log l_x$, being equal to zero.

28. The column $\text{colog}\, p_x$ was formed by differencing $\text{colog}\, l_x$, and checked by addition as above described, and also by comparing it with the column $\log p_x$, the values in these two columns being complementary. Similarly, the column $\Delta\, \text{colog}\, p_x$ was formed from the column $\text{colog}\, p_x$, and checked by addition, and by comparison with the column $\Delta \log p_x$. The absolute correctness of all these columns was thus secured, because

they mutually verify each other. The natural numbers corresponding to $\log p_x$ and $\operatorname{colog} p_x$ were then taken out and inserted in Table No. I.

29. It may be remarked that in order to ensure the correctness of numbers corresponding to logarithms, or of logarithms corresponding to numbers, it is frequently desirable to verify by means of an inverse operation. Thus, for instance, the values of $\log p_x$ having been formed, those of p_x may be obtained by means of a table of antilogarithms, such as Filipowski's or Scott's; and then the values of $\log p_x$ may be re-formed from those of p_x by means of a table of logarithms, and compared with the original values of $\log p_x$. Sometimes in the two sets of logarithms there is an unavoidable difference of a unit in the last place, but wherever that occurs it must be investigated to make sure that a mistake has not been made.

30. The column d_x was formed by differencing l_x, and checked by addition; and from it were obtained $\log d_x$, $\Delta\log d_x$, and $-\Delta\log d_x$, the last bearing the same relation to $\log d_x$ as $\operatorname{colog} p_x$ bears to $\log l_x$.

31. The column q_x was formed from the relation $q_x = 1 - p_x$, and verified by addition, because $q_x + q_{x+1} + \ldots + q_{x+n-1} = n - (p_x + p_{x+1} + \ldots + p_{x+n-1})$. It is, however, a useful check to construct the column of q_x independently by means of a continued operation, as follows:—

$$\log q_x = \log d_x + \operatorname{colog} l_x$$

$$\begin{aligned}\log q_{x+1} &= \log d_{x+1} + \operatorname{colog} l_{x+1}\\ &= \log d_x + \Delta\log d_x + \operatorname{colog} l_x + \operatorname{colog} p_x\\ &= \log q_x + \Delta\log d_x + \operatorname{colog} p_x.\end{aligned}$$

The work can be conveniently arranged in columns, thus

(1) x	(2) $\Delta\log d_x$	(3) $\operatorname{colog} p_x$	(4) $\log q_x$
0	$\bar{1}$·44082	0·05198	$\bar{1}$·05232
1	·77775	·01551	$\bar{2}$·54512
2	·84077	·00957	$\bar{2}$·33838
3	·90579	·00676	$\bar{2}$·18872
4	·90350	·00552	$\bar{2}$·10127
5	·90472	·00447	$\bar{2}$·01029
6	·90398	·00362	$\bar{3}$·91948
7	·91144	·00293	$\bar{3}$·82708
8	·92066	·00240	$\bar{3}$·74145
9	·94414	·00201	$\bar{3}$·66451
*	* *	* *	* *

The values of $\Delta \log d_x$ and $\operatorname{colog} p_x$ are placed in parallel columns on cross-ruled paper, with a vacant column to the right in which to place the values of $\log q_x$ about to be formed. The initial value prepared by the formula $\log q_x = \log d_x + \operatorname{colog} l_x$, is then inserted; and, adding sideways the three numbers in the first line, we have $\log q_{x+1}$, which is to be entered in the second line in column 4; then adding similarly the three numbers in the second line, we have $\log q_{x+2}$; and so on. Verification is obtained at any point by calculating $\log q_x$ by means of the fundamental formula.

32. The values of L_x in Table No. I were formed directly from those of l_x; and regarding them the only point requiring note is, that when $l_x + l_{x+1}$ was an odd number, L_x was put up or down alternately to the nearest unit, so as to avoid fractions. Thus $(l_2 + l_3) = 215551$, and $L_2 = 107776$, being put up to the nearest unit; while on the next occasion where an odd number occurs, namely, $(l_4 + l_5) = 208559$, we have $L_4 = 104279$, which is put down to the nearest unit.

33. The columns N'_x and T_x in Table No. I were formed from the columns l_x and L_x respectively by continued addition from the bottom upwards.

34. In Chap. vi a full explanation is given of the construction of the column μ_x.

35. To construct a table of the values of $\mathring{e}_x$.

By Chap. iii, Art. 16, $\mathring{e}_x = \dfrac{T_x}{l_x}$: whence

$$\log \mathring{e}_x = \log T_x + \operatorname{colog} l_x$$

Also
$$\begin{aligned}\log \mathring{e}_{x+1} &= \log T_{x+1} + \operatorname{colog} l_{x+1} \\ &= \log T_x + \Delta \log T_x + \operatorname{colog} l_x + \operatorname{colog} p_x \\ &= \log \mathring{e}_x + \Delta \log T_x + \operatorname{colog} p_x .\end{aligned}$$

This continued operation is similar to that explained in Art. 31 for forming $\log q_x$, the only difference being that $\Delta \log T_x$ has taken the place of $\Delta \log d_x$. A table of $\log T_x$ and $\Delta \log T_x$ having been first prepared, the work may be arranged in columns, thus

(1) x	(2) $\Delta \log T_x$	(3) $\operatorname{colog} p_x$	(4) $\log \mathring{e}_x$
0	$\bar{1}$·99133	0·05198	1·67928
1	·99185	·01551	·72259
2	·99192	·00957	·72995
3	·99193	·00676	·73144
4	·99190	·00552	·73013
5	·99183	·00447	·72755
6	·99175	·00362	·72385
7	·99166	·00293	·71922
8	·99154	·00240	·71381
9	·99143	·00201	·70775
*	* *	* *	* *

36. Instead of finding $\mathring{e}_x$ directly from T_x, we might form e_x from N'_x, and then obtain $\mathring{e}_x$ from the relation $\mathring{e}_x = \frac{1}{2} + e_x$. This is the way in which the column of $\mathring{e}_x$ in Table No. I was actually constructed.

37. A table of the curtate expectation of life may be formed by means of Gauss's logarithms. By Chap. iii, formula 17, $e_x = p_x(1 + e_{x+1})$:

whence
$$\log e_x = \log p_x + \log\,(1 + e_{x+1})$$
$$= \log p_x + [t]\,\log e_{x+1}\,.$$

38. Gray's tables of Gauss's logarithms provide for calculations to six figures, and cannot with facility be used to only five figures. Therefore it is necessary to assume a sixth figure to the logarithms tabulated in this volume, by adding 0. In appearance the results will be produced to an extra place of decimals, but that must be rejected, the proper correction in the fifth place being made for it.

39. In constructing the table of e_x, commencement must be made at the oldest age, where $e_x = p_x$, x in this case being $\omega - 2$. The following is a type of the calculations, where the numbers to be added together are arranged one below the other. The values of $\log p_x$ for all ages from the oldest to the youngest are first written on ruled paper on every fourth line. Then, commencing with the oldest age, Gauss's logarithmic table is entered with the first logarithm, under which the result is written, and below that again the proportional part. The three lines are then added together, the sum, which is $\log e_{\omega-3}$, being placed in its proper position. This is then used as the argument for entering Gauss's table; and so the work proceeds step by step. Seeing that each successive addition does not include the previous sum, it is well to rule

a line above each $\log p_x$ on the working sheet before proceeding to the further operations.

Age.	logarithms.	e_x
100	$\bar{1}\cdot397940$	0·250
	$\bar{1}\cdot647820$	
	·096902	
	8	
99	$\bar{1}\cdot744730$	·556
	$\bar{1}\cdot675490$	
	·191876	
	11	
98	$\bar{1}\cdot867377$	·737
	$\bar{1}\cdot710550$	
	·239729	
	32	
97	$\bar{1}\cdot950311$	·892
	$\bar{1}\cdot716940$	
	·276891	
	5	
96	$\bar{1}\cdot993826$	·986
	$\bar{1}\cdot740670$	
	·297941	
	13	
95	0·038624	1·093
	$\bar{1}\cdot764240$	
	·320759	
	12	
94	·085011	1·216
	$\bar{1}\cdot782870$	
	* *	*
93		
	$\bar{1}\cdot803810$	
*	* *	*

40. In order to verify the work, the column l_x should be summed from the bottom upwards at stated intervals, thus forming periodical values of N'_x, so that at these points we may check the results by the formula $e_x = \dfrac{N'_x}{l_x}$.

41. The operation when carried out in the form above displayed occupies much space, but it may be compressed into shorter compass without loss of convenience by arranging it in columns, thus:

(1) Age x	(2) $\log p_x$	(3) $[t]\log e_{x+1}$	(4) Prop. Parts.	(5) $\log e_x$	(6) e_x
100	$\bar{1}\cdot397940$	0·000000	00	$\bar{1}\cdot397940$	0·250
99	$\bar{1}\cdot647820$	·096902	08	·744730	0·556
98	$\bar{1}\cdot675490$	·191876	11	·867377	0·737
97	$\bar{1}\cdot710550$	·239729	32	·950311	0·892
96	$\bar{1}\cdot716940$	·276891	05	·993826	0·986
95	$\bar{1}\cdot740670$	·297941	13	0·038624	1·093
94	$\bar{1}\cdot764240$	·320759	12	·085011	1·216
93	$\bar{1}\cdot782870$	* *	*	* *	*
92	$\bar{1}\cdot803810$				
*	* *				

The age and $\log p_x$ being first of all written in retrograde order in the first two columns, the first line in the third and fourth columns is filled up with zeros, because at the oldest age e_{x+1}, and consequently also $\log(1+e_{x+1})$, is zero. In the first line of the fifth column $\log p_x$ for the oldest age is again written, that being also $\log e_x$ at that age. Gauss's table is then entered with this last-mentioned logarithm, and the result written on the second line of the third column, the proportional parts being placed in the fourth column. The numbers in the second line of the second, third, and fourth columns are then added together sideways, and the sum placed in the fifth column. With this sum Gauss's table is again entered, and the result is placed in the next line of the third and fourth columns; and so the work proceeds step by step from the oldest age to the youngest.

42. As will be shown in Arts. 99 to 101, for the purpose of calculating tables of single premiums for contingent assurances the values are required of $\log \frac{1}{2}(p_x^{-1}+1)$ and $\log(p_x^{-1}-1)$; and these are given with their differences in Table No. II. It may be well to show here how they were prepared by means of a continued process.

43. To construct a table of $\log \frac{1}{2}(p_x^{-1}+1)$, we have

$$\tfrac{1}{2}(p_x^{-1}+1)=\frac{1}{2}\left(\frac{l_x}{l_{x+1}}+1\right)=\frac{1}{2}\,\frac{l_x+l_{x+1}}{l_{x+1}}=\frac{\mathrm{L}_x}{l_{x+1}}.$$

Whence $\log \frac{1}{2}(p_x^{-1}+1)=\log \mathrm{L}_x+\operatorname{colog} l_{x+1}$

Also
$$\begin{aligned}\log \tfrac{1}{2}(p_{x+1}^{-1}+1)&=\log \mathrm{L}_{x+1}+\operatorname{colog} l_{x+2}\\&=\log \mathrm{L}_x+\Delta\log \mathrm{L}_x+\operatorname{colog} l_{x+1}+\operatorname{colog} p_{x+1}\\&=\log \tfrac{1}{2}(p_x^{-1}+1)+\Delta\log \mathrm{L}_x+\operatorname{colog} p_{x+1}.\end{aligned}$$

Commencing with the youngest age, the operation may be performed in columnar form. A type is given below which explains itself, it being similar to that for the construction of $\log q_x$, which is illustrated in Art. 31. The first term is calculated, and verification at periodical intervals is obtained, by means of the fundamental formula,

$$\log \tfrac{1}{2}(p_x^{-1}+1)=\log \mathrm{L}_x+\operatorname{colog} l_{x+1}.$$

(1) Age x	(2) $\Delta \log \mathrm{L}_x$	(3) $\operatorname{colog} p_{x+1}$	(4) $\log\frac{1}{2}(p_x^{-1}+1)$
0	$\bar{1}$·96555	0·01551	0·02675
1	·98745	·00957	·00781
2	·99182	·00676	·00483
3	·99384	·00552	·00341
4	·99502	·00447	·00277
5	·99593	·00362	·00226
6	·99672	·00293	·00181
7	·99733	·00240	·00146
8	·99780	·00201	·00119
9	·99811	·00178	·00100
10	·99831	·00161	·00089
*	* *	* *	* *

44. Similarly, to construct a table of $\log(p_x^{-1}-1)$, we have

$$\log(p_x^{-1}-1)=\log\left(\frac{l_x}{l_{x+1}}-1\right)=\log d_x+\operatorname{colog} l_{x+1};$$

and $$\log(p_{x+1}^{-1}-1)=\log(p_x^{-1}-1)+\Delta\log d_x+\operatorname{colog} p_{x+1}.$$

The process here is exactly the same as that in the last preceding article, only that $\Delta\log d_x$ takes the place of $\Delta\log \mathrm{L}_x$. A numerical example is hardly necessary.

45. It will be noticed that in the construction of $\log q_x$ Art. 31, $\log \mathring{e}_x$ Art. 35, $\log\frac{1}{2}(p_x^{-1}+1)$ Art. 43, and $\log(p_x^{-1}-1)$ Art. 44, the third column in each case contains the values of $\operatorname{colog} p_x$. This same function appears in many other continued operations; and if several tables are to be constructed it will be well to enter it on a long slip of paper of the proper width, and of the same intervals of ruling as the paper used for the calculations. This moveable slip may then be placed upon the working sheet of paper, and held in its proper place by weights. In this way the labour will be saved of writing out repeatedly for different operations the values of $\operatorname{colog} p_x$; and at the same time a considerable risk of error will be eliminated. A similar remark applies, as will be seen presently, to other functions much used in the construction of tables.

46. In this connection it may be pointed out that there is a slight difference in the way in which $\operatorname{colog} p_x$ is used in the construction of $\log q_x$ Art. 31, and $\log\frac{1}{2}(p_x^{-1}+1)$ Art. 43. In the former case the value of colog p_x is placed against age x, while in the latter it is placed against age $x-1$. The moveable slip affords facilities for the adjustment of the relative positions of the several functions used in the calculation, because it may be moved up or down the working sheet as may be found necessary.

47. In the construction of tables involving the rate of interest, as well as that of mortality, the function vp_x is of great importance. Thus, for example, $D_{x+1}=vp_xD_x$ and $a_x=vp_x(1+a_{x+1})$.

To construct a table of log vp_x, we have

$$\log vp_x=\log v+\log p_x.$$

Also

$$\log vp_{x+1}=\log v+\log p_{x+1}$$

$$=\log vp_x+\Delta\log p_x.$$

Whence, starting with the initial value log vp_x, by continued addition of $\Delta\log p_x$ the table is completed. Verification may be obtained at stated intervals by means of the fundamental formula, $\log vp_x=\log v+\log p_x$. The following is a type of the calculation arranged in columnar form.

Age x	$\Delta\log p_x$	$\log vp_x$
0	0·03647	$\bar{1}$·93518
1	·00594	·97165
2	·00281	·97759
3	·00124	·98040
4	·00105	·98164
5	·00085	·98269
6	·00069	·98354
7	·00053	·98423
8	·00039	·98476
9	·00023	·98515
10	·00017	·98538
11	·00010	·98555
12	·00002	·98565
13	·00000	·98567
14	$\bar{1}$·99990	·98567
15	·99988	·98557
16	·99981	·98545
17	·99982	·98526
18	·99979	·98508
19	·99980	·98487
20	·99984	·98467
*	* *	* *

48. By the above method, a table of log vp_x will be formed correct in the last place of decimals; and were that table itself the end in view, no alteration in it would be requisite or permissible. But the object of the table is, that it may be used in continued operations for the construction of tables of other functions, such as log D_x; and when so employed, although correct in itself, it will, unless adjusted, produce erroneous results. To take an example, $\Delta \log D_x = \log vp_x$, and a table of $\log D_x$ may consequently be formed by the continued addition to an initial term of the successive values of $\log vp_x$. Now in each value of $\log vp_x$, $\log v$ enters, and that logarithm is only approximately correct in the last place. In $\log p_x$ the last place is also only approximately correct, but the errors are sometimes in one direction and sometimes in the other, and in continued addition tend on the average to counteract each other. With the continued addition of $\log v$ the case is, however, different, because the error in the initial logarithm is repeated in the same direction at each step, and becomes multiplied by the number of operations performed; so that in the construction of a lengthy table, the accumulations become considerable. For example, in the case of the 3 per-cent table above illustrated, $\log v$ correct to five places of decimals is $\bar{1}\cdot98716$, and to seven places of decimals $\bar{1}\cdot9871628$. If, therefore, $\log vp_x$ as formed above to five places be used in a continued operation for constructing 100 values of another function, there will be an error in defect of no less than 28 in the hundredth value, because we have neglected 28 in the sixth and seventh places of the logarithm used in the continued addition, and this becomes multiplied by 100. To obviate this accumulation of error the table of $\log vp_x$ must be adjusted, and the fifth place of certain periodical values increased by a unit, so that there may be 28 such increments in 100 values. By differencing a table of $\log v^x$ to five places of decimals (where of course the difference is $\log v$, and theoretically should be constant), it will be found that the second, sixth, tenth, thirteenth, seventeenth, twentieth, &c., values of $\log vp_x$ must be increased by a unit, if a continued addition into which $\log vp_x$ enters is, throughout, to be absolutely correct. It would be tedious, however, to seek such minute accuracy; and, in fact, the accuracy is not attainable, because the values to be adjusted will depend on the age at which a continued addition commences. In the case of a 3 per-cent table, it is quite sufficient for practical purposes to adjust the second and, subsequently, every fourth value of $\log vp_x$, by increasing it by a unit in the fifth place. In this way the accumulated error in the fifth place, after 100 additions, will be only three, which may be discarded.

Of course the terms to be adjusted in the table of $\log vp_x$ will depend on the rate of interest employed, and the number of decimal places retained. The adjustment discussed above is for 3 per-cent interest, with five places of decimals retained. If the work were being performed to six places of decimals, 23 values in the 100 would have to be adjusted, and the adjustment would have to be in the opposite, or negative, direction. If, again, interest is taken at 4 per-cent, with five places of decimals, 33 values in the 100 require a negative adjustment; and so on.

49. To construct a table of D_x by logarithms.

Since $$D_{x+1}=D_x vp_x.$$

Therefore $$\log D_{x+1}=\log D_x+\log vp_x.$$

Starting with the initial value formed from the fundamental equation $\log D_x=\log l_x+\log v^x$, the required column is formed by the continued addition of $\log vp_x$, adjusted as explained in Art. 48; and is verified by calculating a value at intervals by means of the fundamental formula. The process being so simple, an arithmetical example is not required.

50. The values of $\log vp_x$ used in the construction of the table of $\log D_x$ must be adjusted in the manner shown in Art. 48, but for the reason therein explained there may possibly still remain a trifling error in the last place of decimals. To avoid this error in the tables at the end of the volume, the continued method of construction was not adopted, but each value of $\log D_x$ was formed independently. To ensure absolute accuracy, the work was done in duplicate by different computers, and, moreover, it was further checked by addition of quinary groups, by means of the relation $\log D_x+\log D_{x+1}+\ldots+\log D_{x+4}=\log l_x+\log l_{x+1}+\ldots+\log l_{x+4}+\log v^x+\log v^{x+1}+\ldots+\log v^{x+4}$. By this process, which is laborious, it is hoped that the absolute correctness of the tables of D_x, including the last figure, has been secured.

51. On account of the method by which the tables of $\log D_x$ have been formed, a table of $\log vp_x$ may be obtained simply by differencing, which will contain all the adjustments required for continued operations; and should such a table be desired, this will be the readiest method of constructing it.

52. Having formed the table of $\log D_x$, the values of D_x may be taken out and the column of N_x formed by continued addition from the bottom upwards.

53. The columns of C_x and M_x may be prepared in a precisely similar manner to those of D_x and N_x, merely substituting $\log d_x$ for $\log l_x$, and $\log v^{x+1}$ for $\log v^x$.

54. Having constructed the D and N columns, the annuity-values may with facility be obtained by a continued process.

We have $$\log a_x=\log N_x+\text{colog}\, D_x$$

and $$\log a_{x+1}=\log a_x+\Delta\log N_x+\Delta\,\text{colog}\, D_x.$$

The following is a type of the calculation in columnar form, at 3 per-cent interest. $\Delta \log N_x$ is entered in the second column, $\Delta \operatorname{colog} D_x$ in the third, and in the first line of the fourth column the initial value of $\log a_x$, calculated by the fundamental formula, is inserted. The addition of the three numbers in the first line gives $\log a_{x+1}$, to be entered in the second line; and so on, stage by stage, the work proceeds, verification being obtained at intervals by means of the fundamental formula.

(1) Age x	(2) $\Delta \log N_x$	(3) $\Delta \operatorname{colog} D_x$	(4) $\log a_x$
0	$\bar{1}$·98192	0·06482	1·32466
1	·98235	·02834	·37140
2	·98253	·02241	·38209
3	·98264	·01960	·38703
4	·98266	·01836	·38927
5	·98267	·01730	·39029
6	·98263	·01646	·39026
7	·98258	·01577	·38935
8	·98248	·01524	·38770
9	·98237	·01484	·38542
10	·98223	·01462	·38263
*	* *	* *	* *

55. The values of A_x are formed in a precisely similar manner to those of a_x as shown in Art. 54, $\Delta \log M$ being substituted for $\Delta \log N$.

56. To construct a table of P_x, we have

$$\log P_x = \log M_x + \operatorname{colog} N_{x-1}$$

and

$$\log P_{x+1} = \log P_x + \Delta \log M_x + \Delta \operatorname{colog} N_{x-1}.$$

Therefore the values of $\log P_x$ are constructed in a similar manner to those of $\log A_x$, merely replacing $\Delta \operatorname{colog} D_x$ by $\Delta \operatorname{colog} N_{x-1}$. In each of these cases verification is obtained by calculating periodical values independently by means of the commutation columns.

57. Because of the relation $a_x = vp_x(1 + a_{x+1})$, a table of annuity-values may be formed by means of the logarithms of Gauss. The process is exactly like that illustrated in Art. 39 for preparing a table of the curtate expectations of life, the only difference being that $\log vp_x$, properly adjusted, takes the place of $\log p_x$. It must, however, be noted that verification cannot be conveniently obtained when annuity-values are prepared in this way, and, to ensure correctness, the work should be done in duplicate and carefully examined. This difficulty may, however, be overcome by calculating the annuity-values indirectly through the N column formed by Gauss's logarithms, as explained in the next article.

58. Writing the equation $a_x = vp_x(1+a_{x+1})$ in commutation symbols, we have

$$\frac{N_x}{D_x} = \frac{D_{x+1}}{D_x}\left(1+\frac{N_{x+1}}{D_{x+1}}\right); \text{ or } N_x = D_{x+1}\left(1+\frac{N_{x+1}}{D_{x+1}}\right).$$

Taking the logarithms of both sides this becomes

$$\log N_x = \log D_{x+1} + [t](\log N_{x+1} - \log D_{x+1})$$

In using this formula the work is done in retrograde order from the older ages to the younger. On inspection of the following symbolical example it will be seen that there are two working columns:—

	$\log N_x$	
	$\log D_x$	diff. $=\log a_x$
$[t]\log a_x =$	$\log(1+a_x)$	
sum $=$	$\log N_{x-1}$	
	$\log D_{x-1}$	diff. $=\log a_{x-1}$
$[t]\log a_{x-1} =$	$\log(1+a_{x-1})$	
sum $=$	$\log N_{x-2}$	
	$\log D_{x-2}$	&c.
	&c.	

The preliminary portion of the work is the insertion in the working sheet of the initial value of $\log N_x$, which at the oldest age is the same as $\log D_{x+1}$, and of the successive values of $\log D_x$ in their proper places in reverse order as shown above. This preparatory process having been completed, the subsequent operations consist of, first, the subtraction of $\log D_x$ from $\log N_x$, setting the remainder, which is $\log a_x$, in the adjoining column in line with $\log D_x$; second, the entering of Gauss's table with this remainder, setting the result, which is $\log(1+a_x)$, under $\log D_x$; and third, the addition of this result to $\log D_x$. The sum is $\log N_{x-1}$; and so on. A numerical example will be useful. It is taken at 3 per-cent interest, and, to work with Gray's table of Gauss's logarithms, the values of $\log D_x$ given in Table No. X have been extended to six places by adding 0. Moreover, for convenience and accuracy in working, a line has been assigned to the proportional parts, so that to obtain $\log(1+a_x)$ two lines must be added together; but in extracting the particulars for tabulation this can easily be done by inspection.

100	$\bar{2}$·703440	
	$\bar{1}$·318340 0·094372 00	$\bar{1}$·385100
99	$\bar{1}$·412712	
	$\bar{1}$·683350 0·186438 22	$\bar{1}$·729362
98	$\bar{1}$·869810	
	0·020700 0·232101 4	$\bar{1}$·849110
97	0·252805	
	0·322990 0·267347 7	$\bar{1}$·929815
96	0·590344	
	0·618890 0·286965 26	$\bar{1}$·971454
95	0·905881	
	0·891050 0·308493 16	0·014831
94	1·199559	
	1·139650 0·332012 5	0·059909
93	1·471667	
	1·369620 0·355018 26	0·102047
92	1·724664	
	1·578650 0·380137 8	0·146014
91	1·958795	
	1·771830 0·404457 40	0·186965
90	2·176327	
	1·949480	0·226847
*	* *	* *

For verification, the value of N_x at periodical intervals may be formed by the summation of the column D_x.

59. If only $\log a_x$ were wanted, $\log vp_x$ would take the place of $\log D_x$ in the operation; and the work would be shorter, the several quantities being additive. But even if only $\log a_x$ be at the moment required, the extra trouble in finding $\log N_x$ is well repaid. Not only is verification, as we have shown, more easy and certain, but each value on the working sheet is of use on its own account. Besides the value of $\log D_x$ previously computed, the only other values that appear in the working are those of $\log a_x$, $\log(1+a_x)$, and $\log N_x$, and tables of these functions are always of service.

60. It will be noticed in the preceding type of operation, that the values of $\log N_x$ at a few of the oldest ages differ from those given in Table No. X. The reason is, that in Table No. IX, D_x at the oldest ages is cut down to only three significant figures, and from these N_x is formed, also to only three significant figures, by summation, and that from the curtailed values thus formed, the values of $\log N_x$ given in Table No. X are taken out to five figures. It thus appears that at the oldest ages the fourth and fifth figures of $\log N$ are not to be depended upon, and those calculated by the process described in Art. 58 are not less trustworthy. It will be found that after the point at which N_x in Table No. IX reaches five significant figures, the values of $\log N$ as given in Table No. X, and those as calculated by Gauss's logarithms, do not differ by more than a unit in the fifth place.

61. By means of the arithmometer a table of A_x can be formed with great facility from a table of a_x. We have the relation $A_x = 1 - d(1+a_x)$, and $A_{x+1} = 1 - d(1+a_{x+1})$; whence $A_{x+1} = A_x - d\Delta a_x$. But Δa_x being in almost all cases a negative quantity, it follows that throughout the greater part of the table the differences of A_x are additive. For the purpose of constructing the table of A_x, the values of Δa_x should be written in the second column of suitably ruled paper, the age being placed in the first column. Unity should be placed on the slide of the machine, and the value of d upon the fixed plate. The regulator being set at subtraction, d should be multiplied by $1+a_x$ for the youngest age to be tabulated, and the result will be that the initial value of A_x will thus be thrown up on the slide. The regulator being then set at addition, continued multiplication of d by the successive differences of a_x should be performed, and the results as they appear on the slide, namely the successive values of A_x, should be recorded on the working sheet. The eye will be guided as to the particular value of Δa_x which is to be employed at each stage, because the proper value will be that against the last-recorded value of A_x. If the operations embrace any part of the annuity table in which the differences of the annuities are positive, that will not interfere with the work. It will only be necessary at such points to place the regulator at subtraction instead of at addition. In order to eliminate the risk of error through using the wrong sign for the differences of a_x, it will be convenient, in entering these differences on the working sheet, to write the positive differences in red ink, and the negative differences, which in all tables are the great majority, in black ink.

62. The values of A_x as they appear on the slide of the arithmometer will always consist of many more figures than are required. In order to mark off the figures that are to be recorded, ivory or metal pegs may be fixed in the small holes in the slide which are provided for the purpose. The usual adjustment must be made for the figures rejected, by increasing the last figure retained by unity whenever the first figure rejected is 5 or more. The adjustment will be effected automatically if, before commencing the continued operation, the figure beyond the right-hand peg in the slide be increased by five.

63. All the tables so far discussed have consisted of but one column of values. Certain descriptions of tables, however, have many columns. For instance, complete tables of temporary or deferred annuities have a column for each age. The methods of constructing such tables differ in some respects from those hitherto considered, and they will now be illustrated.

64. To construct a complete table of deferred annuities by means of logarithms, we have, taking as the variable the age $x+n$, at which the annuity is to be entered upon; and keeping x, the present age, constant,

$$\log {}_{n|}a_x = \log N_{x+n} + \text{colog}\, D_x$$

$$\begin{aligned}\log {}_{n+1|}a_x &= \log N_{x+n+1} + \text{colog}\, D_x \\ &= \log N_{x+n} + \Delta \log N_{x+n} + \text{colog}\, D_x \\ &= \log {}_{n|}a_x + \Delta \log N_{x+n}.\end{aligned}$$

To make use of this continued formula the values of $\Delta \log N_x$ should be written on a moveable slip. Starting with age x, the value of $\log N_x + \text{colog}\, D_x$, that is the logarithm of an immediate annuity, or in other words an annuity deferred 0 years, should be entered in the first line on the working sheet in the column for age x. The moveable slip of $\Delta \log \mathrm{N}$ should then be so placed on the left of that column that $\Delta \log \mathrm{N}_x$ is against the first line of the working sheet, and it should then be secured in position by means of weights. The logarithm in the column should now be added sideways to the number against it on the slip, and the sum entered on the second line of the working sheet. This in its turn should be added to the number against it on the slip; and so on the work proceeds step by step. The numbers that are thus entered in the column under age x are the logarithms of the annuities on (x) deferred 0, 1, 2, 3, &c., years. When the column for age x is completed, the slip is moved to the right and placed against the column for age $x+1$; and it is also moved one line up, so that $\Delta \log \mathrm{N}_{x+1}$—instead of $\Delta \log \mathrm{N}_x$—may now be against the first line of the working sheet; and so on from column to column, until the whole table is completed.

65. The verification values may also be formed by a continued process, if we take now as the variable x, the present age; and keep $x+n$, the age at which the annuity is to be entered upon, constant. We have the relation

$$\log {}_{n|}a_x = \log \mathrm{N}_{x+n} + \operatorname{colog} \mathrm{D}_x$$

$$\begin{aligned} \log {}_{n-1|}a_{x+1} &= \log \mathrm{N}_{x+n} + \operatorname{colog} \mathrm{D}_{x+1} \\ &= \log {}_{n|}a_x + \Delta \operatorname{colog} \mathrm{D}_x . \end{aligned}$$

By means of this formula, starting with an initial value, $\log {}_{n|}a_x$, by the continued addition of the successive values of $\Delta \log \mathrm{D}_x$ the values of $\log {}_{n-1|}a_{x+1}$, $\log {}_{n-2|}a_{x+2}$, &c., are formed. These values should be entered on the working sheet before the other operations are commenced, and they will constitute a diagonal line slanting upwards. A series of these diagonal lines—for values, say, of $n=10$, $n=20$, &c.,—having been completed, the correctness of the columns will be secured, if the values from point to point, as the work proceeds downwards, are identical with the verification values already inserted.

66. A numerical example will assist the student to understand the foregoing explanations. Let it be required to construct a table of deferred annuities at 3 per-cent interest, starting at age 10. First, to construct the verification values. Following the principles laid down in Art. 65 we prepare the logarithms of annuities on a life aged 10, deferred 10, 20, &c., years; on a life aged 11, deferred 9, 19, &c., years, and so on. Thus, the type of operation explaining itself,

$n=10$			$n=20$		
Age.	Δ colog D_x	Ver. Vals.	Age.	Δ colog D_x	Ver. Vals.
10	0·01462	1·19785	10	0·01462	0·99472
1	·01445	·21247	1	·01445	1·00934
2	·01434	·22692	2	·01434	·02379
3	·01433	·24126	3	·01433	·03813
4	·01433	·25559	4	·01433	·05246
15	·01443	·26992	15	·01443	·06679
6	·01454	·28435	6	·01454	·08122
7	·01474	·29889	7	·01474	·09576
8	·01492	·31363	8	·01492	·11050
9	·01512	·32855	9	·01512	·12542
20	·01533	·34367	20	·01533	·14054
*	* *	* *	*	* *	* *

It will be seen that as the same values of Δ colog D_x are used in each series of verification values, time will be saved by using a moveable slip for that function.

67. Proceeding now to the calculation of the columns, we have the following type of operation:—

n	10	11	x	Δ log N_x	13	14	15
			10	$\bar{1}$·98223			
			11	·98210			
			12	·98196			
0	1·38263	1·37	13	·98181	1·37233		
1	1·36486	1·36	14	·98161	1·35414		
2	1·34696	1·34	15	·98148	1·33575		
3	1·32892	1·32	16	·98127	1·31723		
4	1·31073	1·30	17	·98110	1·29850		
5	1·29234	1·28	18	·98092	1·27960		1·26992
6	1·27382	1·26	19	·98074	1·26052	1·25559	
7	1·25509	1·25	20	·98059	1·24126		
8	1·23619	1·23	21	·98037	1·22185		
9	1·21711	1·21	22	·98024	1·20222		
10	1·19785	1·19	23	·98000	* *		
1	1·17844	1·17	24	·97982			
2	1·15881	1·15	25	·97962			
3	1·13905	1·13	26	·97940			
4	1·11905	1·11	27	·97918			
15	1·09887	1·09	28	·97895			1·06679
6	1·07849	1·07	29	·97870		1·05246	
7	1·05789	1·05	30	·97844	1·03813		
8	1·03707	1·03	31	·97818			
9	1·01602	1·00	32	·97788			
20	0·99472	0·98	33	·97759			
*	* *	*	*	* *			

In the foregoing illustration the columns for ages 10, 11, and 12 are supposed to have been completed; and the moveable slip, which covers the column for age 12 and part of that for age 11, is against the column for age 13, which is in process of computation. The verification values, which were found above, appear in old-style type, while the other values are in Roman type. It will be seen that, in the columns completed, the verification values fall into their proper positions among the rest, while they stand by themselves in the columns not yet filled in. As it is column for age 13 which is in process of formation, the value for that age on the moveable slip is against the first line of the working sheet.

68. The values of temporary annuities do not lend themselves easily to direct calculation by means of logarithms; but they can be formed without difficulty indirectly, by first preparing a table of deferred annuities. The temporary annuities are then arrived at by means of the relation $|_n a_x = a_x - {}_n|a_x$. It will be seen that from this equation the values of temporary annuities for all durations for present age x may be computed by a continued process, simply by the continued addition of the differences of the deferred annuities to the initial value, which in every case is zero, because $|_0 a_x = 0$.

69. A very valuable verification of tables of deferred and temporary annuities may be obtained by means of commutation columns. If there be $n+1$ values of annuities on (x), deferred 0, 1, 2, &c., n years respectively; then, on adding up the column, the sum should be $\frac{S_x - S_{x+n+1}}{D_x}$. Similarly, if there be $n+1$ values of annuities on (x), temporary for 0, 1, 2, &c., n years respectively; then their sum is $\frac{(n+1)N_x - (S_x - S_{x+n+1})}{D_x}$. The method of verifying tables by summation may with advantage be adopted whenever applicable, because it not only tests the calculations, but if logarithms have been employed it shows whether or not the numbers have been properly taken out; and, moreover, in the case of printed tables it is a check on the work of the compositor.

70. Tables of temporary and deferred annuities may be prepared with very great facility by means of the arithmometer. For temporary annuities we have

$$|_n a_x = D_x^{-1}(D_{x+1} + D_{x+2} + \&c. + D_{x+n}),$$

and
$$|_{n+1} a_x = D_x^{-1}(D_{x+1} + D_{x+2} + \&c. + D_{x+n} + D_{x+n+1})$$
$$= |_n a_x + D_x^{-1} \times D_{x+n+1}.$$

This expression is in the form P + QR, which, as we have seen in Art. 17, is specially adapted to the machine. Placing D_x^{-1} on the fixed plate as the in-factor, by continued multiplication by the successive values of D_x, the out-factor, the column for age x is completed. As the values of D_x are required in the multiplications for each column, they should be written on a moveable slip, to be set in proper position against each column as the work proceeds. For the sake of convenience of reference the values of the reciprocals, D_x^{-1}, may be entered at the head of each column. Verification values cannot conveniently be prepared by a continued process. They should be calculated individually and inserted in their places.

The following is a type of the calculations at 3 per-cent interest. Columns for ages 10, 11, and 12 are supposed to have been completed, and that for age 13 is in progress. Against this column the moveable slip is placed in proper position, and it covers the column for age 12, and part of that for age 11. The verification values appear in old-style type.

n	134391 10	1389 11	x	D_x	148518 13	153501 14	158650 15
			10	74410			
			11	71947			
			12	69592			
0	·000	·00	13	67332	·000		
1	·967	·96	14	65146	·968		
2	1·902	1·90	15	63032	1·904		
3	2·807	2·80	16	60972	2·809		
4	3·683	3·68	17	58964	3·685		
5	4·530	4·53	18	56997	4·531		
6	5·349	5·35	19	55072	5·349		
7	6·141	6·14	20	53188	6·139		
8	6·907	6·90	21	51343	6·902		
9	7·648	7·64	22	49544	7·638		
10	8·362	8·36	23	47791	8·347	8·335	8·320
1	9·052	9·05	24	46090			
2	9·718	9·71	25	44439			
3	10·360	10·35	26	42839			
4	10·980	10·97	27	41291			
15	11·577	11·56	28	39796			
6	12·153	12·14	29	38349			
7	12·708	12·69	30	36949			
8	13·243	13·22	31	35597			
9	13·758	13·74	32	34288			
20	14·254	14·23	33	33022	14·179	14·143	14·103
*	* *	*	*	* *	* *	* *	* *

71. The mode of procedure to form tables of deferred annuities is very similar to the foregoing. The continued formula for the construction of the columns is

$${}_{n+1|}a_x = {}_{n|}a_x - D_x^{-1} \times D_{x+n+1},$$

and therefore the regulator of the machine must be set at subtraction. A numerical example is unnecessary.

In the case of deferred annuities the verification values may be formed on the machine in series, by changing the variable as in Art. 65. The continued formula for these values is

$$_{n-1|}a_{x+1} = {}_{n|}a_x + \mathrm{N}_{x+n} \times \Delta \mathrm{D}_x^{-1}.$$

On the working sheet they will form diagonal lines slanting upwards, as in Art. 67.

72. Several methods have been suggested for forming tables of Policy-Values. From Chap. xviii, formula 13, we have

$$\frac{1 - {}_{n+1}\mathrm{V}_x}{1 - {}_n\mathrm{V}_x} = (1 - {}_1\mathrm{V}_{x+n}) = \frac{1 + a_{x+n+1}}{1 + a_{x+n}};$$

or in logarithms

$$\log(1 - {}_{n+1}\mathrm{V}_x) - \log(1 - {}_n\mathrm{V}_x) = \Delta \log(1 + a_{x+n}).$$

Whence $\log(1 - {}_{n+1}\mathrm{V}_x) = \log(1 - {}_n\mathrm{V}_x) + \Delta \log(1 + a_{x+n})$.

By means of this formula, starting with $n=0$, when, of course, also $\log(1 - {}_n\mathrm{V}_x) = 0$, by the continued addition of the successive values of $\Delta \log(1 + a_{x+n})$, the column of the values of the function $\log(1 - {}_n\mathrm{V}_x)$ for age x is formed; and similarly for other ages. As the values of $\Delta \log(1 + a_x)$ are used for each column, they should be written on a moveable slip, to be employed in the way illustrated in preceding articles.

Verification values for periodical values of n may be formed in series by a continued process by means of the relation

$$\log(1 - {}_{n-1}\mathrm{V}_{x+1}) = \log(1 - {}_n\mathrm{V}_x) - \Delta \log(1 + a_x),$$

the variable having been changed, as in Art. 65. These verification values will appear in diagonal lines slanting upwards, like those in the example given in Art. 67.

73. When the columns of $\log(1 - {}_n\mathrm{V}_x)$ have been formed and verified in the way described above, the natural numbers must be obtained, and their arithmetical complements taken, in order to arrive at the Policy-Values. This process is tedious, and liable to error; but, when completed, the table of Policy-Values may be thoroughly checked by summation, as will be shown in Arts. 80 and 81.

74. A more brief and direct method of forming tables of Policy-Values, and one which affords more efficient checks, is the following.

We have
$$ {}_{n}V_x = \frac{a_x - a_{x+n}}{1+a_x} $$

$$ \begin{aligned} {}_{n+1}V_x &= \frac{a_x - a_{x+n+1}}{1+a_x} \\ &= \frac{a_x - a_{x+n} - \Delta a_{x+n}}{1+a_x} \\ &= {}_{n}V_x + \frac{-\Delta a_{x+n}}{1+a_x}; \end{aligned} $$

the term $\frac{-\Delta a_{x+n}}{1+a_x}$ being almost always positive, because, with rare exceptions, Δa_{x+n} is negative. To make use of this formula we must first prepare a complete table of $\frac{-\Delta a_{x+n}}{1+a_x}$ for all values of x and n; and then the values of policies for each age can be formed by continued addition.

75. To prepare the complete table of $\frac{-\Delta a_{x+n}}{1+a_x}$, we have the continued formula

$$ \log \frac{-\Delta a_{x+n+1}}{1+a_x} = \log \frac{-\Delta a_{x+n}}{1+a_x} + \Delta \log(-\Delta a_{x+n}) $$

whence, commencing with the value for $n=0$, by the continued addition of the successive values of $\Delta \log(-\Delta a_x)$ the column for age x is formed. The first step therefore is to prepare the values of $\Delta \log(-\Delta a_x)$. The following is a specimen of the calculations at 3 per-cent interest:—

x	$-\Delta a_x$	$\log(-\Delta a_x)$	$\Delta \log(-\Delta a_x)$
10	·174	$\bar{1}$·24055	0·03820
1	·190	·27875	·02445
2	·201	·30320	·01902
3	·210	·32222	·01424
4	·217	·33646	·00200
15	·218	·33846	·00198
6	·219	·34044	$\bar{1}$·99401
7	·216	·33445	$\bar{1}$·99596
8	·214	·33041	$\bar{1}$·99387
9	·211	·32428	$\bar{1}$·99169
20	·207	·31597	0·00000
1	·207	·31597	$\bar{1}$·99790
2	·206	·31387	0·00419
3	·208	·31806	·00622
4	·211	·32428	·00816
25	·215	·33244	·00998
6	·220	·34242	·01169
7	·226	·35411	·01138
8	·232	·36549	·00926
9	·237	·37475	·01264
30	·244	·38739	* *
*	*	* *	

The values of $\Delta\log(-\Delta a_x)$ having been prepared, they should be written on a moveable slip, for use in constructing the table of $\log\dfrac{-\Delta a_{x+n}}{1+a_x}$.

76. The next process is to compute verification values of $\log\dfrac{-\Delta a_{x+n}}{1+a_x}$ which may be done in series by changing the variable, thus:—

$$\log\frac{-\Delta a_{x+n}}{1+a_{x+1}}=\log\frac{-\Delta a_{x+n}}{1+a_x}+\Delta\text{colog}\,(1+a_x).$$

When these verification values have been prepared and inserted in the working sheet, they will form diagonal lines slanting upwards, as shown in old-style type in the example below.

n	10	11	*x*	$\Delta\log(-\Delta a_x)$	13	14	15
			10	0·03820			
			11	·02445			
			12	·01902			
0	$\bar{3}$·84029	$\bar{3}$·881	13	·01424	$\bar{3}$·93183	$\bar{3}$·94980	$\bar{3}$·95569
1	$\bar{3}$·87849	$\bar{3}$·905	14	·00200	$\bar{3}$·94607		
2	$\bar{3}$·90294	$\bar{3}$·924	15	·00198	$\bar{3}$·94807		
3	$\bar{3}$·92196	$\bar{3}$·939	16	$\bar{1}$·99401	$\bar{3}$·95005		
4	$\bar{3}$·93620	$\bar{3}$·941	17	$\bar{1}$·99596	$\bar{3}$·94406		
5	$\bar{3}$·93820	$\bar{3}$·943	18	$\bar{1}$·99387	$\bar{3}$·94002		3·93320
6	$\bar{3}$·94018	$\bar{3}$·937	19	$\bar{1}$·99169	$\bar{3}$·93389	$\bar{3}$·92931	
7	$\bar{3}$·93419	$\bar{3}$·933	20	0·00000	$\bar{3}$·92558		
8	$\bar{3}$·93015	$\bar{3}$·927	21	$\bar{1}$·99790	$\bar{3}$·92558		
9	$\bar{3}$·92402	$\bar{3}$·918	22	0·00419	$\bar{3}$·92348		
10	$\bar{3}$·91571	$\bar{3}$·918	23	·00622	* *		
1	$\bar{3}$·91571	$\bar{3}$·916	24	·00816			
2	$\bar{3}$·91361	$\bar{3}$·920	25	·00998			
3	$\bar{3}$·91780	$\bar{3}$·927	26	·01169			
4	$\bar{3}$·92402	$\bar{3}$·935	27	·01138			
15	$\bar{3}$·93218	$\bar{3}$·945	28	·00926			$\bar{2}$·00462
6	$\bar{3}$·94216	$\bar{3}$·956	29	·01264		$\bar{2}$·00073	
7	$\bar{3}$·95385	$\bar{3}$·968	30	·01055	$\bar{3}$·99700		
8	$\bar{3}$·96523	$\bar{3}$·977	31	·01030			
9	$\bar{3}$·97449	$\bar{3}$·990	32	·01172			
20	$\bar{3}$·98713	$\bar{2}$·000	33	·00979			
*	* *	*	*	* *			

The numbers in the foregoing type of calculation are the *logarithms of the differences* of the Policy-Values. On taking out the natural numbers, we have the *differences* of the Policy-Values, and these, by continued addition to the initial value which in every case is zero produce the Policy-Values themselves. They should be computed to one place of decimals more than are to be finally retained, as the last place is not to be relied upon. The following is a specimen of the figures for ages at entry 10 and 11:—

n	$\Delta_n V_{10}$	${}_nV_{10}$	$\Delta_n V_{11}$	${}_nV_{11}$
0	·006923	·000000	·007612	·000000
1	·007559	·006923	·008053	·007612
2	·007997	·014482	·008414	·015665
3	·008355	·022479	·008694	·024079
4	·008634	·030834	·008734	·032773
5	·008674	·039468	·008774	·041507
6	·008713	·048142	·008654	·050281
7	·008594	·056855	·008574	·058935
8	·008514	·065449	·008454	·067509
9	·008395	·073963	·008293	·075963
10	·008236	·082358	·008293	·084256
*	* *	* *	* *	* *

77. The chief advantage of this method of computing tables of Policy-Values is, that a complete check can be applied at the last stage. Thus, for example, on calculating by the formula ${}_nV_x=\dfrac{a_x-a_{x+n}}{1+a_x}$ the value of ${}_{10}V_{10}$, it comes out exactly ·08236, as given above, thus proving that all the work up to that point, *including the turning of logarithms into numbers*, has been correctly performed.

78. By the arithmometer tables of Policy-Values may be prepared with the greatest facility and celerity. One formula is

$${}_{n+1}V_x={}_nV_x+(1+a_x)^{-1}\times(-\Delta a_{x+n})$$

which is in the form P+QR. The values of $-\Delta a_x$ are written on a moveable slip, to be used as the successive out-factors. The reciprocal, $(1+a_x)^{-1}$, is then placed on the plate of the machine as the in-factor, and multiplied successively by the values of $-\Delta a_x$, each product being thrown up on to the slide of the machine and added to the number there already. The results which thus appear in succession on the slide constitute the column of Policy-Values for age at entry x.

By changing the variable, verification values may be prepared by a continued process from the formula

$${}_{n-1}V_{x+1}={}_nV_x-(1+a_{x+n})\times\Delta(1+a_x)^{-1}.$$

On the working sheet they will form diagonal lines slanting upwards, like those shown in previous examples.

79. Formula 15 of Chap. xviii furnishes another method of preparing a table of Policy-Values on the arithmometer by a continued process. If the values of policies of all durations be in the first place calculated for age at entry x, those for age at entry $x+1$ may be derived from them. In this case, $\psi=(P_{x+1}-P_x)(1+a_x)$, and

$$\begin{aligned}{}_0V_{x+1}&={}_1V_x(1+\psi)-\psi\\ {}_1V_{x+1}&={}_2V_x(1+\psi)-\psi\\ &={}_0V_{x+1}+(1+\psi)\times\Delta_1V_x\end{aligned}$$

and generally

$${}_{n+1}V_{x+1}={}_nV_{x+1}+(1+\psi)\times\Delta_{n+1}V_x.$$

Since ${}_0V_{x+1}=0$, by placing $(1+\psi)$ on the plate of the machine and multiplying by Δ_1V_x, we get ${}_1V_{x+1}$, and then by the continued multiplication by the successive values of ΔV_x the column of V_{x+1} is formed. In this operation the differences of V_x are required in the calculation of the next column, but they can be formed very easily by the eye as the work proceeds, the numbers being small.

80. Completed tables of Policy-Values may be verified by addition. From the relation ${}_nV_x=\dfrac{a_x-a_{x+n}}{1+a_x}$, by giving to n every integral value from 0 to n, and taking the sum, we have for $n+1$ Policy-Values

$$\Sigma_0^n V=\frac{(n+1)a_x-\Sigma_0^n a_{x+n}}{1+a_x}.$$

To make use of this formula a column on the commutation principle is formed from a_x exactly as M_x is formed from C_x. The application of this column is obvious.

81. To take a numerical example from the eleven Policy-Values from $n=0$ to $n=10$ given in Art. 76 for age at entry 10:—their sum is ·440953; and applying the formula, with three decimal places in the annuity-values, being the number used in preparing the Policy-Values, the result is ·440956. There is a small discrepancy in the sixth place of decimals, but this is accounted for by the fact that, as mentioned in Art. 76, the sixth figure of the Policy-Value is not to be implicitly relied upon.

82. Tables of the values of Endowment Assurance policies may be prepared in exactly the same way as those for the whole of life, by merely employing the proper annuity-values, as shown in Chap. xviii, Art. 102.

83. In constructing tables of Joint-Life Annuities, the values of $\log vp_{xy}$ come into requisition, and a complete table should be prepared before the annuity tables are commenced. Seeing that $vp_{xy}=vp_{yx}$, it is

not necessary to compute the values for both $x>y$ and $x<y$. For the working formula we have

$$\log vp_{xy}=\log vp_x+\log p_y$$

and

$$\log vp_{xy-1}=\log vp_{xy}+\Delta\mathrm{colog}\,p_{y-1}.$$

Whence, by the continued addition of the successive values of $\Delta\mathrm{colog}\,p_{y-1}$ to an initial value, we form the values of $\log vp_{xy}$ for the column in which x is constant, and y proceeds from the older to the younger ages. The initial term may conveniently be taken as $\log vp_{xx}$, where the ages are equal. To construct the table, the values of $\log vp_x$, adjusted as explained in Art. 48, should be written in retrograde order on the first line of the working sheet, and on the next line in similar order the values of $\log p_x$. The addition of these two lines will produce $\log vp_{xx}$. Then, starting from these as initial values, by means of a moveable slip containing in retrograde order the values of $\Delta\mathrm{colog}\,p_{x-1}$, the continued addition of these differences is effected. Verification, in diagonal lines slanting upwards, may be obtained by the formula

$$\log vp_{x-1:y}=\log vp_{xy}+\Delta\mathrm{colog}\,vp_{x-1},$$

where x is treated as the variable, and y is constant.

It will be observed here that we write $\Delta\mathrm{colog}\,vp_{x-1}$ and not $\Delta\mathrm{colog}\,p_{x-1}$, although, theoretically, these differences are identical. This course is adopted in order that the adjustment for the error in $\log v$ may fall in proper place. The following specimen of the calculations at 3 per-cent interest will more clearly explain the operations.

$x-y$	100	99	98		97	96	95
			x	$\Delta\mathrm{colog}\,p_{x-1}$			
	38510	63	100	24988	69771	70410	72784
	39794	64	99	02767	71055	71694	74067
			98	03506			
0	78304	28	97	00639	40826	42104	46851
1	03292	31	96	02373	41465		
2	06059	34	95	02357	43838		
3	09565	35	94	01863	46195		
4	10204	37	93	02094	* *		
5	12577	39	92	01584			56303
6	14934	41	91	01554		53929	
7	16797	43	90	01437	53290		
8	18891	45	89	01284			
9	20475	47	88	01182			
10	22029	48	87	01066			
*	* *	*	*	* *			

84. In the example, the columns for ages 100, 99, and 98, are supposed to have been finished, and that for 97 is in progress, the moveable slip being against it in proper position. A diagonal line of verification values in old-style type appears. It will be noticed that x remains constant in the columns, and the difference $x-y$ remains constant in the lines. It is the values for these constant differences $x-y$ that are generally required in constructing tables of other functions, and they can be conveniently extracted from the lines at small risk of error.

85. To construct a complete table of D_{xy} by means of logarithms. If a table of $\log vp_{xy}$ have already been formed, then to form $\log D_{xy}$ we may proceed exactly as for the construction of D_x, explained in Art. 49, because $\log D_{x+1:y+1} = \log D_{xy} + \log vp_{xy}$.

If the commutation column is to be in Davies's form, then the initial value will be $\log l_x + \log l_y + \log v^x$, where $x>y$; but if the table is to be in De Morgan's form, then the initial value will be $\log l_x + \log l_y + \log v^{\frac{x+y}{2}}$. In either case verification values at intervals will be obtained by means of the fundamental formula.

86. That this method of constructing the table of $\log D_{xy}$ may be most conveniently carried out, the table of $\log vp_{xy}$ should be so arranged that in the columns we have the values for constant differences of $x-y$. The sheet containing these values may then be folded perpendicularly, so that the column for a particular difference, $x-y$, may appear at the right-hand edge. This column may then be placed against the corresponding column of the working sheet of $\log D_{xy}$, and the additions effected sideways without difficulty. By this arrangement the values of D_{xy}, when taken out, will come in such order that the column N_{xy} can be at once formed by continued addition.

87. It will be noticed that, as in Art. 83 the working sheet for the construction of the table of $\log vp_{xy}$ was so planned that the constant differences of $x-y$ appeared in the lines and not in the columns, therefore the table of $\log vp_{xy}$ must be re-arranged, in order that the construction of $\log D_{xy}$, as here explained, may be conveniently carried out.

88. Another and, perhaps, a preferable method of constructing a table of $\log D_{xy}$, in Davies's form, may be derived from the formula

$$\log D_{x:y-1} = \log D_{xy} + \operatorname{colog} p_{y-1}.$$

We proceed exactly as we did in Art. 83 for $\log vp_{xy}$, starting, however, with different initial values. The first line in the working sheet will consist of the successive values of $\log(v^x l_x)$, or $\log \mathrm{D}_x$; and the second line, of the successive values of $\log l_x$. The addition of these two lines will give the values of $\log \mathrm{D}_{xx}$, and then, by means of a moveable slip of $\operatorname{colog} p_{x-1}$, (not $\Delta \operatorname{colog} p_{x-1}$ as in Art. 83), the whole table is computed. Similarly, verification values may be formed in series as in Art. 83, and entered on the working sheet, where they will appear in diagonal lines slanting upwards, before the other operations are commenced. A numerical example is not required, as the type of operation is exactly like that in Art. 83.

89. If the joint-life commutation table is to be in De Morgan's form, then the only difference in the mode of construction from that explained in the last preceding article is to be found in the values on the moveable slip. These must now be the values of $\operatorname{colog} v^{\frac{1}{2}}p_{y-1}$, because in De Morgan's form of column

$$\log \mathrm{D}_{x:y-1} = \log \mathrm{D}_{xy} + \operatorname{colog} v^{\frac{1}{2}}p_{y-1}.$$

Of course, the values of $\operatorname{colog} v^{\frac{1}{2}}p_{y-1}$ on the moveable slip must be adjusted in the last place of decimals, for the error in $\operatorname{colog} v^{\frac{1}{2}}$ which accumulates in continued operations. The reason for this, and the method of adjustment, are fully set out in Art. 48, where the similar function $\log vp_x$ was dealt with.

90. In Art. 58 a full description was given, with a numerical illustration, of the method of constructing a table of $\log \mathrm{N}_x$ by means of Gauss's logarithms, whereby at the same time are formed the values of $\log a_x$, and $\log(1+a_x)$. Exactly the same process may be adopted for joint-life functions, because

$$\log \mathrm{N}_{xy} = \log \mathrm{D}_{x+1:y+1} + [t](\log \mathrm{N}_{x+1:y+1} - \log \mathrm{D}_{x+1:y+1}).$$

Using this method, the joint-life annuity-table is constructed in columns for the successive constant differences of $x-y$; and, as explained in Art. 58 already referred to, a complete verification can be obtained at intervals, by taking out the values of D_{xy} in numbers, and summing them to form periodical values of N_{xy}.

91. Another method of forming tables of joint-life annuities is derived from the relation

$$a_{xy} = vp_{xy}(1 + a_{x+1:y+1}),$$

whence, by Gauss's logarithms,

$$\log a_{xy} = \log vp_{xy} + [t] \log a_{x+1:y+1}.$$

To make use of this equation, the values of $\log vp_{xy}$ must first be calculated; and moreover there is no check on the continued operation, so that to ensure accuracy the entire work must be done in duplicate. Seeing that it is as easy to construct a table of $\log D_{xy}$ as one of $\log vp_{xy}$, and that, as explained in Art. 90, the preparation therefrom of $\log N_{xy}$, $\log a_{xy}$, and $\log(1+a_{xy})$, *with* verification, is scarcely more troublesome than that of $\log a_{xy}$ from $\log vp_{xy}$, *without* verification, there can be no doubt that the construction of joint-life annuity-values through the N_{xy} column is the more advantageous method.

92. The values of D_{xy} in Davies's form may be very easily computed on the arithmometer. When $x=$ or $>y$, then $D_{xy} = D_x \times l_y$; and

$$D_{x:y-1} = D_x \times l_{y-1} = D_x \times l_y + D_x \times d_{y-1}.$$

To make use of this formula, the operation will be:—Place D_x on the fixed plate of the machine, and multiply by l_y where $y=x$. The result will be to throw D_{xx} on to the slide of the machine. The continued multiplication of D_x, which remains constant on the fixed plate, by the successive values in retrograde order of d_{y-1}, will give in retrograde order the values of D_{xy}, for all values of y less than x. After each multiplication, the result must be recorded, but not effaced from the slide. The following is a type of the operation when the record is made in *columns*. It represents the construction of D_{xy} for $x=60$, 59, &c., at 3 per-cent interest. It will be observed that the values of d_{y-1} are written on a moveable slip: also that verification values at periodical intervals appear in old-style type. These verification values are calculated by continued process by means of the equation $D_{x-1:y} = D_{xy} - l_y \Delta D_{x-1}$, wherein it must be remembered that the term apparently subtractive is really additive, because ΔD_x is essentially negative. It may be remarked that the radix of the table has been reduced, by dividing l_x by 100; as otherwise the number of figures to record would be superabundant.

$x-y$	60	59	y	d_{y-1}	57	56
			61	17·55		
			60	16·82		
			59	16·12		
			58	15·41		
0	58769	64040	57	14·75	75203	
1	60449	65746	56	14·14	76945	
2	62059	67377	55	13·53	78614	
3	63598	68937	54	12·96	80212	
4	65071	70433	53	12·43	81743	
5	66483	71865	52	11·93	83211	
6	67835	73236	51	11·44	84620	90608
7	69129	74552	50	11·01	85971	
8	70371	75814	49	10·61	87271	
9	71562	77024	48	10·21	* *	
10	72705	78189	47	9·86		
1	73804	79312	46	9·54		
2	74864	80392	45	9·24		
3	75884	81436	44	8·95		
4	76869	82445	43	8·71		
15	77821	83423	42	8·46		
6	78744	84370	41	8·23		102410
7	79638	85291	40	8·06	97169	
8	80508	86186	39	7·86		
9	81353	87057	38	7·70		
20	82175	87910	37	7·56		
*	* *	* *	36	7·42		
			*	*		

93. According to the above arrangement, the values of D_{xy} for constant differences of $x-y$ appear in the lines, and not in the columns; and they are therefore not conveniently situated for the formation of N_{xy} from them. To obviate this difficulty the working sheet may be arranged somewhat differently. The successive values of D_{xy} as they are formed by the process above explained may be placed in the lines, instead of in the columns, thus interchanging the order. The columns will then contain the values of D_{xy} for constant differences of $x-y$, and the passage to N_{xy} will be easy. The inconvenience of this method, and it is a serious one, is that there is nothing to guide the eye as to the particular value of d_y which should be used in each multiplication. This difficulty may be in part overcome by placing a pencil mark against each d_y as it is used, the pencil marks to be effaced after each line of the working sheet has been finished.

94. The modes of constructing survivorship commutation columns, C^1_{xy} and M^1_{xy}, are very similar to those already described for the joint life D and N columns.

95. To construct by logarithms a table of C^1_{xy}. When the function is to be in De Morgan's form, we have

$$\log C^1_{xy} = \log v + \log v^{\frac{x+y}{2}} + \log d_x + \log l_{y+\frac{1}{2}}$$

and

$$\log C^1_{x:y-1} = \log v + \log v^{\frac{x+y-1}{2}} + \log d_x + \log l_{y-\frac{1}{2}}$$

$$= \log C^1_{xy} + \operatorname{colog} v^{\frac{1}{2}} p_{y-\frac{1}{2}}.$$

It thus appears that, as a preliminary process, a table of colog $v^{\frac{1}{2}} p_{y-\frac{1}{2}}$ should be prepared, the last place of which should be adjusted in accordance with Art. 48. In the case of the survivorship commutation table, if it is to be complete, every value of x must be combined with every value of y, and not only with those where y is not greater than x as in the case of the D_{xy} table. Therefore on the first line of the working sheet for the survivorship table, log $l_{y+\frac{1}{2}}$ must be repeated at the head of each column, y being the oldest age in the mortality table, namely, $(\omega-1)$. On the second line, the successive values of log d_x must be entered in retrograde order; and on the third line, log $v^{\frac{x+\omega+1}{2}}$ for the same values of x as in the second line. The addition of these three lines will give the successive values of log $C^1_{x:\omega-1}$. Each column may then be completed by the continued addition of colog $v^{\frac{1}{2}} p_{y-\frac{1}{2}}$, for all values of y in retrograde order, from the oldest age to the youngest. The values of these continued addends may be written on a slip, which will be moved from left to right as the columns are completed one after the other; but which in the present instance will not be moved upwards, as precisely the same values of log $l_{y+\frac{1}{2}}$ appear in the same positions in all the columns.

96. To form the verification series, we have, making x instead of y the variable,

$$\log C^1_{xy} = \log v + \log v^{\frac{x+y}{2}} + \log d_x + \log l_{y+\frac{1}{2}}$$

and

$$\log C^{1}_{\overline{x-1}:y} = \log C^1_{xy} + \operatorname{colog} v^{\frac{1}{2}} + \Delta \operatorname{colog} d_{x-1}.$$

In the first of these expressions, giving to x its greatest value, namely, $(\omega-1)$, and to y a set of convenient equidistant values, we have the initial terms of the verification series, which are to be entered in their proper places in the first column of the working sheet. Then, proceeding with the continued addition of $\operatorname{colog} v^{\frac{1}{2}} + \Delta\operatorname{colog} d_{x-1}$, for the successive values of x from the oldest to the youngest, the whole of the verification series are prepared. These, when completed and entered on the working sheet, will form horizontal lines, and not diagonal lines as in previous examples.

97. If the table is to be in Davies's form, the principles of construction will be the same, but the details will be different. It will be necessary to prepare the table in two sections, namely, for $x>y$ and $x<y$.

When $x>y$

$$\log \mathrm{C}^1_{xy} = \log v^{x+1} + \log d_x + \log l_{y+\frac{1}{2}}$$

and
$$\log \mathrm{C}^1_{x:y-1} = \log \mathrm{C}^1_{x.y} + \Delta\operatorname{colog} l_{y-\frac{1}{2}}$$

and, when $x<y$

$$\log \mathrm{C}^1_{x.y} = \log v^{y+1} + \log d_x + \log l_{y+\frac{1}{2}}$$

and
$$\log \mathrm{C}_{\overset{1}{\overline{x-1}}:y} = \log \mathrm{C}^1_{xy} + \Delta\operatorname{colog} d_{x-1}.$$

By means of these continued formulas the table may be conveniently prepared; and the student will find no difficulty in arranging the types of calculation, and in deducing continued formulas for verification values.

98. The construction of survivorship commutation columns on the arithmometer is conducted on the same principles as that of the D_{xy} columns when the tables are in Davies's form. We have the continued formulas

for $x>y$
$$\mathrm{C}^1_{x:y-1} = \mathrm{C}^1_{xy} - \mathrm{C}_x \times \Delta l_{y-\frac{1}{2}}$$

$$= \mathrm{C}^1_{xy} + \mathrm{C}_x \times d_{y-\frac{1}{2}}$$

and, for $x<y$

$$\mathrm{C}_{\overset{1}{\overline{x-1}}:y} = \mathrm{C}^1_{xy} - v^{y+1} l_{y+\frac{1}{2}} \times \Delta d_{x-1}$$

Therefore, placing C_x or $v^{y+1} l_{y+\frac{1}{2}}$ on the fixed plate, by the continued multiplication by $d_{y-\frac{1}{2}}$, or Δd_{x-1}, as the case may be, the columns are computed.

Verification values may also be prepared much in the same manner as for the table of D_{xy}.

99. A table of A^1_{xy} may be constructed by means of Gauss's logarithms without the intervention of commutation tables. We have the obvious relation

$$\begin{aligned}A^1_{xy}&=vq^1_{xy}+vp_{xy}A^{\,1}_{x+1:\overline{y+1}}\\&=vp_{xy}\{\tfrac{1}{2}(p_x^{-1}-1)(p_y^{-1}+1)+A^{\,1}_{x+1:\overline{y+1}}\}\,;\end{aligned}$$

or, in logarithms, when for conciseness we write Π_{xy} for $\frac{1}{2}(p_x^{-1}-1)(p_y^{-1}+1)$,

$$\begin{aligned}\log A^1_{xy}&=\log vp_{xy}+\log\{\Pi_{xy}+A^{\,1}_{x+1:\overline{y+1}}\}\\&=\log vp_{xy}+\log\Pi_{xy}+[t]\{\log A^{\,1}_{x+1:\overline{y+1}}-\log\Pi_{xy}\}\end{aligned}$$

The following is a symbolical illustration of the working of the formula.

α	$\log A^{\,1}_{x+1:\overline{y+1}}$	$\alpha-\beta$
β	$\log \Pi_{xy}$	
γ	$\log vp_{xy}$	
δ	$[t](\alpha-\beta)$	
$\beta+\gamma+\delta=\alpha_{-1}$	$\log A^1_{xy}$	$\alpha_{-1}-\beta_{-1}$
β_{-1}	$\log \Pi_{x-1:y-1}$	
γ_{-1}	$\log vp_{x-1:y-1}$	
δ_{-1}	$[t](\alpha_{-1}-\beta_{-1})$	
$\beta_{-1}+\gamma_{-1}+\delta_{-1}=\alpha_{-2}$	$\log A^{\,1}_{x-1:\overline{y-1}}$	&c.
	&c. &c.	

100. Before proceeding to further operations, the values of $\log\Pi_{xy}$ and $\log vp_{xy}$ must be written in retrograde order at suitable intervals on the working sheet, a line being left for the proportional parts in $[t](\beta-\alpha)$, which line is not shown in the above symbolical example. The initial value—which is easily computed independently—will be A^1_{xy}, where either x or y, as the case may be, is the oldest age in the mortality table, and the continued formula, working from that initial value, will produce, in retrograde order, the successive values of $\log A^1_{xy}$ for a constant difference $x-y$.

101. In order to use this formula, complete tables are required in the first instance of $\log vp_{xy}$ and $\log\frac{1}{2}(p_x^{-1}-1)(p_y^{-1}+1)$. The construction

of the first of these has already been described; and that of the second is very similar, except that for it every age of x must be combined with every age of y. With this end in view, $\log(p_x^{-1}-1)$ should be repeated at the head of each column on the first line of the working sheet, x being the oldest age in the mortality table, namely, $(\omega-1)$. On the second line should then be written, in retrograde order, the successive values of $\log\frac{1}{2}(p_y^{-1}+1)$. The addition of these two lines will give the values of $\log \Pi_{xy}$, for all values of y combined with the greatest value of x. Each column may then be completed by the continued addition of $-\Delta\log(p_{x-1}^{-1}-1)$ by means of a moveable slip.

102. To construct commutation columns for sickness benefits. In Chap. xx, Arts. 3 and 4, it was shown that the function for sickness benefits corresponding to C_x for assurance benefits is $D_x v^{\frac{1}{2}} z_x$, which for convenience we may here write C'_x. The construction of sickness commutation columns therefore resolves itself into the construction of a table of C'_x.

We have

$$\log C'_x = \log D_x + \log z_x + \log v^{\frac{1}{2}}.$$

and

$$\log C'_{x+1} = \log D_{x+1} + \log z_{x+1} + \log v^{\frac{1}{2}}.$$

Therefore

$$\log C'_{x+1} = \log C'_x + \log vp_x + \Delta\log z_x.$$

To use this formula, the process will be the same as was illustrated in Art. 31 when the construction of $\log q_x$ was described. Taking the type of operation in that article, we have merely to write $\Delta\log z_x$ for $\Delta\log d_x$, $\log vp_x$ for $\operatorname{colog} p_x$, and $\log C'_x$ for $\log q_x$.

The z_x here employed may represent the rate of sickness of unlimited duration, or that limited to the first six months of illness, or the second six months, or any other period. Whatever the period of sickness may be, the principles of the construction of the tables are the same.

103. As an illustration of a method of constructing certain classes of tables not hitherto discussed, it may be shown how to prepare a table of the values of μ_x from the formula

$$\mu_x = \frac{7(d_{x-1}+d_x)-(d_{x-2}+d_{x+1})}{12l_x}.$$

The age being entered in the first column of the working sheet, in the second column the successive values of $(d_{x-1}+d_x)$ are written. These are very easily formed, by inspection, from the mortality table, the numbers being always small. In the third column the product by 7 of the number in the second column is entered, thus forming the first part of the numerator of the fraction. In the fourth column are placed the values of $(d_{x-2}+d_{x+1})$. These can very conveniently be formed from the mortality table by cutting a narrow slip of card, of such a width that it will cover two lines of the table. To form the number for column 4 opposite age x, the card must be so placed on the mortality table as to cover d_{x-1} and d_x, and to exhibit d_{x-2} just above the card, and d_{x+1} just below it. These two numbers being small can be added together by inspection, and the sum placed in column 4. For column 5 the difference is taken between columns 3 and 4; and we thus have the numerator of the fraction expressing μ_x.

All these columns can be verified by summation. The sum of column 2, increased by d_0, is equal to $2l_0$. The sum of column 3 is of course seven times the sum of column 2. The sum of column 4, increased by $d_0+d_1+d_2$, is equal to $2l_0$; and the sum of column 5 is equal to the difference between the sums of columns 3 and 4.

Writing for the moment n_x for the numerator of the fraction representing μ_x, we have

$$\log \mu_x = \log n_x + \text{colog}\, l_x + \text{colog}\, 12$$

and

$$\begin{aligned}\log \mu_{x+1} &= \log n_{x+1} + \text{colog}\, l_{x+1} + \text{colog}\, 12 \\ &= \log \mu_x + \Delta \log n_x + \text{colog}\, p_x.\end{aligned}$$

Therefore, arranging the values of $\Delta \log n_x$ and $\text{colog}\, p_x$ side by side in columns, and then starting with the initial value, which is $\log \mu_2$ because the formula does not admit of earlier values of the function being calculated, by continued addition the column of $\log \mu_x$ is completed.

CHAPTER XXII.

FORMULAS OF FINITE DIFFERENCES.

1. It would be beyond the scope of the present work to enter deeply on the subjects of Interpolation and Summation, but it will be useful in this chapter and the next to deduce briefly some of the more elementary formulas of Finite Differences which are frequently employed in the science of Life Contingencies, and show their application in the process, important to the Actuary, of Interpolation. In Chap. xxiv formulas of summation, for the most part involving the simpler principles of the Differential and Integral Calculus, will also be discussed, and it will be assumed that the reader already knows the elements of that science. An acquaintance with the rudiments of the Differential and Integral Calculus is much more common than is the case with the Calculus of Finite Differences.

2. As its name implies, the Calculus of Finite Differences deals with the changes in value which take place in a function, consequent on finite (as distinguished from infinitesimal) changes in the value of the variable. In practice the successive values of the function form a series which has either to be completed or summed.

3. If we have a series of successive values of a function arranged in a column, and if we take each value from the value immediately following it, and place the result of the subtractions in a parallel column, the values so formed are called the *first differences* of the function, and are denoted by the symbol Δ. Thus, if u_x represent a function of x, then Δu_x represents the first difference of u_x; that is, if the increment of x, the variable, be h, then $\Delta u_x = u_{x+h} - u_x$. For example, if $u_x = x^2$, then $\Delta u_x = (x+h)^2 - x^2 = 2xh + h^2$; or if $u_x = b^x$, $\Delta u_x = b^{x+h} - b^x = b^x(b^h - 1)$.

4. The difference of u_x, Δu_x, is generally also a function of x, which may be differenced exactly in the same way as u_x itself was differenced; and to this difference of the difference of u_x the symbol $\Delta^2 u_x$ is assigned, where the index of the power over Δ does not mean that a quantity, Δ, is

to be multiplied by itself, but that an operation represented by Δ is to be repeated. $\Delta^2 u_x$ is called the *second difference* of u_x, or the difference of the *second order*. The second difference of a function of x, being generally also a function of x, may likewise be differenced, thus producing the third difference, $\Delta^3 u_x$, of the function; and so on *ad infinitum*, unless a stage is reached when the differences cease to be functions of the variable, and remain constant when the variable is changed, and when consequently all differences of higher orders vanish.

5. The series of values of the function, with the successive orders of differences, may be conveniently arranged in tabular form, thus,

Variable x	Function u	First Diff. Δ	Second Diff. Δ^2	Third Diff. Δ^3	Fourth Diff. Δ^4	&c.
x $x+h$ $x+2h$ $x+3h$ $x+4h$ $x+5h$ &c.	u_x u_{x+h} u_{x+2h} u_{x+3h} u_{x+4h} u_{x+5h} &c.	Δu_x Δu_{x+h} Δu_{x+2h} Δu_{x+3h} Δu_{x+4h} &c.	$\Delta^2 u_x$ $\Delta^2 u_{x+h}$ $\Delta^2 u_{x+2h}$ $\Delta^2 u_{x+3h}$ &c.	$\Delta^3 u_x$ $\Delta^3 u_{x+h}$ $\Delta^3 u_{x+2h}$ &c.	$\Delta^4 u_x$ $\Delta^4 u_{x+h}$ &c.	&c.

Here each difference is placed between the values of which it is the difference; but it is sometimes convenient somewhat to alter the arrangement, and to keep the value of the function and all its differences on one line, thus,

Variable	Function u	First Diff. Δ	Second Diff. Δ^2	Third Diff. Δ^3	&c.
x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	&c.
$x+h$	u_{x+h}	Δu_{x+h}	$\Delta^2 u_{x+h}$	$\Delta^3 u_{x+h}$	&c.
$x+2h$	u_{x+2h}	Δu_{x+2h}	$\Delta^2 u_{x+2h}$	$\Delta^3 u_{x+2h}$	&c.
&c.	&c.	&c.	&c.	&c.	&c.

6. The first term of a series, and the differences of that term, are called the *leading term* and *leading differences* of the series. Thus, in the foregoing table, u_x, Δu_x, $\Delta^2 u_x$, $\Delta^3 u_x$, &c., are the leading term and the leading differences of the series.

7. We have in the foregoing statements taken the increment of x, the independent variable, as h; but to obtain as much simplicity as possible in the mathematical formulas, it is usual to adopt that increment as the unit of measurement. Thus, if the function be the numbers living, l_y, by a

mortality table, and if the successive values supplied be l_y, l_{y+1}, l_{y+2}, &c., we shall have $u_x = l_y$, $u_{x+1} = l_{y+1}$, $u_{x+2} = l_{y+2}$, &c., where the unit of measurement is one year. But if the successive values of the function given be l_y, l_{y+5}, l_{y+10}, &c., we shall have $u_x = l_y$, $u_{x+1} = l_{y+5}$, $u_{x+2} = l_{y+10}$, &c., where the unit of measurement is five years. In this latter case, if we wished to represent, for instance, l_{y+1}, we should write $u_{x+\frac{1}{5}}$. The suffixes to the symbol for the function are often taken to represent the order of the terms of a series, and the initial term from which the series starts is represented by u_0. Thus, if the series given were l_{20}, l_{25}, l_{30}, l_{35}, &c., we should have $l_{20} = u_0$, $l_{25} = u_1$, $l_{30} = u_2$, $l_{35} = u_3$, &c. Throughout this chapter it will be assumed that the increment of the variable is unity (that is, the unit of measurement), unless the contrary be distinctly stated.

8. The values of a function corresponding to equidifferent values of the variable are called equidistant, whatever their numerical values may be. Thus, x^m, $(x+1)^m$, $(x+2)^m$, $(x+3)^m$, &c., are said to be equidistant, although the differences between the terms are not equal.

9. Numerical illustrations of the foregoing principles may be of assistance to the student. In the first, we have the cubes of the natural numbers; u_0 being 0^3, u_1 being 1^3; and so on, u_x being x^3. It will be noticed that the third differences are constant, so that the fourth differences and differences of all higher orders vanish.

In the second example the differences are alternately positive and negative, but although in numerical value they diminish rapidly, $\Delta^5 u_0$ being less than one-twentieth part of Δu_0, they do not vanish. If we had an unlimited number of terms of the series, carried out to a sufficient number of decimal places, we could form as many orders of differences as we pleased, and they would never become zero, although by going far enough we should reach quantities indefinitely small.

Example 1.

u	Δ	Δ^2	Δ^3	Δ^4
$u_0 = 0$	1	6	6	0
$u_1 = 1$	7	12	6	0
$u_2 = 8$	19	18	6	&c.
$u_3 = 27$	37	24	&c.	
$u_4 = 64$	61	&c.		
$u_5 = 125$	&c.			
&c.				

Example 2.

u	Δ	Δ^2	Δ^3	Δ^4	Δ^5
$u_0=74556$	-18618	$+5476$	-2342	$+1286$	-804
$u_1=55938$	-13142	$+3134$	-1056	$+482$	-253
$u_2=42796$	-10008	$+2078$	-574	$+229$	-113
$u_3=32788$	-7930	$+1504$	-345	$+116$	-47
$u_4=24858$	-6426	$+1159$	-229	$+69$	
$u_5=18432$	-5267	$+930$	-160		
$u_6=13165$	-4337	$+770$			
$u_7=8828$	-3567				
$u_8=5261$					

10. In the first of the above examples the third difference is constant. We shall now prove, generally, that if u_x be a rational integral function of x of the nth degree, then the nth difference is constant. The function being rational and integral and of the nth degree, it may be developed in the form $u_x=ax^n+bx^{n-1}+cx^{n-2}+\&c.+rx+s$, where a, b, c, &c. are independent of x, and where there are $n+1$ finite terms. Differencing, we have

$$\begin{aligned}\Delta u_x&=a(x+1)^n+b(x+1)^{n-1}+c(x+1)^{n-2}+\&c.+r(x+1)+s\\&\quad-ax^n-bx^{n-1}-cx^{n-2}-\&c.-rx-s\\&=anx^{n-1}+b_1x^{n-2}+c_1x^{n-3}+\&c.+r_1x+s_1\end{aligned}$$

where the new coefficients b_1, c_1, &c. are determinable, and independent of x. It is seen that Δu_x is a rational integral function of the degree $(n-1)$. Repeating the process, we have

$$\Delta^2u_x=an(n-1)x^{n-2}+b_2x^{n-3}+c_2x^{n-4}+\&c.+r_2x+s_2,$$

$$\Delta^3u_x=an(n-1)(n-2)x^{n-3}+b_3x^{n-4}+c_3x^{n-5}+\&c.+r_3x+s_3,$$

each difference being a rational integral function of a degree less by one than the difference immediately preceding it. Continuing the operation, we shall finally have

$$\Delta^nu_x=an(n-1)(n-2)\,....\,3.2.1 \quad . \quad . \quad . \quad . \quad (1)$$

a constant quantity.

$$\left.\begin{array}{ll}\text{Also} & \Delta^{n+1}u_x=0,\\ \text{and as a special case,} & \Delta^nx^n=1.2.3\,....\,n\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

11. The differences of the powers of the natural numbers occur very frequently in questions connected with series, and it is useful for purposes of calculation to have a table of the leading term and the leading differences, that is, of 0^n, $\Delta 0^n$, $\Delta^2 0^n$, &c., for the early values of n. The figures in following table, giving the values up to $\Delta^{10}0^{10}$, are taken from De Morgan's Differential Calculus, the arrangement of the table being that adopted by Boole in his work on Finite Differences. It will be of benefit to the student to recalculate at least part of the table for himself. By doing so he will understand better the principles on which it is based, and gain useful practical acquaintance with the operation of differencing.

For instance, taking the differences of 0^4, we calculate the values of 0^4, 1^4, 2^4, 3^4, and 4^4, and place them in a column, and successively form the columns of first, second, third, and fourth differences, thus

	Δ	Δ^2	Δ^3	Δ^4	
$0^4=$ 0	1	14	36	24	0
$1^4=$ 1	15	50	60	24	
$2^4=$ 16	65	110	84		
$3^4=$ 81	175	194			
$4^4=$256	369				
$5^4=$ 625					

It will be seen that the leading differences are the same as those given in the table against 0^4.

	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8	Δ^9	Δ^{10}
0	1	0	0	0	0	0	0	0	0	0
0^2	1	2	0	0	0	0	0	0	0	0
0^3	1	6	6	0	0	0	0	0	0	0
0^4	1	14	36	24	0	0	0	0	0	0
0^5	1	30	150	240	120	0	0	0	0	0
0^6	1	62	540	1560	1800	720	0	0	0	0
0^7	1	126	1806	8400	16800	15120	5040	0	0	0
0^8	1	254	5796	40824	126000	191520	141120	40320	0	0
0^9	1	510	18150	186480	834120	1905120	2328480	1451520	362880	0
0^{10}	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

12. In the process of differencing, a constant coefficient of the function remains as a constant coefficient of the difference. Thus, if c be independent of x,

$$\Delta c u_x = c u_{x+1} - c u_x$$

$$= c(u_{x+1} - u_x)$$

$$= c \Delta u_x.$$

Similarly, $$\Delta^n c u_x = c \Delta^n u_x.$$

13. In the process of differencing a constant term vanishes. Thus,

$$\begin{aligned}\Delta(u_x+c)&=(u_{x+1}+c)-(u_x+c)\\&=u_{x+1}-u_x\\&=\Delta u_x.\end{aligned}$$

14. It follows self-evidently from the definition of the symbols that

$$\Delta^m\Delta^n u_x=\Delta^{m+n}u_x.$$

15. The student can easily prove for himself that

$$u_{x+n}=u_x+\Delta u_x+\Delta u_{x+1}+\Delta u_{x+2}+\&c.+\Delta u_{x+n-1},$$

so that u_{x+n} consists of an initial term, u_x, and the sum of a series of n terms, consisting of the successive values of the function Δu_x, beginning with the value Δu_x, and ending with the value Δu_{x+n-1}.

16. From the last preceding paragraph it follows that, if we have as data the first term of a series and all the first differences of the series, we can by simple addition construct the whole series. Taking u_0 as the first term, we have

$$\begin{aligned}u_1&=u_0+\Delta u_0\\u_2&=u_{1_0}\overset{2}{+}\Delta u_{1_0}+\Delta^2 u_0\\u_3&=u_{2_0}\overset{3}{+}\Delta u_{2_0}+3\Delta^2 u_0+\Delta^3 u_0\\\&c.&=\&c.\end{aligned}$$

Thus the terms are found, each from the one preceding it, by adding in succession the terms in the column of differences. This is a process which is very often adopted in preparing tables for actuarial purposes. Frequently it is easier to calculate the column of differences, and so construct the series of the values of a function, than it would be to calculate directly the successive values of the function itself. (See, for instance, Chap. viii, Art. 18.)

17. If we have for data the first term of a series, and all the leading differences, that is, u_0, Δu_0, $\Delta^2 u_0$, $\Delta^3 u_0$, &c. . . $\Delta^n u_0$, we can construct the whole series by a process of simple addition. By hypothesis, $\Delta^n u$ is constant; and therefore, starting from $\Delta^{n-1}u_0$, and adding continuously to it the constant $\Delta^n u$, we shall form the whole column $\Delta^{n-1}u$. Making use then of this column in the way explained in the last preceding article, we can form the column $\Delta^{n-2}u$; and so on, until we have formed the column Δu, from which the column of u can be constructed. This process can be conveniently adopted to form a table of any function, of which the differences vanish after a limited number of orders. Thus, to form a table of the cubes of the natural numbers, instead of calculating each term separately, we may calculate the first four terms, and difference them, thus getting all the leading differences; and then proceed with the table as follows:—

x	$\Delta^3 u_x$	$\Delta^2 u_x$	Δu_x	u_x
0	6	6	1	0
1	6	12	7	1
2		18	19	8
3	Constant	24	37	27
4		30	61	64
5		*	91	125
6		*	*	216
*		*	*	*

Here we have calculated by actual multiplication the values of 0^3, 1^3, 2^3, and 3^3, and, by continued addition of the differences derived from these values, we have built up the table. Instead of calculating by multiplication the values of the first four terms, we might have taken the values of the leading differences from the table of the differences of nothing given in Art. 11.

18. To express $\Delta^n u_0$ in terms of the successive values of the function u.

Writing down the successive values of the function and differencing them out, we have

$$\begin{array}{cccccc}
u_0 & & & & & \\
 & \Delta u_0 & & & & \\
u_1 & & \Delta^2 u_0 & & & \\
 & \Delta u_1 & & \Delta^3 u_0 & & \\
u_2 & & \Delta^2 u_1 & & \Delta^4 u_0 & \\
 & \Delta u_2 & & \Delta^3 u_1 & & \&c. \\
u_3 & & \Delta^2 u_2 & & \Delta^4 u_1 & \\
 & \Delta u_3 & & \Delta^3 u_2 & & \&c. \\
u_4 & & \Delta^2 u_3 & & * & \\
 & \Delta u_4 & & * & & \&c. \\
u_5 & & * & & * & \\
 & * & & * & & \&c.
\end{array}$$

and

$$\begin{aligned}
\Delta u_0 &= u_1 - u_0 & \Delta^2 u_0 &= \Delta u_1 - \Delta u_0 = u_2 - 2u_1 + u_0 \\
\Delta u_1 &= u_2 - u_1 & \Delta^2 u_1 &= \Delta u_2 - \Delta u_1 = u_3 - 2u_2 + u_1 \\
\Delta u_2 &= u_3 - u_2 & \Delta^2 u_2 &= \Delta u_3 - \Delta u_2 = u_4 - 2u_3 + u_2
\end{aligned}$$

$$\Delta^3 u_0 = \Delta^2 u_1 - \Delta^2 u_0 = u_3 - 3u_2 + 3u_1 - u_0$$

$$\Delta^3 u_1 = \Delta^2 u_2 - \Delta^2 u_1 = u_4 - 3u_3 + 3u_2 - u_1$$

It is evident that, so far, the coefficients follow the law of the Binomial Theorem. Assuming, then, that that law holds for the nth difference, we shall prove that it must also hold for the $(n+1)$th. We have by hypothesis

$$\Delta^n u_0 = u_n - nu_{n-1} + \frac{n(n-1)}{\underline{|2}} u_{n-2} - \&\text{c.} \pm \frac{n(n-1)}{\underline{|2}} u_2 \mp nu_1 \pm u_0 \quad . \quad (3)$$

Similarly

$$\Delta^n u_1 = u_{n+1} - nu_n + \frac{n(n-1)}{\underline{|2}} u_{n-1} - \&\text{c.} \pm \frac{n(n-1)}{\underline{|2}} u_3 \mp nu_2 \pm u_1$$

Deducting the first line from the second, we have

$$\Delta^{n+1} u_0 = u_{n+1} - (n+1)u_n + \frac{(n+1)n}{\underline{|2}} u_{n-1} - \&\text{c.} \pm \frac{(n+1)n}{\underline{|2}} u_2 \mp (n+1)u_1 \pm u_0$$

which is of precisely similar form, replacing n by $(n+1)$. Therefore, if the law hold for Δ^n, it must hold for Δ^{n+1}. But it holds for Δ^3, therefore it must hold for Δ^4, and for all higher orders of differences.

19. If we use the symbol D prefixed to a function to signify the value of that function when the value of the variable is increased by unity, so that $Du_x = u_{x+1}$; we have $Du_{x+1} = u_{x+2} = DDu_x$. Just as we used the symbol Δ^2 to represent $\Delta\Delta$, that is, to signify a repetition of the operation expressed by Δ, so we may use D^2 to represent DD, that is, to signify a repetition of the operation expressed by D, which is the operation of increasing the value of the variable by unity, and finding the value of the function corresponding to the value of the variable so increased. In this notation we therefore have $u_{x+2} = D^2 u_x$. Similarly $u_{x+3} = Du_{x+2} = D^3 u_x$, and generally $u_{x+n} = D^n u_x$. Substituting these symbolical values in the equation for $\Delta^n u_0$, instead of u_n, u_{n-1}, &c., we have

$$\Delta^n u_0 = D^n u_0 - nD^{n-1} u_0 + \frac{n(n-1)}{\underline{|2}} D^{n-2} u_0 - \&\text{c.}$$

This equation may be put in compact form by treating the symbol of operation, D, in the same way as we treat symbols of quantity, and separating it from the functions to which it is attached. Proceeding in this fashion we have

$$\Delta^n u_0 = \left\{ D^n - nD^{n-1} + \frac{n(n-1)}{\underline{|2}} D^{n-2} - \&\text{c.} \right\} u_0$$

$$= (D-1)^n u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

seeing that the expression between the brackets, { }, follows the law of the Binomial Theorem. Taking the symbols Δ and D by themselves, we have the general equation

$$\Delta^n = (D-1)^n \quad \dots \dots \dots \quad (5)$$

whence

$$D = 1 + \Delta \quad \dots \dots \dots \quad (6)$$

and

$$D^n = (1+\Delta)^n \quad \dots \dots \dots \quad (7)$$

20. These last three equations express the relation which exists between the operations signified by the symbols Δ and D, and are of great importance. Throughout the most intricate analyses these symbols may be treated as if they were symbols of quantity; and on the final results being interpreted according to the definitions of the symbols, they will be found to be correct. As an example, we may express u_n in a series consisting of u_0 and its successive differences. By definition,

$$u_n = D^n u_0 = (1+\Delta)^n u_0.$$

$$= \left\{1 + n\Delta + \frac{n(n-1)}{\underline{|2}}\Delta^2 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3 + \text{\&c.}\right\} u_0$$

$$= u_0 + n\Delta u_0 + \frac{n(n-1)}{\underline{|2}}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3 u_0 + \text{\&c.} \quad \dots \quad (8)$$

21. We may prove this result by finding the same expression without resorting to the method of separation of symbols. Thus,

$$u_1 = u_0 + \Delta u_0$$

$$u_2 = u_1 + \Delta u_1$$

$$= (u_0 + \Delta u_0) + (\Delta u_0 + \Delta^2 u_0) = u_0 + 2\Delta u_0 + \Delta^2 u_0$$

$$u_3 = u_2 + \Delta u_2$$

$$= (u_0 + 2\Delta u_0 + \Delta^2 u_0) + \Delta(u_0 + 2\Delta u_0 + \Delta^2 u_0)$$

$$= u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0.$$

As far as we have gone, the coefficients follow the law of the Binomial Theorem; and, assuming this law for u_n we shall prove that it must also hold for u_{n+1}. Thus, by hypothesis,

$$u_n = u_0 + n\Delta u_0 + \frac{n(n-1)}{\underline{|2}}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3 u_0 + \text{\&c.}$$

But

$$\begin{aligned}u_{n+1}&=u_n+\Delta u_n\\&=u_0+n\Delta u_0+\frac{n(n-1)}{\underline{|2}}\Delta^2u_0+\frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3u_0+\ \&c.\\&\quad+\Delta\left\{u_0+n\Delta u_0+\frac{n(n-1)}{\underline{|2}}\Delta^2u_0+\frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3u_0+\ \&c.\right\}\\&=u_0+(n+1)\Delta u_0+\frac{(n+1)n}{\underline{|2}}\Delta^2u_0+\frac{(n+1)n(n-1)}{\underline{|3}}\Delta^3u_0+\ \&c.\end{aligned}$$

Here we have the same law as before, and therefore, if the equation hold for u_n, it must also hold for u_{n+1}. But we have proved that it holds for u_3, and therefore it must hold for u_4 and for all values of the function corresponding to higher values of the variable.

22. As illustrations of the operation of differencing we may find the successive differences of a few elementary functions.

If $\qquad u_x=b^x$, then $\Delta u_x=b^{x+1}-b^x=b^x(b-1)$.

Repeating the operation, we have

$$\Delta^2b^x=b^x(b-1)^2;$$

and generally

$$\Delta^nb^x=b^x(b-1)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Again

$$\begin{aligned}\Delta\log u_x&=\log u_{x+1}-\log u_x=\log\frac{u_{x+1}}{u_x}\\&=\log\left(1+\frac{\Delta u_x}{u_x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)\end{aligned}$$

23. When factors of a continued product increase or decrease by a constant difference, or when they are similar functions of a variable which in passing from one to the other increases or decreases by a constant difference, as in the expression $u_x\times u_{x+h}\times u_{x+2h}\times \&c.\times u_{x+(m-1)h}$ the factors are sometimes called *factorials*, and the term in which they are involved a *factorial term.*

24. Let $u_x=x(x-1)(x-2)\ldots(x-m+1)$; then

$$\begin{aligned}\Delta u_x&=(x+1)x(x-1)\ldots(x-m+2)-x(x-1)(x-2)\ldots(x-m+1)\\&=mx(x-1)(x-2)\ldots(x-m+2).\end{aligned}$$

25. It is sometimes convenient to express the factorial term

$$x(x-1)(x-2)\ldots(x-m+1),$$

which contains m factors, by the symbol $x^{(m)}$. In this notation we have

$$\Delta x^{(m)} = mx^{(m-1)}.$$

The reader acquainted with the differential calculus will see the analogy between this last result and the equation $\dfrac{dx^m}{dx} = mx^{m-1}$.

Repeating the operation of differencing on $\Delta x^{(m)}$, we have generally

$$\Delta^n x^{(m)} = m(m-1) \ldots (m-n+1) x^{(m-n)} \quad . \quad . \quad . \quad (11)$$

26. If $u_x = \dfrac{1}{x(x+1)(x+2) \ldots (x+m-1)}$, which by an extension of the notation we may write $x^{(-m)}$, we have

$$\begin{aligned}
\Delta u_x &= \frac{1}{(x+1)(x+2)(x+3) \ldots (x+m)} - \frac{1}{x(x+1)(x+2) \ldots (x+m-1)} \\
&= \frac{x-(x+m)}{x(x+1)(x+2) \ldots (x+m)} \\
&= -\frac{m}{x(x+1)(x+2) \ldots (x+m)} \\
&= -mx^{(-m-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)
\end{aligned}$$

which again we see to be analogous to the equation $\dfrac{dx^{-m}}{dx} = -mx^{-(m+1)}$.

27. Taking the most general form of factorial in which the increment of the variable is unity, we have

$$\left.\begin{aligned}
\Delta\, u_x u_{x-1} \ldots u_{x-m+1} &= (u_{x+1} - u_{x-m+1}) \times u_x u_{x-1} \ldots u_{x-m+2} \\
\Delta \frac{1}{u_x u_{x+1} \ldots u_{x+m-1}} &= \frac{u_x - u_{x+m}}{u_x u_{x+1} \ldots u_{x+m}}
\end{aligned}\right\} \quad . \quad (13)$$

and in particular if $u_x = ax + b$,

$$\left.\begin{aligned}
\Delta\, u_x u_{x-1} \ldots u_{x-m+1} &= amu_x u_{x-1} \ldots u_{x-m+2} \\
\Delta \frac{1}{u_x u_{x+1} \ldots u_{x+m-1}} &= \frac{-am}{u_x u_{x+1} \ldots u_{x+m}}
\end{aligned}\right\} \quad . \quad (14)$$

[28] To develop $\phi(x)$, a supposed rational integral function of x of the mth degree, in a series of factorials. Assume

$$\phi(x) = a + bx + cx^{(2)} + dx^{(3)} \ldots + kx^{(m)}$$

where, in the most general way $x^{(m)}$ is taken to represent the factorial term $x(x-h)(x-2h)\ldots(x-\overline{m-1}.h)$, the difference of x being h instead of 1 as previously. Differencing $\phi(x)$ for the increment h of x, we have

$$\Delta\phi(x)=hb+2hcx+3hdx^{(2)}+\&\text{c.}+mhkx^{(m-1)}$$

or, dividing the left-hand member of the equation by Δx and the right-hand member by the equal quantity h, we have

$$\frac{\Delta\phi(x)}{\Delta x}=b+2cx+3dx^{(2)}+\&\text{c.}+mkx^{(m-1)}$$

Repeating the operation successively, we obtain

$$\frac{\Delta^2\phi(x)}{(\Delta x)^2}=2c+3.2.dx+\ \ldots\ +m(m-1)kx^{(m-2)}$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

and finally $\dfrac{\Delta^m\phi(x)}{(\Delta x)^m}=m(m-1)\ldots 2.1.k.$

Now, making $x=0$ in the last series of equations, and representing by $\{\phi(x)\}_0$, $\left\{\dfrac{\Delta\phi(x)}{\Delta x}\right\}_0$, $\left\{\dfrac{\Delta^2\phi(x)}{(\Delta x)^2}\right\}_0$, &c., what the functions enclosed in the brackets $\{\ \}_0$ become when, after reduction, $x=0$; we have

$$\{\phi(x)\}_0=a,\ \left\{\frac{\Delta\phi(x)}{\Delta x}\right\}_0=b,\ \left\{\frac{\Delta^2\phi(x)}{(\Delta x)^2}\right\}_0=2c,$$

$$\&\text{c.} \qquad \&\text{c.} \qquad \&\text{c.}$$

$$\left\{\frac{\Delta^m\phi(x)}{(\Delta x)^m}\right\}_0=1.2.3\ldots mk;$$

whence

$$a=\{\phi(x)\}_0, \qquad d=\frac{1}{\underline{|3}}\left\{\frac{\Delta^3\phi(x)}{(\Delta x)^3}\right\}_0,$$

$$b=\left\{\frac{\Delta\phi(x)}{\Delta x}\right\}_0, \qquad \&\text{c.}= \qquad \&\text{c.}$$

$$c=\frac{1}{\underline{|2}}\left\{\frac{\Delta^2\phi(x)}{(\Delta x)^2}\right\}_0, \qquad k=\frac{1}{\underline{|m}}\left\{\frac{\Delta^m\phi(x)}{(\Delta x)^m}\right\}_0.$$

Therefore, substituting these values of the coefficients a, b, c, &c., in the expansion of $\phi(x)$, we have, writing for $x^{(m)}$ its value in terms of x and h,

$$\phi(x)=\{\phi(x)\}_0+\left\{\frac{\Delta\phi(x)}{\Delta x}\right\}_0 x+\left\{\frac{\Delta^2\phi(x)}{(\Delta x)^2}\right\}_0\frac{x(x-h)}{1.2}$$

$$+\left\{\frac{\Delta^3\phi(x)}{(\Delta x)^3}\right\}_0\frac{x(x-h)(x-2h)}{1.2.3}+\&\text{c.} \quad .\ .\ .\ \quad (15)$$

[29] If in this last expression we make Δx and h equal to unity, we have

$$\phi(x)=\phi(0)+x\Delta\phi(0)+\frac{x(x-1)}{\lfloor\underline{2}}\Delta^2\phi(0)+\frac{x(x-1)(x-2)}{\lfloor\underline{3}}\Delta^3\phi(0)+\&c., \quad . \; . \; . \; . \; . \quad (16)$$

when we represent by $\phi(0)$, $\Delta\phi(0)$, $\Delta^2\phi(0)$, &c., what these functions become when in them $x=0$. It will be noticed that formula 16 agrees with formula 8. If on the other hand we make h, the increment of x, infinitely small, and introduce accordingly the notation of the differential calculus, according to which the symbols ϕ', ϕ'', &c., represent the successive differential coefficients, we have

$$\phi(x)=\phi(0)+x\phi'(0)+\frac{x^2}{\lfloor\underline{2}}\phi''(0)+\frac{x^3}{\lfloor\underline{3}}\phi'''(0)+\&c. \quad . \; . \; . \quad (17)$$

which is Maclaurin's modification of Taylor's theorem of the differential calculus.

We have here a good illustration of the analogy which exists between the differential calculus and the calculus of finite differences.

[30] The following investigation will be of use later on in Chap. xxiv. Arts. 22 and 23.

To expand $\dfrac{x}{(1+x)^t-1}$ in a series of the ascending powers of x.

Let the function be represented by $\phi(x)$; and let $\phi'(x)$, $\phi''(x)$, &c., $\phi^{(\kappa)}(x)$, represent its successive differential coefficients.

Then because $\phi(x)=\dfrac{x}{(1+x)^t-1}$, therefore

$$\phi(x)\times(1+x)^t=x+\phi(x).$$

Differentiating successively both sides of the last equation, we have

$$\phi'(x)(1+x)^t+t\phi(x)(1+x)^{t-1}=1+\phi'(x)$$

$$\phi''(x)(1+x)^t+2t\phi'(x)(1+x)^{t-1}+t(t-1)\phi(x)(1+x)^{t-2}=\phi''(x)$$

$$\begin{aligned}\phi'''(x)(1+x)^t+3t\phi''(x)(1+x)^{t-1}+3t(t-1)\phi'(x)(1+x)^{t-2}&\\ +t(t-1)(t-2)\phi(x)(1+x)^{t-3}&=\phi'''(x)\,;\end{aligned}$$

and generally, the law of the terms being manifest,

$$\begin{aligned}\phi^{(\kappa)}(x)(1+x)^t+\kappa t\phi^{(\kappa-1)}(x)(1+x)^{t-1}+\&c.&\\ +t(t-1)\ldots(t-\kappa+1)\phi(x)(1+x)^{t-\kappa}&=\phi^{(\kappa)}(x).\end{aligned}$$

Writing now $x=0$, and putting $\phi(0)$, $\phi'(0)$, &c., for the consequent values of $\phi(x)$, $\phi'(x)$, &c., we have

$$\begin{aligned}
&\phi'(0)+t\phi(0)=1+\phi'(0)\\
&2t\phi'(0)+t(t-1)\phi(0)=0\\
&3t\phi''(0)+3t(t-1)\phi'(0)+t(t-1)(t-2)\phi(0)=0\\
&4t\phi'''(0)+6t(t-1)\phi''(0)+4t(t-1)(t-2)\phi'(0)\\
&\qquad\qquad\qquad\qquad+t(t-1)(t-2)(t-3)\phi(0)=0,
\end{aligned}$$

and so on. Whence we have, after reduction,

$$\begin{aligned}
\phi(0)&=\frac{1}{t}\\
\phi'(0)&=-\frac{t-1}{2t}\\
\phi''(0)&=\frac{(t-1)(t+1)}{6t}\\
\phi'''(0)&=-\frac{(t-1)(t+1)}{4t}\\
\phi^{\text{iv}}(0)&=\frac{(t-1)(t+1)(19-t^2)}{30t}\\
\phi^{\text{v}}(0)&=-\frac{(t-1)(t+1)(9-t^2)}{4t}\\
\phi^{\text{vi}}(0)&=\frac{(t-1)(t+1)(863-145t^2+2t^4)}{84t}\\
\text{\&c.}\quad&=\qquad\qquad\text{\&c.}
\end{aligned}$$

Applying now Maclaurin's theorem (formula 17), we finally have

$$\begin{aligned}
\frac{x}{(1+x)^t-1}&=\frac{1}{t}-\frac{t-1}{2t}x+\frac{t^2-1}{\underline{|2}\times 6t}x^2-\frac{t^2-1}{\underline{|3}\times 4t}x^3\\
&+\frac{(t^2-1)(19-t^2)}{\underline{|4}\times 30t}x^4-\frac{(t^2-1)(9-t^2)}{\underline{|5}\times 4t}x^5\\
&+\frac{(t^2-1)(863-145t^2+2t^4)}{\underline{|6}\times 84t}x^6-\text{\&c.}\ .\ .\ .\ .\qquad(18)
\end{aligned}$$

CHAPTER XXIII.

INTERPOLATION.

1. If there be a series of which one or more terms are missing, a process whereby these missing terms are ascertained and inserted is called *Interpolation.* Or, if there be a complete series of the values of a function for certain equidifferent values of the variable, when intermediate terms are inserted for intermediate values of the variable, these intermediate terms are said to be *Interpolated.*

The word *interpolate* is, however, frequently used in a wider sense. Primarily it meant to insert terms *between* given terms, but its meaning has been extended to include the completion of a series of which a certain number of terms are given, whether the added terms fall between or beyond those supplied in the data.

2. If the mathematical law of the function be known, then, in order to interpolate missing terms, these terms need only be calculated by the mathematical formula supplied by the law. But it is not often that such a process is possible. Generally the law of the series is unknown; or, if known, it may be too complicated to admit of easy calculation; or perhaps it is not such as to admit of expression in an algebraical formula. Under these circumstances it is usual virtually to assume a mathematical law which will approximately represent the function, and by means of which the required terms may be computed. The nature of the law assumed will depend upon the sequence of the given values of the function: but it is usually most convenient to assume that the function is a rational integral function of the variable x, and by means of the given values of the function, to determine either directly or indirectly the coefficients of the powers of x. If the differences of a function vanish after a finite number of orders, then, as we have proved in Chap. xxii, Arts. 10 and 20, the assumption is strictly correct; but, generally the differences do not vanish, and under such circumstances the assumption is only approximately

true. In the case, however, of almost all tabulated functions, the differences of the higher orders rapidly diminish, and become insignificant; and if the terms sought are not far distant from the given terms, the error in the assumption may be disregarded.

3. If there be any n values of a function given, namely, u_a, u_b, u_c, &c., u_n, to find other terms.

Taking first the most obvious method, assume

$$u_x = A + Bx + Cx^2 + \&c. + Kx^{n-1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From the given values of the function we have n simple equations by means of which to determine the n coefficients A, B, C, &c.; and when these are determined, we can, by means of the assumed law, find the value of the function for any desired value of the variable.

4. To take an example:

Having given $l_{10}=2844$, $l_{30}=2501$,

$l_{20}=2705$, $l_{40}=2236$,

to find the intermediate values of l. Here we may assume

$$l_{10}=u_0 \qquad l_{30}=u_{2x}$$

$$l_{20}=u_x \qquad l_{40}=u_{3x}$$

so that $x=10$, and $a=0$, $b=10$, $c=20$, $d=30$. Then assuming $u_x=A+Bx+Cx^2+Dx^3$, we have

$$\begin{aligned}
2844 &= u_0 = A\\
2705 &= u_x = A+10B+100C+1000D\\
2501 &= u_{2x} = A+20B+400C+8000D\\
2236 &= u_{3x} = A+30B+900C+27000D.
\end{aligned}$$

In this example the given values of the function are equidistant, and the values of the coefficients may most easily be found by differencing the equations, thus,

$$\begin{aligned}
-139 &= 10B+100C+1000D\\
-204 &= 10B+300C+7000D\\
-265 &= 10B+500C+19000D.
\end{aligned}$$

Repeating the operation,

$$\begin{aligned}
-65 &= 200C+6000D\\
-61 &= 200C+12000D.
\end{aligned}$$

Finally, $4 = 6000D.$

Hence,
$$\begin{aligned} A &= +2844 \\ B &= -\ \ 10{\cdot}516\dot{6} \\ C &= -\ \ \ 0{\cdot}3450 \\ D &= +\ \ \ 0{\cdot}000\dot{6}. \end{aligned}$$

and,
$$\begin{aligned} l_{11} &= A + B + C + D \\ l_{12} &= A + 2B + 4C + 8D \\ l_{13} &= A + 3B + 9C + 27D. \end{aligned}$$

5. Here the successive coefficients of B are the first powers of the natural numbers; those of C are the second powers, and those of D the third powers. Before commencing the calculations, it would therefore be convenient to form tables of the squares and cubes of the natural numbers up to 30, and tables of the multiples of B, C, and D. It was by means of the formula here illustrated, that Griffith Davies constructed his Equitable Experience Table, and the example is based upon his book.

6. In the process illustrated in Art. 4, it is not necessary that the given values of the function should be equidistant. When the values given and the value sought constitute a series of equidistant terms, whatever may be the position of the value sought in that series a simpler process may be followed. Let the complete series consist of $(n+1)$ terms of which n are given. Then, assuming the series to be u_0, u_1, u_2, &c. u_n, and that the nth difference of the series vanishes; we have by Chap. xxii, formula 3,

$$\Delta^n u_0 = u_n - n u_{n-1} + \frac{n(n-1)}{\underline{|2}} u_{n-2} - \&c. + (-1)^n u_0 = 0,$$

an equation from which any one of the quantities $u_0, u_1, \ldots u_n$ may be found.

7. As an example, let the function be

$$\begin{aligned} u_0 &= \log 350 = 2{\cdot}54407 \\ u_1 &= \log 351 = 2{\cdot}54531 \\ u_2 &= \log 352 = 2{\cdot}54654 \\ u_4 &= \log 354 = 2{\cdot}54900 \end{aligned}$$

to find log 353, that is u_3.

We have by the general equation, when $n=4$,

$$\Delta^4 u_0 = u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$$

whence $$u_3 = \frac{u_4 + 6u_2 - 4u_1 + u_0}{4} = 2{\cdot}54777$$

which is log 353 correct in the last place.

8. If the complete series consist of $(n+2)$ terms, of which two are missing, we may form two equations thus,

$$\Delta^n u_0 = u_n - nu_{n-1} + \frac{n(n-1)}{\underline{|2}} u_{n-2} - \&c. + (-1)^n u_0 = 0$$

$$\Delta^n u_1 = u_{n+1} - nu_n + \frac{n(n-1)}{\underline{|2}} u_{n-1} - \&c. + (-1)^n u_1 = 0$$

from which any two of the quantities $u_0, u_1, \ldots . u_{n+1}$ may be found. Generally if the complete series consist of $(n+r)$ terms, of which r are missing, we may form in a similar way r equations, and solve for the r missing terms.

9. Cases may occur where, by some simple algebraical artifice, the missing terms may be interpolated; but it is impossible to lay down a general rule. The following is an example, derived from the construction of the first ten values of l_x of the Institute H^{M} Mortality Table:

$$u_0 = 96779$$
$$u_1 = 97245$$
$$u_2 = 97624$$
$$u_9 = 100000,$$

to find the values u_3 to u_8 inclusive. Here, four values being given, we can form the leading differences up to three orders: thus

$$u_9 = u_0 + 9\Delta u_0 + \frac{9.8}{\underline{|2}} \Delta^2 u_0 + \frac{9.8.7}{\underline{|3}} \Delta^3 u_0$$
$$= u_0 + 9\Delta u_0 + 36\Delta^2 u_0 + 84\Delta^3 u_0;$$

whence $$\Delta^3 u_0 = \frac{u_9 - \{u_0 + 9\Delta u_0 + 36\Delta^2 u_0\}}{84}.$$

All the values on the right-hand side of the equation are given in the data, because we have

$$\begin{array}{lrr} & \Delta & \Delta^2 \\ u_0 = 96779 & 466 & -87 \\ u_1 = 97245 & 379 & \\ u_2 = 97624 & & \end{array}$$

whence $$\Delta^3 u_0 = \frac{100000-\{96779+4194-3132\}}{84} = 25{\cdot}7$$

Possessing now the initial term and the leading differences of the series, we can complete the series as explained in Chap. xxii, Art. 20.

10. Lagrange has given the following theorem for interpolation, which is perfectly general, and by means of which, when any n values of a function are given, any other values may be approximately found. Let u_a, u_b, u_c, &c., u_n, be the given values. Then, if, as in Art. 3, we assume the function to be rational and integral and of the degree $(n-1)$, there will be n constant coefficients, the values of which are to be found. Instead of assuming that the function has the form given in Art. 3 we may assume that

$$\left.\begin{aligned} u_x = {} & \mathrm{A}(x-b)(x-c) \ldots\ldots (x-n) \\ & +\mathrm{B}(x-a)(x-c) \ldots\ldots (x-n) \\ & +\mathrm{C}(x-a)(x-b) \ldots\ldots (x-n) \\ & +\&\text{c.} \end{aligned}\right\} \quad \ldots \quad (2)$$

to n terms; each of the n terms in the right-hand member having $(n-1)$ of the factors $(x-a)$, $(x-b)$, &c., and wanting one of these factors. If the right-hand member of the equation were developed in powers of x, it is evident that it would be of the degree $(n-1)$, and that each power of x would have a constant coefficient, thus fulfilling the hypothesis.

Making $x=a$, we have

$$u_a = \mathrm{A}(a-b)(a-c) \ldots\ldots (a-n),$$

therefore $$\mathrm{A} = \frac{u_a}{(a-b)(a-c) \ldots\ldots (a-n)}.$$

Similarly, by making $x=b$ we have

$$\mathrm{B} = \frac{u_b}{(b-a)(b-c) \ldots\ldots (b-n)},$$

and so on.

Hence finally

$$u_x = u_a \frac{(x-b)(x-c) \ldots (x-n)}{(a-b)(a-c) \ldots (a-n)} + u_b \frac{(x-a)(x-c) \ldots (x-n)}{(b-a)(b-c) \ldots (b-n)} + \ldots$$

$$\ldots + u_n \frac{(x-a)(x-b)(x-c) \ldots\ldots}{(n-a)(n-b)(n-c) \ldots\ldots} \quad \ldots \quad (3)$$

11. This formula of Lagrange is convenient where only one or two unknown values of a function are required, but it would involve lengthy calculations where a number of values are sought. As an example we may again find log 353 from the data in Art. 7. We have

$$\begin{array}{lll} u_a=2{\cdot}54407 & \text{and} & a=0 \text{ also } x=3 \\ u_b=2{\cdot}54531 & ,, & b=1 \\ u_c=2{\cdot}54654 & ,, & c=2 \\ u_d=2{\cdot}54900 & ,, & d=4 \end{array}$$

whence

$$u_x=2{\cdot}54407\,\frac{2.1.(-1)}{(-1)(-2)(-4)}+2{\cdot}54531\,\frac{3.1.(-1)}{1(-1)(-3)}$$

$$+2{\cdot}54654\,\frac{3.2.(-1)}{2.1.(-2)}+2{\cdot}54900\,\frac{3.2.1}{4.3.2}$$

$$=2{\cdot}54407(\tfrac{1}{4})+2{\cdot}54531(-1)+2{\cdot}54654(\tfrac{3}{2})+2{\cdot}54900(\tfrac{1}{4})$$

$$={\cdot}63602-2{\cdot}54531+3{\cdot}81981+{\cdot}63725$$

$$=2{\cdot}54777 \text{ as before.}$$

12. Although it has been remarked of Lagrange's Theorem that if a number of values be sought lengthy calculations are necessary, yet by proceeding in a somewhat different manner the theorem may sometimes be conveniently employed. For instance, we may take again the example of Art. 9, and by calculating directly u_3 by Lagrange's Theorem, we may, by subtraction, form the three leading differences, and by means of them interpolate all the other required values. Thus

$$\begin{array}{lll} u_a=\ 96779 & \text{and} & a=0 \text{ also } x=3 \\ u_b=\ 97245 & ,, & b=1 \\ u_c=\ 97624 & ,, & c=2 \\ u_d=100000 & ,, & d=9 \end{array}$$

$$u_3=96779\cdot\frac{2.1(-6)}{(-1)(-2)(-9)}+97245\cdot\frac{3.1(-6)}{1.(-1)(-8)}$$

$$+97624\,\frac{3.2.(-6)}{2.1.(-7)}+100000\,\frac{3.2.1}{9.8.7}$$

$$=96779(\tfrac{2}{3})-97245(\tfrac{9}{4})+97624(\tfrac{18}{7})+100000(\tfrac{1}{84})$$

$$=64519{\cdot}33-218801{\cdot}25+251033{\cdot}14+1190{\cdot}48$$

$$=97941{\cdot}7$$

To find the differences we now have

u	Δ	Δ^2	Δ^3
$u_0=96779$	466	-87	25·7
$u_1=97245$	379	$-61·3$	
$u_2=97624$	317·7		
$u_3=97941·7$			

We have thus, as before, the first term and the three leading differences of the series, by means of which we can fill in the remaining terms.

13. Where the given terms form a series of equidistant values of the function, we may conveniently employ formula 8 of Chap. xxii. We there found that

$$u_x=u_0+x\Delta u_0+\frac{x(x-1)}{\underline{|2}}\Delta^2 u_0+\&\text{c.}$$

When x is integral, the formula will reproduce the original terms of the series; and when x is fractional, it will give approximately an intermediate term corresponding to the fractional value of the variable. Thus, if $x=\frac{s}{t}$ we shall have

$$u_{\frac{s}{t}}=u_0+\frac{s}{t}\Delta u_0+\frac{\frac{s}{t}\left(\frac{s}{t}-1\right)}{\underline{|2}}\Delta^2 u_0+\frac{\frac{s}{t}\left(\frac{s}{t}-1\right)\left(\frac{s}{t}-2\right)}{\underline{|3}}\Delta^3 u_0+\&\text{c.}$$

$$=u_0+\frac{s}{t}\Delta u_0+\frac{s(s-t)}{t^2\underline{|2}}\Delta^2 u_0+\frac{s(s-t)(s-2t)}{t^3\underline{|3}}\Delta^3 u_0+\&\text{c.} \quad . \quad (4)$$

and when s is less than t, this may be conveniently written for purposes of calculation

$$u_{\frac{s}{t}}=u_0+\frac{s}{t}\Delta u_0-\frac{s(t-s)}{t^2\underline{|2}}\Delta^2 u_0+\frac{s(t-s)(2t-s)}{t^3\underline{|3}}\Delta^3 u_0-\&\text{c.} \quad . \quad (5)$$

14. To take an example, using the data of Art. 4, let it be required to find l_{12} of Davies's Equitable Experience Table. We have

	Δ	Δ^2	Δ^3
$u_0=2844$	-139	-65	$+4$
$u_1=2705$	-204	-61	
$u_2=2501$	-265		
$u_3=2236.$			

Also $x=\frac{2}{10}$ or $\frac{1}{5}$.

In formula 5, making $s=1$ and $t=5$, we have

$$u_{\frac{1}{5}}=u_0+\frac{1}{5}\Delta u_0-\frac{5-1}{5^2\,\underline{|2}}\Delta^2u_0+\frac{(5-1)(2\times5-1)}{5^3\,\underline{|3}}\Delta^3u_0$$

$$=u_0+\frac{1}{5}\Delta u_0-\frac{2}{25}\Delta^2u_0+\frac{6}{125}\Delta^3u_0$$

$$=2844-27{\cdot}8+5{\cdot}2+{\cdot}2$$

$$=2821{\cdot}6.$$

The value as given by Griffith Davies is 2822.

15. When only one or two interpolated values are required, or when the given values do not constitute a series of equidistant terms, one or other of the methods explained in the foregoing articles may be conveniently adopted. But in practice a problem very often met with is,—Having given a series of equidistant terms, to insert $m-1$ terms between each pair of those given; that is, to subdivide each interval into m parts, so as to produce a series of terms distant from each other by only $\frac{1}{m}$ of the interval between the given terms. When such is the problem, the best course to follow is,—from the differences of the series composed of the given terms only, to calculate, by the method immediately to be explained, the differences which shall correspond to the series when the intervals are subdivided; and then, by means of these differences, which may be called subdivided differences, to construct by addition (as explained in Chap. xxii, Art. 17), the series with subdivided intervals. It will be convenient to consider as unity the interval between the terms of the series of given values; so that the interval will be $\frac{1}{m}$ between the terms of the series when the intervals have been subdivided. Also it will be convenient to use Δ to signify the differences of the given values, and δ to signify the differences when the intervals have been subdivided. Thus, if u_0, u_1, u_2, &c., u_n, represent the series of given values, then u_0, $u_{\frac{1}{m}}$, $u_{\frac{2}{m}}$, $u_{\frac{3}{m}}$, &c., $u_{\frac{mn-1}{m}}$, u_n, will represent the series when the intervals have been subdivided. Also we shall have $\Delta u_0=u_1-u_0$ and $\delta u_0=u_{\frac{1}{m}}-u_0$. The problem now is, to find δu_0, δ^2u_0, δ^3u_0, &c., $\delta^n u_0$ in terms of Δu_0, Δ^2u_0, Δ^3u_0, &c., $\Delta^n u_0$.

16. By means of formula 5 we can express $u_0, u_{\frac{1}{m}}, u_{\frac{2}{m}}, \ldots u_{\frac{n}{m}}$ in terms of u_0, Δu_0, Δ^2u_0, &c., $\Delta^n u_0$; and then by differencing out the series of equations so found, we shall get the values of δu_0, δ^2u_0, δ^3u_0, &c., $\delta^n u_0$

required. For practical purposes, it will in most cases be more than sufficient if we stop at the fifth difference. It should be noted that, for five differences we must have, to begin with, the values of u from u_0 to u_5 inclusive, and, generally, for n differences the data must consist of $n+1$ equidistant values of u.

17. Applying formula 5, and multiplying out the numerators,

$$u_0=u_0+0\cdot\frac{\Delta u_0}{m}-\frac{0\times m-0^2}{\underline{|2}}\cdot\frac{\Delta^2 u_0}{m^2}+\frac{2\times 0\times m^2-3\times 0^2\times m+0^3}{\underline{|3}}\cdot\frac{\Delta^3 u_0}{m^3}$$
$$-\frac{6\times 0\times m^3-11\times 0^2\times m^2+6\times 0^3\times m-0^4}{\underline{|4}}\cdot\frac{\Delta^4 u_0}{m^4}$$
$$+\frac{24\times 0\times m^4-50\times 0^2\times m^3+35\times 0^3\times m^2-10\times 0^4\times m+0^5}{\underline{|5}}\cdot\frac{\Delta^5 u_0}{m^5}$$

$$u_{\frac{1}{m}}=u_0+1\cdot\frac{\Delta u_0}{m}-\frac{1\times m-1^2}{\underline{|2}}\cdot\frac{\Delta^2 u_0}{m^2}+\frac{2\times 1\times m^2-3\times 1^2\times m+1^3}{\underline{|3}}\cdot\frac{\Delta^3 u_0}{m^3}$$
$$-\frac{6\times 1\times m^3-11\times 1^2\times m^2+6\times 1^3\times m-1^4}{\underline{|4}}\cdot\frac{\Delta^4 u_0}{m^4}$$
$$+\frac{24\times 1\times m^4-50\times 1^2\times m^3+35\times 1^3\times m^2-10\times 1^4\times m+1^5}{\underline{|5}}\cdot\frac{\Delta^5 u_0}{m^5}$$

&c.=&c. for $u_{\frac{2}{m}}$, $u_{\frac{3}{m}}$, &c. Now, differencing, we have

$$\delta u_0=\frac{\Delta u_0}{m}-\frac{m\Delta 0-\Delta 0^2}{\underline{|2}}\cdot\frac{\Delta^2 u_0}{m^2}+\frac{2m^2\Delta 0-3m\Delta 0^2+\Delta 0^3}{\underline{|3}}\cdot\frac{\Delta^3 u_0}{m^3}$$
$$-\frac{6m^3\Delta 0-11m^2\Delta 0^2+6m\Delta 0^3-\Delta 0^4}{\underline{|4}}\cdot\frac{\Delta^4 u_0}{m^4}$$
$$+\frac{24m^4\Delta 0-50m^3\Delta 0^2+35m^2\Delta 0^3-10m\Delta 0^4+\Delta 0^5}{\underline{|5}}\cdot\frac{\Delta^5 u_0}{m^5}$$

$$\delta^2 u_0=\frac{\Delta^2 0^2}{\underline{|2}}\cdot\frac{\Delta^2 u_0}{m^2}-\frac{3m\Delta^2 0^2-\Delta^2 0^3}{\underline{|3}}\cdot\frac{\Delta^3 u_0}{m^3}+\frac{11m^2\Delta^2 0^2-6m\Delta^2 0^3+\Delta^2 0^4}{\underline{|4}}\cdot\frac{\Delta^4 u_0}{m^4}$$
$$-\frac{50m^3\Delta^2 0^2-35m^2\Delta^2 0^3+10m\Delta^2 0^4-\Delta^2 0^5}{\underline{|5}}\cdot\frac{\Delta^5 u_0}{m^5}$$

$$\delta^3 u_0=\frac{\Delta^3 0^3}{\underline{|3}}\cdot\frac{\Delta^3 u_0}{m^3}-\frac{6m\Delta^3 0^3-\Delta^3 0^4}{\underline{|4}}\cdot\frac{\Delta^4 u_0}{m^4}+\frac{35m^2\Delta^3 0^3-10m\Delta^3 0^4+\Delta^3 0^5}{\underline{|5}}\cdot\frac{\Delta^5 u_0}{m^5}$$

$$\delta^4 u_0=\frac{\Delta^4 0^4}{\underline{|4}}\cdot\frac{\Delta^4 u_0}{m^4}-\frac{10m\Delta^4 0^4-\Delta^4 0^5}{\underline{|5}}\cdot\frac{\Delta^5 u_0}{m^5}$$

$$\delta^5 u_0=\frac{\Delta^5 0^5}{\underline{|5}}\cdot\frac{\Delta^5 u_0}{m^5}$$

18. It may assist the student in understanding the foregoing operations to remind him that for all values of t, $\Delta 0^t = 1$; and $\Delta^{t+1}0^t$, and all higher orders of differences, vanish.

19. Inserting now the numerical values of the differences of 0 taken from the table in Chap. xxii, Art. 11, we have

$$\delta u_0 = \frac{\Delta u_0}{m} - \frac{m-1}{2} \cdot \frac{\Delta^2 u_0}{m^2} + \frac{2m^2-3m+1}{6} \cdot \frac{\Delta^3 u_0}{m^3} - \frac{6m^3-11m^2+6m-1}{24} \cdot \frac{\Delta^4 u_0}{m^4}$$
$$+ \frac{24m^4-50m^3+35m^2-10m+1}{120} \cdot \frac{\Delta^5 u_0}{m^5}$$

$$\delta^2 u_0 = \frac{\Delta^2 u_0}{m^2} - (m-1)\frac{\Delta^3 u_0}{m^3} + \frac{11m^2-18m+7}{12} \cdot \frac{\Delta^4 u_0}{m^4}$$
$$- \frac{10m^3-21m^2+14m-3}{12} \cdot \frac{\Delta^5 u_0}{m^5}$$

$$\delta^3 u_0 = \frac{\Delta^3 u_0}{m^3} - \frac{3m-3}{2} \cdot \frac{\Delta^4 u_0}{m^4} + \frac{7m^2-12m+5}{4} \cdot \frac{\Delta^5 u_0}{m^5}$$

$$\delta^4 u_0 = \frac{\Delta^4 u_0}{m^4} - 2(m-1)\frac{\Delta^5 u_0}{m^5}$$

$$\delta^5 u_0 = \frac{\Delta^5 u_0}{m^5}$$

20. From these equations the subdivided differences can easily be calculated for any value of m. In practice the most common values of m are 5 and 10; and to aid numerical calculations, it will be well if we here tabulate the subdivided differences for each of these values, as follows:—

21. To subdivide the interval into 5 parts.

$$\delta u_0 = {\cdot}2\Delta u_0 - {\cdot}08\Delta^2 u_0 + {\cdot}048\Delta^3 u_0 - {\cdot}0336\Delta^4 u_0 + {\cdot}025536\Delta^5 u_0$$
$$\delta^2 u_0 = {\cdot}04\Delta^2 u_0 - {\cdot}032\Delta^3 u_0 + {\cdot}0256\Delta^4 u_0 - {\cdot}02112\Delta^5 u_0$$
$$\delta^3 u_0 = {\cdot}008\Delta^3 u_0 - {\cdot}0096\Delta^4 u_0 + {\cdot}0096\Delta^5 u_0$$
$$\delta^4 u_0 = {\cdot}0016\Delta^4 u_0 - {\cdot}00256\Delta^5 u_0$$
$$\delta^5 u_0 = {\cdot}00032\Delta^5 u_0$$

22. To subdivide the interval into 10 parts.

$$\delta u_0 = {\cdot}1\Delta u_0 - {\cdot}045\Delta^2 u_0 + {\cdot}0285\Delta^3 u_0 - {\cdot}0206625\Delta^4 u_0 + {\cdot}01611675\Delta^5 u_0$$
$$\delta^2 u_0 = {\cdot}01\Delta^2 u_0 - {\cdot}009\Delta^3 u_0 + {\cdot}007725\Delta^4 u_0 - {\cdot}0066975\Delta^5 u_0$$
$$\delta^3 u_0 = {\cdot}001\Delta^3 u^0 - {\cdot}00135\Delta^4 u_0 + {\cdot}0014625\Delta^5 u_0$$
$$\delta^4 u_0 = {\cdot}0001\Delta^4 u_0 - {\cdot}00018\Delta^5 u_0$$
$$\delta^5 u_0 = {\cdot}00001\Delta^5 u_0$$

23. In the foregoing investigation we have found the values of δu_0, $\delta^2 u_0$, &c., by writing u_0, $u_{\frac{1}{m}}$, $u_{\frac{2}{m}}$, &c., in terms of Δu_0, $\Delta^2 u_0$, &c. Another process may be followed, which is equally convenient for any given value of m, but by which we cannot so readily obtain a general expression. Writing u_1, u_2, &c., in terms of u_0 and the subdivided differences, we have, when $m=5$,

$$\begin{aligned}
u_0 &= u_0 \\
u_1 &= u_0 + 5\delta u_0 + 10\delta^2 u_0 + 10\delta^3 u_0 + 5\delta^4 u_0 + \delta^5 u_0 \\
u_2 &= u_0 + 10\delta u_0 + 45\delta^2 u_0 + 120\delta^3 u_0 + 210\delta^4 u_0 + 252\delta^5 u_0 \\
u_3 &= u_0 + 15\delta u_0 + 105\delta^2 u_0 + 455\delta^3 u_0 + 1365\delta^4 u_0 + 3003\delta^5 u_0 \\
u_4 &= u_0 + 20\delta u_0 + 190\delta^2 u_0 + 1140\delta^3 u_0 + 4845\delta^4 u_0 + 15504\delta^5 u_0 \\
u_5 &= u_0 + 25\delta u_0 + 300\delta^2 u_0 + 2300\delta^3 u_0 + 12650\delta^4 u_0 + 53130\delta^5 u_0
\end{aligned}$$

Differencing now successively both sides of the equation, we have

$$\begin{aligned}
\Delta u_0 &= 5\delta u_0 + 10\delta^2 u_0 + 10\delta^3 u_0 + 5\delta^4 u_0 + \delta^5 u_0 \\
\Delta u_1 &= 5\delta u_0 + 35\delta^2 u_0 + 110\delta^3 u_0 + 205\delta^4 u_0 + 251\delta^5 u_0 \\
\Delta u_2 &= 5\delta u_0 + 60\delta^2 u_0 + 335\delta^3 u_0 + 1155\delta^4 u_0 + 2751\delta^5 u_0 \\
\Delta u_3 &= 5\delta u_0 + 85\delta^2 u_0 + 685\delta^3 u_0 + 3480\delta^4 u_0 + 12501\delta^5 u_0 \\
\Delta u_4 &= 5\delta u_0 + 110\delta^2 u_0 + 1160\delta^3 u_0 + 7805\delta^4 u_0 + 37626\delta^5 u_0 \\
\Delta^2 u_0 &= 25\delta^2 u_0 + 100\delta^3 u_0 + 200\delta^4 u_0 + 250\delta^5 u_0 \\
\Delta^2 u_1 &= 25\delta^2 u_0 + 225\delta^3 u_0 + 950\delta^4 u_0 + 2500\delta^5 u_0 \\
\Delta^2 u_2 &= 25\delta^2 u_0 + 350\delta^3 u_0 + 2325\delta^4 u_0 + 9750\delta^5 u_0 \\
\Delta^2 u_3 &= 25\delta^2 u_0 + 475\delta^3 u_0 + 4325\delta^4 u_0 + 25125\delta^5 u_0 \\
\Delta^3 u_0 &= 125\delta^3 u_0 + 750\delta^4 u_0 + 2250\delta^5 u_0 \\
\Delta^3 u_1 &= 125\delta^3 u_0 + 1375\delta^4 u_0 + 7250\delta^5 u_0 \\
\Delta^3 u_2 &= 125\delta^3 u_0 + 2000\delta^4 u_0 + 15375\delta^5 u_0 \\
\Delta^4 u_0 &= 625\delta^4 u_0 + 5000\delta^5 u_0 \\
\Delta^4 u_1 &= 625\delta^4 u_0 + 8125\delta^5 u_0 \\
\Delta^5 u_0 &= 3125\delta^5 u_0
\end{aligned}$$

Whence

$$\begin{aligned}
\delta^5 u_0 &= {\cdot}00032\Delta^5 u_0 \\
\delta^4 u_0 &= {\cdot}0016\Delta^4 u_0 - 8\delta^5 u_0 \\
\delta^3 u_0 &= {\cdot}008\Delta^3 u_0 - 6\delta^4 u_0 - 18\delta^5 u_0 \\
\delta^2 u_0 &= {\cdot}04\Delta^2 u_0 - 4\delta^3 u_0 - 8\delta^4 u_0 - 10\delta^5 u_0 \\
\delta u_0 &= {\cdot}2\Delta u_0 - 2\delta^2 u_0 - 2\delta^3 u_0 - \delta^4 u_0 - {\cdot}2\delta^5 u_0
\end{aligned}$$

Here, having first found $\delta^5 u_0$, we work backwards and find successively $\delta^4 u_0$, $\delta^3 u_0$, $\delta^2 u_0$, and δu_0.

Similarly, when $m=10$, we have

$$\begin{aligned}
u_0&=u_0\\
u_1&=u_0+10\delta u_0+45\delta^2 u_0+120\delta^3 u_0+210\delta^4 u_0+252\delta^5 u_0\\
u_2&=u_0+20\delta u_0+190\delta^2 u_0+1140\delta^3 u_0+4845\delta^4 u_0+15504\delta^5 u_0\\
u_3&=u_0+30\delta u_0+435\delta^2 u_0+4060\delta^3 u_0+27405\delta^4 u_0+142506\delta^5 u_0\\
u_4&=u_0+40\delta u_0+780\delta^2 u_0+9880\delta^3 u_0+91390\delta^4 u_0+658008\delta^5 u_0\\
u_5&=u_0+50\delta u_0+1225\delta^2 u_0+19600\delta^3 u_0+230300\delta^4 u_0+2118760\delta^5 u_0
\end{aligned}$$

Differencing now successively both sides of the equations, we have

$$\begin{aligned}
\Delta u_0&=10\delta u_0+45\delta^2 u_0+120\delta^3 u_0+210\delta^4 u_0+252\delta^5 u_0\\
\Delta u_1&=10\delta u_0+145\delta^2 u_0+1020\delta^3 u_0+4635\delta^4 u_0+15252\delta^5 u_0\\
\Delta u_2&=10\delta u_0+245\delta^2 u_0+2920\delta^3 u_0+22560\delta^4 u_0+127002\delta^5 u_0\\
\Delta u_3&=10\delta u_0+345\delta^2 u_0+5820\delta^3 u_0+63985\delta^4 u_0+515502\delta^5 u_0\\
\Delta u_4&=10\delta u_0+445\delta^2 u_0+9720\delta^3 u_0+138910\delta^4 u_0+1460752\delta^5 u_0
\end{aligned}$$

$$\begin{aligned}
\Delta^2 u_0&=100\delta^2 u_0+900\delta^3 u_0+4425\delta^4 u_0+15000\delta^5 u_0\\
\Delta^2 u_1&=100\delta^2 u_0+1900\delta^3 u_0+17925\delta^4 u_0+111750\delta^5 u_0\\
\Delta^2 u_2&=100\delta^2 u_0+2900\delta^3 u_0+41425\delta^4 u_0+388500\delta^5 u_0\\
\Delta^2 u_3&=100\delta^2 u_0+3900\delta^3 u_0+74925\delta^4 u_0+945250\delta^5 u_0
\end{aligned}$$

$$\begin{aligned}
\Delta^3 u_0&=1000\delta^3 u_0+13500\delta^4 u_0+96750\delta^5 u_0\\
\Delta^3 u_1&=1000\delta^3 u_0+23500\delta^4 u_0+276750\delta^5 u_0\\
\Delta^3 u_2&=1000\delta^3 u_0+33500\delta^4 u_0+556750\delta^5 u_0
\end{aligned}$$

$$\begin{aligned}
\Delta^4 u_0&=10000\delta^4 u_0+180000\delta^5 u_0\\
\Delta^4 u_1&=10000\delta^4 u_0+280000\delta^5 u_0
\end{aligned}$$

$$\Delta^5 u_0=100000\delta^5 u_0$$

Whence

$$\begin{aligned}
\delta^5 u_0&=\cdot 00001\Delta^5 u_0\\
\delta^4 u_0&=\cdot 0001\Delta^4 u_0-18\delta^5 u_0\\
\delta^3 u_0&=\cdot 001\Delta^3 u_0-13\cdot 5\delta^4 u_0-96\cdot 75\delta^5 u_0\\
\delta^2 u_0&=\cdot 01\Delta^2 u_0-9\delta^3 u_0-44\cdot 25\delta^4 u_0-150\delta^5 u_0\\
\delta u_0&=\cdot 1\Delta u_0-4\cdot 5\delta^2 u_0-12\delta^3 u_0-21\delta^4 u_0-25\cdot 2\delta^5 u_0
\end{aligned}$$

24. As a numerical example of the last formula, suppose that we have $\log p_{20}=\bar{1}\cdot9969724$, $\log p_{30}=\bar{1}\cdot9964260$, $\log p_{40}=\bar{1}\cdot9959051$, and $\log p_{50}=\bar{1}\cdot9943048$, and that it is required to interpolate the intervening values.

Here

	Δ	Δ^2	Δ^3
$u_0=\bar{1}\cdot9969724$	$-\cdot0005464$	$+\cdot0000255$	$-\cdot0011049$
$u_1=\bar{1}\cdot9964260$	$-\cdot0005209$	$-\cdot0010794$	
$u_2=\bar{1}\cdot9959051$	$-\cdot0016003$		
$u_3=\bar{1}\cdot9943048$			

Inserting these values in the formulas, and treating all differences of higher orders than the third as zero, we have

$$\delta^3=-\cdot00000110490$$

$$\delta^2=+\cdot00001019910$$

$$\delta=-\cdot00008727715$$

25. In making the succeeding calculations it will be convenient to omit the negative characteristic of u, and to carry the decimal point eleven places to the right. The operations will then be as follows:—

δ^3	δ^2	δ	u
-110490	$+1019910$	-8727715	$99697240000=\log p_{20}$
	909420	7707805	99688512285
	798930	6798385	99680804480
	688440	5999455	99674006095
	577950	5311015	99668006640
	467460	4733065	$99662695625=\log p_{25}$
	356970	4265605	99657962560
	246480	3908635	99653696955
	135990	3662155	99649788320
	&c.	3526165	99646126165
		&c.	$99642600000=\log p_{30}$
			&c.

The decimal point may now be inserted in its proper place, and the results cut down to seven figures.

The illustration is taken from the construction of Farr's Life Table (Males) of the Healthy Districts of England (*J.I.A.*, ix, 137).

CENTRAL DIFFERENCES.

26. Using formula 8 of Chap. xxii, and assuming x to lie between 0 and 1, we have

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{\underline{|2}}\Delta^2 u_0 + \frac{x(x-1)(x-2)}{\underline{|3}}\Delta^3 u_0 + \frac{x(x-1)(x-2)(x-3)}{\underline{|4}}\Delta^4 u_0.$$

The differences involved in this expression depend upon the five values of the function, u_0, u_1, u_2, u_3 and u_4; and therefore the interpolated value, u_x, is derived from these values alone, without having any regard to the values which precede u_0. Looking upon the values of the function as ordinates of a curve, the process of interpolation is the same as that of finding points of the curve intermediate to those given. If the form of the curve corresponded exactly to the equation assumed,—in other words, if, in the example above, the fourth difference were absolutely and not only approximately constant,—then it would not matter from which set of given values the interpolated values were derived, because from every set exactly the same result would be obtained. But the form of the curve being only approximately indicated by the values of the function, the interpolated values are only approximate, and they vary slightly according to the set of original values selected for the purpose of the calculations. Supposing that the portion of the curve in the neighbourhood of u_x were drawn by hand, it would manifestly be more consistently and more accurately accomplished if the points on both sides of it were taken into account, than if attention were exclusively directed to the points situated on one side alone. So, if the curve in the vicinity of u_x be completed by means of a mathematical formula instead of by a graphic process, it will also be well to make use of points on both sides instead of points on only one side. Keeping still to fourth differences, we shall therefore attain to greater accuracy if for u_0, u_1, u_2, u_3 and u_4, we substitute either u_{-2}, u_{-1}, u_0, u_1 and u_2, or u_{-1}, u_0, u_1, u_2 and u_3, as in either event, x being less than 1, we shall have u_x situated as nearly centrally as possible, having three points on one side of it and two on the other.

27. Using the values u_{-2}, u_{-1}, u_0, u_1, and u_2, the interpolated value becomes

$$u_x = u_{-2} + (2+x)\Delta u_{-2} + \frac{(2+x)(1+x)}{\underline{|2}}\Delta^2 u_{-2}$$
$$+\frac{(2+x)(1+x)x}{\underline{|3}}\Delta^3 u_{-2} + \frac{(2+x)(1+x)x(x-1)}{\underline{|4}}\Delta^4 u_{-2} \quad . \; . \quad (6)$$

28. It will be convenient here to introduce a new system of notation. Let the differences be designated as in the following scheme, in which

$$(a_0) = \tfrac{1}{2}(a_{-1} + a_{+1}), \text{ and } (c_0) = \tfrac{1}{2}(c_{-1} + c_{+1}),$$

these arithmetical means being supposed to be inserted after the differencing is completed :—

$$\begin{array}{ccccc}
u_{-2} & & & & \\
& a_{-2} & & & \\
u_{-1} & & b_{-1} & & \\
& a_{-1} & & c_{-1} & \\
u_0 & (a_0) & b_0 & (c_0) & d_0 \\
& a_{+1} & & c_{+1} & \\
u_{+1} & & b_{+1} & & \\
& a_{+2} & & & \\
u_{+2} & & & &
\end{array}$$

In this changed notation formula 6 becomes

$$u_x = u_{-2} + (2+x)a_{-2} + \frac{(2+x)(1+x)}{\underline{|2}} b_{-1}$$
$$+\frac{(2+x)(1+x)x}{\underline{|3}} c_{-1} + \frac{(2+x)(1+x)x(x-1)}{\underline{|4}} d_0 \quad . \quad (7)$$

29. To reduce u_{-2}, a_{-2}, b_{-1} and c_{-1} to the central quantities, u_0, a_0, b_0, and c_0, we have

$$c_{-1} = c_0 - \tfrac{1}{2}d_0$$

$$b_{-1} = b_0 - c_{-1} = b_0 - c_0 + \tfrac{1}{2}d_0$$

$$a_{-2} = a_{-1} - b_{-1} = a_0 - \tfrac{1}{2}b_0 - b_0 + c_0 - \tfrac{1}{2}d_0$$
$$= a_0 - \tfrac{3}{2}b_0 + c_0 - \tfrac{1}{2}d_0$$

$$u_{-2} = u_{-1} - a_{-2} = u_0 - a_{-1} - a_0 + \tfrac{3}{2}b_0 - c_0 + \tfrac{1}{2}d_0$$
$$= u_0 - a_0 + \tfrac{1}{2}b_0 - a_0 + \tfrac{3}{2}b_0 - c_0 + \tfrac{1}{2}d_0$$
$$= u_0 - 2a_0 + 2b_0 - c_0 + \tfrac{1}{2}d_0$$

Hence
$$\begin{aligned}u_x=u_0&-2a_0+2b_0-c_0+\tfrac{1}{2}d_0\\&+(2+x)(a_0-\tfrac{3}{2}b_0+c_0-\tfrac{1}{2}d_0)\\&+\frac{(2+x)(1+x)}{\underline{|2}}(b_0-c_0+\tfrac{1}{2}d_0)\\&+\frac{(2+x)(1+x)x}{\underline{|3}}(c_0-\tfrac{1}{2}d_0)\\&+\frac{(2+x)(1+x)x(x-1)}{\underline{|4}}d_0\end{aligned}$$

$$=u_0+x\left(a_0+\frac{x}{\underline{|2}}b_0\right)+\frac{x(x^2-1)}{\underline{|3}}(c_0+\frac{x}{4}d_0)\quad .\ .\ (8)$$

Or, rearranging the terms,

$$u_x=u_0+\left(a_0-\frac{c_0}{\underline{|3}}\right)x+\left(\frac{b_0}{\underline{|2}}-\frac{d_0}{\underline{|4}}\right)x^2+\frac{c_0}{\underline{|3}}x^3+\frac{d_0}{\underline{|4}}x^4\quad .\ .\ (9)$$

If in equation 8 we substitute

$$a_0=a_{+1}-\tfrac{1}{2}b_0 \text{ and } c_0=c_{+1}-\tfrac{1}{2}d_0,$$

we have

$$\begin{aligned}u_x=u_0+xa_{+1}&+\frac{x(x-1)}{\underline{|2}}b_0+\frac{(x+1)x(x-1)}{\underline{|3}}c_{+1}\\&+\frac{(x+1)x(x-1)(x-2)}{\underline{|4}}d_0\quad .\ .\ .\ .\ .\ (10)\end{aligned}$$

30. If in either equation 9 or 10 we give x, as already supposed, a fractional value between 0 and 1, we shall have a corresponding interpolated value of the function; and we may very easily and conveniently calculate such interpolated value by working backwards from term to term of the equation, eliminating one difference after another as follows:—

Let a, b, and c (without suffixes) represent the respective differences after the succeeding differences in the equation have been eliminated. Then taking equation 10,

$$\left.\begin{aligned}c&=c_{+1}-\frac{2-x}{4}d_0\\b&=b_0+\frac{1+x}{3}c\\a&=a_{+1}-\frac{1-x}{2}b\\u_x&=u_0+xa\end{aligned}\right\}\quad .\ .\ .\ .\ .\ (11)$$

31. When the formula is such that the fourth difference is actually constant, the foregoing formulas will give absolutely accurate results; but when the fifth and succeeding differences do not actually vanish, but only become so small as to be insignificant, the formulas will give results not exact but only approximate. To secure greater accuracy it will be well to bring into account u_3 by substituting for d_0 the mean fourth difference,

$$(d)=\tfrac{1}{2}(d_0+d_{+1}).$$

When the mean fourth difference is used, the data must consist of six values of u, two of which are on the negative side of u_0 and three on the positive; and this is in accordance with the concluding words of Art. 26, because, instead of using only one set of values of the function, we practically use both sets $u_{-2}, u_{-1}, u_0, u_{+1}$, and u_{+2}, and $u_{-1}, u_0, u_{+1}, u_{+2}$, and u_{+3}, taking three values on the one side of the interpolated value u_x, and three on the other.

32. To take a numerical example, the logarithms of the numbers 220, 230, 240, 250, 260, and 270 being given, let it be required to find log 245. Arranging the data in tabular form, and differencing, we have,

$$\begin{array}{llrrrr}
\text{No.} & \text{Log.} & & & & \\
220 & 3424227 & & & & \\
 & & +193051 & & & \\
230 & 3617278 & & -8217 & & \\
 & & 184834 & & +671 & \\
240 & u_0=3802112 & & b_0=-7546 & & d_0=-80 \\
 & & a_{+1}=177288 & & c_{+1}=591 & \\
250 & 3979400 & & 6955 & & d_{+1}=-64 \\
 & & 170333 & & 527 & \\
260 & 4149733 & & 6428 & & \overline{} \\
 & & 163905 & & & -144 \\
270 & 4313638 & & & & (d)=-\ \ 72
\end{array}$$

Here $x=\frac{1}{2}$. Therefore by formula 11

$$c=591+\tfrac{3}{8}\times 72=618$$

$$b=-7546+\tfrac{1}{2}\times 618=-7237$$

$$a=177288+\tfrac{1}{4}\times 7237=179097$$

$$\log 245=u_x=3802112+\tfrac{1}{2}\times 179097=3891661,$$

which is correct in the last place of decimals.

33. If, instead of only one intermediate value, we require the whole series, the method of central differences does not offer great facilities for the calculation. Woolhouse has given (*J.I.A.*, xi, 74) tables to help in the work, but in the absence of such tables, and when the interval is to be divided into not less than 10, then the calculations may with advantage be performed by a continued process by forming the leading differences of the interpolated series. In order to do this, the most convenient course will be to difference equation 9, giving to x the fractional values $\frac{0}{m}$, $\frac{1}{m}$, &c., and for symmetry writing $u_0 = u_{\frac{0}{m}}$. Thus,

$$\delta u_{\frac{0}{m}} = \frac{1}{m}\left(a_0 - \frac{c_0}{6}\right)\Delta 0 + \frac{1}{m^2}\left(\frac{b_0}{2} - \frac{d_0}{24}\right)\Delta 0^2 + \frac{1}{m^3}\cdot\frac{c_0}{6}\Delta 0^3 + \frac{1}{m^4}\cdot\frac{d_0}{24}\Delta 0^4$$

$$\delta^2 u_{\frac{0}{m}} = \frac{1}{m^2}\left(\frac{b_0}{2} - \frac{d_0}{24}\right)\Delta^2 0^2 + \frac{1}{m^3}\cdot\frac{c_0}{6}\Delta^2 0^3 + \frac{1}{m^4}\cdot\frac{d_0}{24}\Delta^2 0^4$$

$$\delta^3 u_{\frac{0}{m}} = \frac{1}{m^3}\frac{c_0}{6}\Delta^3 0^3 + \frac{1}{m^4}\cdot\frac{d_0}{24}\Delta^3 0^4$$

$$\delta^4 u_{\frac{0}{m}} = \frac{1}{m^4}\cdot\frac{d_0}{24}\Delta^4 0^4$$

34. Inserting the values of the differences of 0 given in the table on page 424, we have

$$\delta u_{\frac{0}{m}} = \frac{1}{m}\left(a_0 - \frac{c_0}{6}\right) + \frac{1}{m^2}\left(\frac{b_0}{2} - \frac{d_0}{24}\right) + \frac{1}{m^3}\cdot\frac{c_0}{6} + \frac{1}{m^4}\cdot\frac{d_0}{24}$$

$$\delta^2 u_{\frac{0}{m}} = \frac{2}{m^2}\left(\frac{b_0}{2} - \frac{d_0}{24}\right) + \frac{6}{m^3}\cdot\frac{c_0}{6} + \frac{14}{m^4}\cdot\frac{d_0}{24}$$

$$\delta^3 u_{\frac{0}{m}} = \frac{6}{m^3}\cdot\frac{c_0}{6} + \frac{36}{m^4}\cdot\frac{d_0}{24}$$

$$\delta^4 u_{\frac{0}{m}} = \frac{24}{m^4}\cdot\frac{d_0}{24};$$

or, reducing arithmetically the fractions to their lowest terms,

$$\left.\begin{aligned}
\delta u_{\frac{0}{m}} &= \frac{1}{m}\left(a_0 - \frac{c_0}{6}\right) + \frac{1}{m^2}\left(\frac{b_0}{2} - \frac{d_0}{24}\right) + \frac{1}{m^3}\cdot\frac{c_0}{6} + \frac{1}{m^4}\cdot\frac{d_0}{24} \\
\delta^2 u_{\frac{0}{m}} &= \frac{1}{m^2}\left(b_0 - \frac{d_0}{12}\right) + \frac{c_0}{m^3} + \frac{7}{m^4}\cdot\frac{d_0}{12} \\
\delta^3 u_{\frac{0}{m}} &= \frac{c_0}{m^3} + \frac{3}{m^4}\cdot\frac{d_0}{2} \\
\delta^4 u_{\frac{0}{m}} &= \frac{d_0}{m^4}
\end{aligned}\right\} \quad . \; . \; (12)$$

As mentioned in Art. 32, the mean value, (d), may advantageously be substituted for d_0.

35. Having now got u_0 and its leading differences, we can by simple addition supply all the interpolated values. If only the interval u_0 to u_1 is to be subdivided, the operation can be performed quite as easily by the central differences used in formula 12, as by the ordinary differences given in Art. 19. If, however, we are also to subdivide the intervals u_1 to u_2, u_2 to u_3, &c., by central differences, a fresh set of differences must be calculated by formula 12 for each interval to be subdivided, and the work becomes comparatively laborious.

36. In order to illustrate the last three articles, we may extend the numerical example of Art. 32, by finding from the data there given the logarithms of 241, 242, &c., to 249.

From the original data we have

$$a_0 = 181061\,; \qquad c_0 = 631\,; \qquad m = 10\,;$$

$$b_0 = -\ 7546\,; \qquad (d) = -\ 72\,;$$

$$\frac{1}{m}\left(a_0 - \frac{c_0}{6}\right) = 18095{\cdot}5833\,; \qquad \frac{1}{m^3}\cdot\frac{c_0}{6} = {\cdot}1052\,;$$

$$\frac{1}{m^2}\left(\frac{b_0}{2} - \frac{(d)}{24}\right) = -37{\cdot}7000\,; \qquad \frac{1}{m^4}\cdot\frac{(d)}{24} = -{\cdot}0003.$$

Whence, performing the calculations,

$$\delta u_{\frac{0}{m}} = 18057{\cdot}9882 \qquad \delta^3 u_{\frac{0}{m}} = {\cdot}6204$$

$$\delta^2 u_{\frac{0}{m}} = -\ 74{\cdot}7730 \qquad \delta^4 u_{\frac{0}{m}} = -{\cdot}0072$$

Arranging the calculations in tabular form, and completing the work by adding the successive differences, we have the following scheme :—

δ^4	δ^3	δ^2	δ	u
−·0072	+·6204	−74·7730	+18057·9882	3802112·0000
	·6132	74·1526	17983·2152	3820169·9882
	·6060	73·5394	17909·0626	3838153·2034
	·5988	72·9334	17835·5232	3856062·2660
	·5916	72·3346	17762·5898	3873897·7892
	·5844	71·7430	17690·2552	3891660·3790
	·5772	71·1586	17618·5122	3909350·6342
	·5700	70·5814	17547·3536	3926969·1464
		70·0114	17476·7722	3944516·5000
			17406·7608	3961993·2722
				3979400·0330

The results are never wrong by so much as a unit in the seventh place, but in order to obtain this degree of accuracy, it is necessary to work to a good many places of decimals.

37. In the case of ordinary logarithms one law holds throughout the table, and therefore the same advantage is not to be gained by the use of central differences as if the table were less regular. If the law of the table be not uniform throughout, then, interpolation by means of central differences will give much better results than the more ordinary method.

DIFFERENTIAL COEFFICIENTS.

[38] For purposes of calculation in connection with the formulas which will be found in next chapter, it is often useful to be able to express the differential coefficients of a function in terms of its differences, whether the differences be taken for the interval of unity or for the interval of m. For differences taken for intervals of unity we shall use the symbol δ, and for differences taken for the interval of m, the symbol Δ.

[39] By definition, $\frac{du_x}{dx}=\frac{u_{x+h}-u_x}{h}$ when h approaches the limit 0, and by Chap. xxii, formula 8

$$\frac{u_{x+h}-u_x}{h}=\frac{\left\{u_x+h\delta u_x+\frac{h(h-1)}{\underline{|2}}\delta^2 u_x+\&c.\right\}-u_x}{h}$$

whence, when $h=0$

$$\frac{du_x}{dx}=\delta u_x-\tfrac{1}{2}\delta^2 u_x+\tfrac{1}{3}\delta^3 u_x-\tfrac{1}{4}\delta^4 u_x+\tfrac{1}{5}\delta^5 u_x-\&c. \quad . \quad . \quad (13)$$

[40] Formula 13 is that usually employed to obtain the first differential coefficient in terms of the differences; but, by substitution, another more convergent series may be obtained. Thus, by Chap. xxii, formula 8, making $n=-\frac{1}{2}$ and reducing the fractions

$$u_{x-\frac{1}{2}}=u_x-\tfrac{1}{2}\delta u_x+\tfrac{3}{8}\delta^2 u_x-\tfrac{5}{16}\delta^3 u_x+\tfrac{35}{128}\delta^4 u_x-\&c.$$

Hence $$\delta u_{x-\frac{1}{2}}=\delta u_x-\tfrac{1}{2}\delta^2 u_x+\tfrac{3}{8}\delta^3 u_x-\tfrac{5}{16}\delta^4 u_x+\tfrac{35}{128}\delta^5 u_x-\&c.$$

Also, in a similar way,

$$\delta^3 u_{x-\frac{3}{2}} = \delta^3 u_x - \tfrac{3}{2}\delta^4 u_x + \tfrac{15}{8}\delta^5 u_x - \&c.$$

Now, if we deduct one twenty-fourth part of the last expression from the expression immediately preceding it, we have

$$\delta u_{x-\frac{1}{2}} - \tfrac{1}{24}\delta^3 u_{x-\frac{3}{2}} = \delta u_x - \tfrac{1}{2}\delta^2 u_x + \tfrac{1}{3}\delta^3 u_x - \tfrac{1}{4}\delta^4 u_x + \tfrac{25}{128}\delta^5 u_x - \&c.$$

which agrees as far as differences of the fourth order with equation 13, and which differs as regards the fifth order only in that the coefficient of $\delta^5 u_0$ is $\frac{25}{128}$ instead of $\frac{25}{125}$. We may, therefore, write approximately,

$$\frac{du_x}{dx} = \delta u_{x-\frac{1}{2}} - \tfrac{1}{24}\delta^3 u_{x--\frac{3}{2}} \quad \dots\dots \quad (14)$$

[41] To find the second differential coefficient, we have by equation 13,

$$\begin{aligned}\frac{d^2 u_x}{dx^2} &= \delta\frac{du_x}{dx} - \tfrac{1}{2}\delta^2\frac{du_x}{dx} + \tfrac{1}{3}\delta^3\frac{du_x}{dx} - \&c.\\ &= \delta^2 u_x - \tfrac{1}{2}\delta^3 u_x + \tfrac{1}{3}\delta^4 u_x - \&c.\\ &\qquad - \tfrac{1}{2}\delta^3 u_x + \tfrac{1}{4}\delta^4 u_x - \&c.\\ &\qquad\qquad + \tfrac{1}{3}\delta^4 u_x - \&c.\end{aligned}$$

Whence, adding the three lines together,

$$\frac{d^2 u_x}{dx^2} = \delta^2 u_x - \delta^3 u_x + \tfrac{11}{12}\delta^4 u_x - \&c. \quad \dots\dots \quad (15)$$

Similarly, from equation 15,

$$\begin{aligned}\frac{d^3 u_x}{dx^3} &= \delta^3 u_x - \delta^4 u_x + \&c. - \tfrac{1}{2}\delta^4 u_x + \&c.\\ &= \delta^3 u_x - \tfrac{3}{2}\delta^4 u_x + \&c. \quad \dots\dots \quad (16)\end{aligned}$$

[42] We have stopped at fourth differences, because in general the higher orders are insignificant. If we assume that they absolutely vanish, that is, that the fourth difference is constant, then from equation 16

$$\begin{aligned}\delta^3 u_x - \tfrac{3}{2}\delta^4 u_x &= \delta^3 u_{x-1} - \tfrac{1}{2}\delta^4 u_{x-2}\\ &= \delta^3 u_{x-\frac{3}{2}} = \tfrac{1}{2}(\delta^3 u_{x-2} + \delta^3 u_{x-1}).\end{aligned}$$

That is,

$$\frac{d^3 u_x}{dx^3} = \tfrac{1}{2}(\delta^3 u_{x-2} + \delta^3 u_{x-1}) \quad \dots\dots \quad (17)$$

or,

$$\frac{d^3 u_x}{dx^3} = \delta^3 u_{x-\frac{3}{2}} \quad \dots\dots \quad (18)$$

[43] In Articles 38 to 42 the differences have been taken for intervals of unity. If they be taken for longer intervals, say of m, as is the case in the majority of the formulas of summation in Chap. xxiv, we can obtain the differential coefficients by expressing δ in terms of Δ by means of the formulas given in Art. 19 of this chapter.

[44] Thus, by formula 13,

$$\frac{du_x}{dx}=\delta u_x-\tfrac{1}{2}\delta^2u_x+\tfrac{1}{3}\delta^3u_x-\tfrac{1}{4}\delta^4u_x+\&c.$$

$$\begin{array}{rllll}
= & \dfrac{\Delta u_x}{m} & -\dfrac{m-1}{2}\cdot\dfrac{\Delta^2u_x}{m^2} & +\dfrac{2m^2-3m+1}{6}\cdot\dfrac{\Delta^3u_x}{m^3} & -\dfrac{6m^3-11m^2+6m-1}{24}\cdot\dfrac{\Delta^4u_x}{m^4} \\
 & & -\tfrac{1}{2}\dfrac{\Delta^2u_x}{m^2} & +\dfrac{m-1}{2}\cdot\dfrac{\Delta^3u_x}{m^3} & -\dfrac{11m^2-18m+7}{24}\cdot\dfrac{\Delta^4u_x}{m^4} \\
 & & & +\tfrac{1}{3}\dfrac{\Delta^3u_x}{m^3} & -\dfrac{3m-3}{6}\cdot\dfrac{\Delta^4u_x}{m^4} \\
 & & & & -\tfrac{1}{4}\dfrac{\Delta^4u_x}{m^4}
\end{array}$$

$$=\frac{1}{m}(\Delta u_x-\tfrac{1}{2}\Delta^2u_x+\tfrac{1}{3}\Delta^3u_x-\&c.)\quad\ldots\ldots\ldots\ldots\quad(19)$$

Corresponding to equation 14, we may also write approximately,

$$\frac{du_x}{dx}=\frac{1}{m}(\Delta u_{x-\frac{1}{2}}-\tfrac{1}{24}\Delta^3u_{x-\frac{3}{2}})\quad\ldots\ldots\quad(20)$$

[45] Similarly, from equation 15,

$$\frac{d^2u_x}{dx^2}=\delta^2u_x-\delta^3u_x+\tfrac{11}{12}\delta^4u_x-\&c.$$

$$\begin{array}{rlll}
= & \dfrac{\Delta^2u_x}{m^2} & -(m-1)\dfrac{\Delta^3u_x}{m^3} & +\dfrac{11m^2-18m+7}{12}\cdot\dfrac{\Delta^4u_x}{m^4}-\&c. \\
 & & -\dfrac{\Delta^3u_x}{m^3} & +\dfrac{3m-3}{2}\cdot\dfrac{\Delta^4u_x}{m^4}-\&c. \\
 & & & +\tfrac{11}{12}\dfrac{\Delta^4u_x}{m^4}
\end{array}$$

That is,

$$\frac{d^2u_x}{dx^2}=\frac{1}{m^2}(\Delta^2u_x-\Delta^3u_x+\tfrac{11}{12}\Delta^4u_x-\&c.)\quad\ldots\quad(21)$$

[46] Also, in precisely the same manner from equations 16, 17 and 18,

$$\frac{d^3u_x}{dx^3}=\frac{1}{m^3}(\Delta^3u_x-\tfrac{3}{2}\Delta^4u_x)\quad .\ .\ .\ .\ .\ .\quad (22)$$

$$=\frac{1}{2m^3}(\Delta^3u_{x-2}+\Delta^3u_{x-1})\quad .\ .\ .\ .\ .\quad (23)$$

$$=\frac{1}{m^3}\Delta^3u_{x-\frac{3}{2}}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (24)$$

[47] By means of central differences, a very useful formula for the first differential coefficient may be deduced. By formula 8, stopping at second differences,

$$u_t=u_0+t\left(a_0+\frac{t}{2}b_0\right);$$

whence

$$\frac{u_t-u_0}{t}=a_0+\frac{t}{2}b_0.$$

Diminishing now *t* without limit, this becomes

$$\frac{du_0}{dt}=a_0=\tfrac{1}{2}(a_{-1}+a_{+1})=\tfrac{1}{2}(u_{+1}-u_{-1})$$

Or, more generally,

$$\frac{du_x}{dx}=\tfrac{1}{2}(u_{x+1}-u_{x-1})\quad .\ .\ .\ .\ .\ .\ .\quad (25)$$

[48] The same result may be arrived at from formula 13, by treating the second difference as constant. Thus,

$$\begin{aligned}\frac{du_x}{dx}&=\delta u_x-\tfrac{1}{2}\delta^2u_x\\&=u_{x+1}-u_x-\tfrac{1}{2}(\delta u_{x+1}-\delta u_x)\\&=u_{x+1}-u_x-\tfrac{1}{2}(\delta u_x-\delta u_{x-1})\\&=u_{x+1}-u_x-\tfrac{1}{2}(u_{x+1}-2u_x+u_{x-1})\\&=\tfrac{1}{2}(u_{x+1}-u_{x-1}).\end{aligned}$$

[49] By formula 19, we may also write approximately,

$$\frac{du_x}{dx}=\frac{1}{2m}(u_{x+m}-u_{x-m})\quad .\ .\ .\ .\ .\ .\ .\quad (26)$$

[50] Applying formula 25 to continuous annuities, and reversing the order of the terms because the annuity decreases in value with the increase in age of the life, we have

$$-\frac{d\bar{a}_x}{dx}=\tfrac{1}{2}(\bar{a}_{x-1}-\bar{a}_{x+1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$-\frac{d\bar{a}_{xy}}{dx}=\tfrac{1}{2}(\bar{a}_{x-1:y}-\bar{a}_{x+1:y}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$-\frac{d\bar{a}_{xy}}{dy}=\tfrac{1}{2}(\bar{a}_{x:y-1}-\bar{a}_{x:y+1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Also

$$-\frac{d\bar{a}_{xy}}{dt}=\tfrac{1}{2}(\bar{a}_{x-1:y-1}-\bar{a}_{x+1:y+1}) \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Or

$$-\frac{d\bar{a}_{xy}}{dt}=-\left(\frac{d\bar{a}_{xy}}{dx}+\frac{d\bar{a}_{xy}}{dy}\right)$$

$$=\tfrac{1}{2}(\bar{a}_{x-1:y}+\bar{a}_{x:y-1}-\bar{a}_{x+1:y}-\bar{a}_{x:y+1}) \quad . \quad (31)$$

[51] By formula 26, we may also write approximately,

$$-\frac{d\bar{a}_x}{dx}=\frac{1}{2m}(\bar{a}_{x-m}-\bar{a}_{x+m}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In case of a mortality table not well graduated, formula 32 will frequently give better results than formula 27.

Formulas similar to No. 32 hold for Nos. 28 to 31.

CHAPTER XXIV.

SUMMATION.

1. By *Summation* is meant, the finding of the sum of all the terms of a given series by means of a mathematical formula. The problem may be regarded as presenting two cases, according as the series to be summed is complete or incomplete. In the first case, we have only to find the total value of a set of given terms; or, of a set of terms the law of which is known, and which may therefore be considered to be given. In the second case the data consist only of a set of representative terms, corresponding to certain equidifferent values of the variable; and the problem consists in finding the sum of either a finite or an infinite number of similar terms, corresponding to a completed series of the values of the variable. In this last problem a process of interpolation is presupposed, to be followed by that of summation; but the interpolated values are not required for their own sake, and they need not be actually computed. By means of the differences or the differential coefficients of the terms of the original data, an approximate sum of the series may be obtained, differing generally by only an insignificant percentage from what the sum would be if the series were complete in the first instance. This process of approximate summation is of great value to the actuary. In many questions involving complicated benefits, the arithmetical labour of computing the exact value would be prohibitive. Such, for instance, are the compound survivorship annuities and assurances of Chap. xv. But by means of the formulas of this chapter, the values of these and similar benefits may be found with very little labour, and with accuracy abundantly sufficient for all practical purposes. Before, however, proceeding to this branch of the subject, the question of the summation of complete series may, with advantage, be discussed.

2. In differencing the function u_x we obtain Δu_x, which generally is also a function of x. Inversely, we may perform such an operation on the function Δu_x, as will give the function u_x. A suitable symbol for this inverse operation would be Δ^{-1}, and it would be analogous to such symbols as $\log^{-1}$, $\tan^{-1}$, &c., appearing in other branches of mathematics. A more distinctive symbol, Σ, is however generally employed, which suggests the principle underlying the operation inverse to differencing. In arithmetic, addition is inverse to subtraction; and so in the calculus now under discussion, summation is inverse to differencing. This fact readily appears from the following considerations.

3. We have seen, Chap. xxii, Art. 18, that

$$u_{x+n}=u_x+\Delta u_x+\Delta u_{x+1}+\&c.+\Delta u_{x+n-1}.$$

Writing $x=0$, and $n=x$, this becomes $u_x=u_0+\Delta u_0+\Delta u_1+\&c.+\Delta u_{x-1}$. Here it appears that u_x is the sum of a series, consisting of an initial term u_0 (which is a constant and may be represented by c) and a number of the terms of the function Δu_x, ending with the term which corresponds to the value $x-1$ of the variable. We may therefore write

$$u_x=\Sigma(\Delta u_x)+c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where Σ means the sum of a series of values of the function under the symbol, the last of such values being that corresponding to the value $x-1$ of the variable. Viewing the operation as the inverse of differencing, equation 1 may also be written

$$u_x=\Delta^{-1}(\Delta u_x)+c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now if U_x be such a function that u_x is its difference; that is, if $U_{x+1}-U_x=u_x$, then by equation 1,

$$U_x=\Sigma u_x+c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where c, as already explained, is a constant depending on the term at which the summation commences, and not on the number of terms summed.

4. From this it follows that we can sum a series, u_0, u_1, u_2, &c. to u_{x-1}, if we can find the form of the function of which u_x is the difference; because, U_x being that function, $u_0+u_1+u_2+\&c.+u_{x-1}=U_x-U_0$, or more generally

$$u_a+u_{a+1}+\&c.+u_{a+x-1}=U_{a+x}-U_a \quad . \quad . \quad . \quad . \quad . \quad (4)$$

5. *Integration* is the name given to the operation of finding the function of which u_x is the difference, and that function is called the *Finite Integral* of u_x, or simply the *Integral* when there is no danger of confusion with integrals where the increment of x is infinitesimal.

6. We saw, Chap. xxii, Art. 13, that in the process of differencing a constant term vanishes: so now, integration being the inverse of differencing, we see that in the process of integration a constant is introduced. Also, since, Chap. xxii, Art. 12, a constant coefficient of a function remains as a constant coefficient of the difference, it follows that a constant coefficient of a function remains as a constant coefficient of the integral.

7. From equation 4 we learn that, in order to find the sum of a series, we must integrate the term next beyond those to be summed, and from the result deduct the integral of the first term. In the process of subtraction the constant introduced by integration disappears.

8. By observing the form of U_x which produces a difference, u_x, of a given form, many problems in integration may be solved. For instance, to find Σb^x. By Chap. xxii, Art. 22, $\Delta b^x = b^x(b-1)$, or $b^x = \frac{\Delta b^x}{b-1}$. Whence, writing $u_x = \Delta b^x$, we have $b^x = \frac{u_x}{b-1}$, and $\Sigma b^x = \frac{\Sigma u_x}{b-1} = \frac{b^x}{b-1} + c$. In order from this integral to find the sum of the series $b^r + b^{r+1} + \&c. + b^{r+x-1}$, we must in accordance with Art. 7 take the difference between two integrals. It is convenient to represent the sum of a definite number of terms of a series, from say $x=m$ to $x=n$ inclusive (that is, a *Definite Integral*), by the symbol $\Sigma_m^n u_x$. In the case in question the sum will be $\Sigma_r^{r+x-1} b^x$; and by equation 4, $\Sigma_r^{r+x-1} b^x = \frac{b^{r+x} - b^r}{b-1} = b^r \frac{b^x - 1}{b-1}$, which, from the usual formula for the sum of a geometrical series, we know to be correct.

9. Again, adhering to the notation of Chap. xxii, Art. 25, to find the integral of $x^{(m)}$. We have by that article $\Delta x^{(m)} = m x^{(m-1)}$, or, writing $m+1$ for m, and integrating both sides, and dividing by $m+1$,

$$\Sigma x^{(m)} = \frac{x^{(m+1)}}{m+1} + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

As an example, let it be required to find the sum, from $x=20$ to $x=39$, of the series $x(x-1)(x-2) + (x+1)x(x-1) + \&c.$ In the present notation the series is $x^{(3)}+(x+1)^{(3)}+\&c.+(x+19)^{(3)}$, and the sum is $\Sigma_{20}^{39}x^{(3)}$. By formula 5,

$$\Sigma_{20}^{39}x^{(3)}=\frac{40^{(4)}}{4}-\frac{20^{(4)}}{4}=\frac{40.39.38.37-20.19.18.17}{4}$$

which is 519270.

As another example, let it be required to find the sum of n terms of the series $2.5.8+5.8.11+8.11.14+\&c.$ Here it can be seen that the xth term, u_x, of the series is $(3x-1)(3x+2)(3x+5)$, and that the problem is, to find $\Sigma_1^n(3x-1)(3x+2)(3x+5)$. The series is one of factorials, and may be treated in a similar manner to the simpler factorials considered in the last preceding article. If we add another factor, we have $(3x-4)(3x-1)(3x+2)(3x+5)$, the difference of which is $12(3x-1)(3x+2)(3x+5)$. Whence, u_x is the difference of $\frac{1}{12}(3x-4)(3x-1)(3x+2)(3x+5)$. We have thus found the integral of the general term of the series; and to find the sum of n terms of the series we must in that integral give n successively the values $(n+1)$ and 1, and take the algebraical difference of the results. That is, the sum of the series is $\frac{1}{12}(3n-1)(3n+2)(3n+5)(3n+8)-\frac{(-1).2.5.8}{12}$.

10. The last example is a special case of the integration of $u_xu_{x+1}\ldots u_{x+m-1}$, where $u_x=ax+b$. Taking this more general case,

$$\begin{aligned}&u_xu_{x+1}\ldots u_{x+m}-u_{x-1}u_xu_{x+1}\ldots u_{x+m-1}\\&=(u_{x+m}-u_{x-1})u_xu_{x+1}\ldots u_{x+m-1}\\&=a(m+1)u_xu_{x+1}\ldots u_{x+m-1}.\end{aligned}$$

Therefore

$$\Sigma u_xu_{x+1}\ldots u_{x+m-1}=\frac{u_{x-1}u_xu_{x+1}\ldots u_{x+m-1}}{a(m+1)}+c \quad . \quad . \quad (6)$$

If in this expression we make both a and m equal to 3, we have the series and the integral of the last preceding article.

11. If the terms of the series have the form $u_xu_{x-1}\ldots u_{x-m+1}$, where $u_x=ax+b$, the integration may be easily effected by Chap. xxii, formula 14. From that we see that

$$\Sigma u_xu_{x-1}\ldots u_{x-m+1}=\frac{u_xu_{x-1}\ldots u_{x-m}}{a(m+1)}+c \quad . \quad . \quad (7)$$

Formulas 6 and 7 are identical, relating merely to different terms of the series. If in No. 6 we write $(x-m+1)$ for x, we at once get No. 7.

[**12**] By means of formula 16 of Chap. xxii, and formula 5 of this chapter, we can integrate any rational integral function of x. If $\phi(x)$ be that function we have

$$\phi(x)=\phi(0)+x\Delta\phi(0)+\frac{x^{(2)}}{\underline{|2}}\Delta^2\phi(0)+\&c.$$

Integrating both sides we have

$$\Sigma\phi(x)=c+x\phi(0)+\frac{x^{(2)}}{\underline{|2}}\Delta\phi(0)+\frac{x^{(3)}}{\underline{|3}}\Delta^2\phi(0)+\&c. \quad . \quad . \quad (8)$$

[**13**] As an example, we may find the sum of the mth powers of the first n natural numbers. Here $\phi(x)=x^m$; and $\phi(0)=0^m=0$. Therefore

$$\Sigma x^m=c+\frac{x^{(2)}}{\underline{|2}}\Delta 0^m+\frac{x^{(3)}}{\underline{|3}}\Delta^2 0^m+\&c.+\frac{x^{(m+1)}}{\underline{|m+1}}\Delta^m 0^m.$$

whence

$$\Sigma_1^n x^m=\frac{(n+1)n}{\underline{|2}}\Delta 0^m+\frac{(n+1)n(n-1)}{\underline{|3}}\Delta^2 0^m+\&c.$$

Making $m=3$, we have the cubes of the natural numbers. Taking $\Delta 0^3$, $\Delta^2 0^3$, and $\Delta^3 0^3$ from the table in Chap. xxii, Art. 11,

$$\begin{aligned}\Sigma_1^n x^3&=\frac{(n+1)n}{2}+(n+1)n(n-1)+\frac{(n+1)n(n-1)(n-2)}{4}\\&=\frac{n^2(n+1)^2}{4}.\end{aligned}$$

14. To integrate a fractional expression of the form $\dfrac{1}{u_x u_{x+1}\ldots u_{x+m-1}}$, when $u_x=ax+b$. By Chap. xxii, formula 14,

$$\frac{1}{u_x u_{x+1}\ldots u_{x+m-1}}=-\frac{1}{a(m-1)}\Delta\frac{1}{u_x u_{x+1}\ldots u_{x+m-2}}$$

whence

$$\Sigma\frac{1}{u_x u_{x+1}\ldots u_{x+m-1}}=-\frac{1}{a(m-1)}\,\frac{1}{u_x u_{x+1}\ldots u_{x+m-2}}+c \quad . \quad (9)$$

Also

$$\Sigma\frac{1}{u_x u_{x-1}\ldots u_{x-m+1}}=-\frac{1}{a(m-1)}\,\frac{1}{u_{x-1}u_{x-2}\ldots u_{x-m+1}}+c \quad (10)$$

The denominator of the integral has one fewer factors than the denominator of the original function; and hence, for the function to be integrable, there must be at least two factors in the denominator.

15. If we write x, instead of $ax+b$, for u_x, we have, in the notation of Chap. xxii, Art. 26,

$$\Sigma x^{(-m)} = -\frac{x^{(-m+1)}}{m-1} + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For example, let it be required to find the sum of n terms of the series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \&c.$ By equation 11, writing $m=3$, we have

$$\Sigma_1^n \frac{1}{x(x+1)(x+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

It is clear that if there be an infinite number of terms in the series, the sum is $\frac{1}{4}$.

16. As another example, let it be required to find the sum to n terms of the series $\frac{1}{2.5.8} + \frac{1}{5.8.11} + \frac{1}{8.11.14} + \&c.$ As in Art. 9 the general term of the expression is $\frac{1}{(3x-1)(3x+2)(3x+5)}$, which is equivalent to $\frac{1}{u_x u_{x+1} u_{x+2}}$, when for u_x we write $(3x-1)$. Therefore, in formula 9, making both a and m equal to 3, we have

$$\Sigma \frac{1}{u_x u_{x+1} u_{x+2}} = -\frac{1}{6} \cdot \frac{1}{u_x u_{x+1}} + c$$

$$= -\frac{1}{6(3x-1)(3x+2)} + c$$

and $$\Sigma_1^n \frac{1}{u_x u_{x+1} u_{x+2}} = \frac{1}{60} - \frac{1}{6(3n+2)(3n+5)}$$

If the series be infinite, the sum is $\frac{1}{60}$.

17. If in Chap. xxii, formula 8, we write x for n, we have

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{\lfloor 2} \Delta^2 u_0 + \&c.$$

If now U_x be such a function that $\Delta U_x = u_x$, then $\Delta^2 U_x = \Delta u_x$, $\Delta^3 U_x = \Delta^2 u_x$, &c., and the expression becomes

$$U_x = U_0 + x u_0 + \frac{x(x-1)}{\lfloor 2} \Delta u_0 + \&c.$$

and $$\mathrm{U}_x-\mathrm{U}_0=xu_0+\frac{x(x-1)}{2}\Delta u_0+\&\text{c}.$$

But from the fundamental relation between the functions we have

$$\mathrm{U}_x-\mathrm{U}_0=u_0+u_1+\&\text{c}.+u_{x-1}.$$

Therefore $$u_0+u_1+\&\text{c}.+u_{x-1}=xu_0+\frac{x(x-1)}{\underline{\lvert 2}}\Delta u_0+\&\text{c}. \quad . \ . \ (12)$$

or more generally

$$\Sigma_a^{a+n-1}u_x=nu_a+\frac{n(n-1)}{\underline{\lvert 2}}\Delta u_a+\frac{n(n-1)(n-2)}{\underline{\lvert 3}}\Delta^2 u_a+\&\text{c}. \ . \ (13)$$

18. Formula 13 affords the means of summing a series the differences of which vanish after a finite number of orders, even although we may not know the form of the function; or, if we know the form, we can sum the series by calculating the numerical values of the first few terms. As an example, let it be required to sum to 20 terms the series 6840, 7980, 9240, 10626, 12144, &c. We have

	u	Δ	Δ^2	Δ^3
$u_0=$	6840	1140	120	6
$u_1=$	7980	1260	126	6
$u_2=$	9240	1386	132	
$u_3=$	10626	1518		
$u_4=$	12144			

Therefore,

$$\Sigma_0^{19}u_x=20\times 6840+\frac{20\times 19}{2}\times 1140+\frac{20\times 19\times 18}{6}\times 120$$
$$+\frac{20\times 19\times 18\times 17}{24}\times 6$$
$$=519270.$$

This is the same result as in Art. 9, and in fact the two series are identical.

As another example, let it be required to find, by means of formula 13, the sum of the cubes of the first n natural numbers. We have

	u	Δ	Δ^2	Δ^3
$u_0=1^3=$	1	7	12	6
$u_1=2^3=$	8	19	18	
$u_2=3^3=$	27	37		
$u_3=4^3=$	64			

Therefore,

$$\Sigma_1^n x^3 = n + 7\frac{n(n-1)}{2} + 12\frac{n(n-1)(n-2)}{6} + 6\frac{n(n-1)(n-2)(n-3)}{24}$$

$$= \frac{n^4+2n^3+n^2}{4} = \frac{n^2(n+1)^2}{4}$$

This is the same result as we obtained by formula 8, and in fact formula 8 is the special case of formula 13, in which $u_a = \phi(0)$. In practice it will generally be found that when the form of the function is known, formula 8 is the more convenient; but when only certain terms of the series, and not the form, are known, formula 13 must be used.

[19] An interesting case of the combination of summation with interpolation occurs in the construction of the Peerage Mortality Table by G. W. Berridge (*J.I.A.*, xii, 220, and xiv, 244).

[20] Let there be a function of x, of which the successive values for integral values of x are u_0, u_1, u_2, &c. Let the sum of the first ten values of the function from $x=0$ to $x=9$ be denoted by S_1; that of the second ten values from $x=10$ to $x=19$, by S_2, and so on. Let the differences of the series u_0, u_1, u_2, &c., be denoted by δ, δ^2, δ^3, &c., and the differences of the series S_1, S_2, S_3, &c., by Δ, Δ^2, Δ^3, &c. Then having as data the values of S from S_1 to S_6, it is required by means of a constant fifth difference to construct the series u_0, u_1, u_2, &c., from u_0 to u_{59}.

By formula 13,

$$\begin{aligned}
S_1 &= 10u_0 + 45\delta + 120\delta^2 + 210\delta^3 + 252\delta^4 + 210\delta^5 \\
S_2 &= 20u_0 + 190\delta + 1{,}140\delta^2 + 4{,}845\delta^3 + 15{,}504\delta^4 + 38{,}760\delta^5 \\
&\qquad - S_1 \\
S_3 &= 30u_0 + 435\delta + 4{,}060\delta^2 + 27{,}405\delta^3 + 142{,}506\delta^4 + 593{,}775\delta^5 \\
&\qquad - (S_1+S_2) \\
S_4 &= 40u_0 + 780\delta + 9{,}880\delta^2 + 91{,}390\delta^3 + 658{,}008\delta^4 + 3{,}838{,}380\delta^5 \\
&\qquad - (S_1+S_2+S_3) \\
S_5 &= 50u_0 + 1{,}225\delta + 19{,}600\delta^2 + 230{,}300\delta^3 + 2{,}118{,}760\delta^4 + 15{,}890{,}700\delta^5 \\
&\qquad - (S_1+S_2+S_3+S_4) \\
S_6 &= 60u_0 + 1{,}770\delta + 34{,}220\delta^2 + 487{,}635\delta^3 + 5{,}461{,}512\delta^4 + 50{,}063{,}860\delta^5 \\
&\qquad - (S_1+S_2+S_3+S_4+S_5)
\end{aligned}$$

Transferring now the values of S that appear on the right of the above equations to the other side, and differencing continuously, we have,

$$\left.\begin{aligned}
S_1&=10u_0+45\delta+120\delta^2+210\delta^3+252\delta^4+210\delta^5\\
S_2&=10u_0+145\delta+1{,}020\delta^2+4{,}635\delta^3+15{,}252\delta^4+38{,}550\delta^5\\
S_3&=10u_0+245\delta+2{,}920\delta^2+22{,}560\delta^3+127{,}002\delta^4+555{,}015\delta^5\\
S_4&=10u_0+345\delta+5{,}820\delta^2+63{,}985\delta^3+515{,}502\delta^4+3{,}244{,}605\delta^5\\
S_5&=10u_0+445\delta+9{,}720\delta^2+138{,}910\delta^3+1{,}460{,}752\delta^4+12{,}052{,}320\delta^5\\
S_6&=10u_0+545\delta+14{,}620\delta^2+257{,}335\delta^3+3{,}342{,}752\delta^4+34{,}173{,}160\delta^5
\end{aligned}\right\}\tag{A}$$

$$\left.\begin{aligned}
\Delta&=100\delta+900\delta^2+4{,}425\delta^3+15{,}000\delta^4+38{,}340\delta^5\\
&100\delta+1{,}900\delta^2+17{,}925\delta^3+111{,}750\delta^4+516{,}465\delta^5\\
&100\delta+2{,}900\delta^2+41{,}425\delta^3+388{,}500\delta^4+2{,}689{,}590\delta^5\\
&100\delta+3{,}900\delta^2+74{,}925\delta^3+945{,}250\delta^4+8{,}807{,}715\delta^5\\
&100\delta+4{,}900\delta^2+118{,}425\delta^3+1{,}882{,}000\delta^4+22{,}120{,}840\delta^5
\end{aligned}\right\}\quad .\tag{B}$$

$$\left.\begin{aligned}
\Delta^2&=1{,}000\delta^2+13{,}500\delta^3+96{,}750\delta^4+478{,}125\delta^5\\
&1{,}000\delta^2+23{,}500\delta^3+276{,}750\delta^4+2{,}173{,}125\delta^5\\
&1{,}000\delta^2+33{,}500\delta^3+556{,}750\delta^4+6{,}118{,}125\delta^5\\
&1{,}000\delta^2+43{,}500\delta^3+936{,}750\delta^4+13{,}313{,}125\delta^5
\end{aligned}\right\}\quad \ldots\ldots\tag{C}$$

$$\left.\begin{aligned}
\Delta^3&=10{,}000\delta^3+180{,}000\delta^4+1{,}695{,}000\delta^5\\
&10{,}000\delta^3+280{,}000\delta^4+3{,}945{,}000\delta^5\\
&10{,}000\delta^3+380{,}000\delta^4+7{,}195{,}000\delta^5
\end{aligned}\right\}\quad \ldots\ldots\ldots\tag{D}$$

$$\left.\begin{aligned}
\Delta^4&=100{,}000\delta^4+2{,}250{,}000\delta^5\\
&100{,}000\delta^4+3{,}250{,}000\delta^5
\end{aligned}\right\}\quad \ldots\ldots\ldots\ldots\tag{E}$$

$$\Delta^5=1{,}000{,}000\delta^5\quad \ldots\ldots\ldots\ldots\ldots\ldots\tag{F}$$

Now, the values of the differences, Δ, are supplied in the original data, and working backwards through the successive equations, we obtain the following values of the differences δ.

$$\left.\begin{aligned}
\delta^5&=\frac{1}{10^6}\Delta^5\\
\delta^4&=\frac{1}{10^5}\Delta^4-22\cdot5\delta^5\\
\delta^3&=\frac{1}{10^4}\Delta^3-18\cdot0\delta^4-169\cdot50\delta^5\\
\delta^2&=\frac{1}{10^3}\Delta^2-13\cdot5\delta^3-96\cdot75\delta^4-478\cdot125\delta^5\\
\delta&=\frac{1}{10^2}\Delta-9\cdot0\delta^2-44\cdot25\delta^3-150\cdot000\delta^4-383\cdot4\delta^5
\end{aligned}\right\}\quad \ldots\tag{G}$$

Having now found all the values of δ, we can calculate by means of equation (A) the value of u_0.

$$u_0 = \frac{1}{10}\left\{S_1 - \frac{10\times 9}{\underline{|2}}\delta - \frac{10\times 9\times 8}{\underline{|3}}\delta^2 - \&c.\right\} \quad . \quad . \quad . \quad . \qquad (H)$$

In equations (H) and (G) we have, therefore, the leading term and the leading differences of the series u_0, u_1, u_2, &c., and by continuous addition the series can be constructed.

Berridge applied the formula by taking $u_x = \log p_x$.

APPROXIMATE SUMMATION

✓ *Lubbock's Formula.*

[21] The first formula of approximate summation which we shall discuss, is that which was given by the late Sir J. W. Lubbock, Bart., in the Cambridge Philosophical Transactions for the year 1829. It involves only the Calculus of Finite Differences, and it can be most conveniently employed to obtain approximate values of benefits of many descriptions.

[22] Let u_0, u_t, u_{2t}, &c., u_{nt}, $u_{(n+1)t}$, &c., up to u_{mnt}, be the successive values of any function u, where t is fractional and n integral, and where $nt=1$, so that u_{nt}, u_{2nt}, &c., may otherwise be written u_1, u_2, &c. Then expressing by means of Chap. xxii, formula 8, the successive values of u, from u_0 to $u_{(n-1)t}$ in terms of u_0 and its differences, we have

$$u_0 = u_0$$

$$u_t = u_0 + t\Delta u_0 + \frac{t(t-1)}{\underline{|2}}\Delta^2 u_0 + \frac{t(t-1)(t-2)}{\underline{|3}}\Delta^3 u_0 + \&c.$$

$$u_{2t} = u_0 + 2t\Delta u_0 + \frac{2t(2t-1)}{\underline{|2}}\Delta^2 u_0 + \frac{2t(2t-1)(2t-2)}{\underline{|3}}\Delta^3 u_0 + \&c.$$

$$\&c. = \&c.$$

$$u_{(n-1)t} = u_0 + \overline{n-1}t\Delta u_0 + \frac{\overline{n-1}t(\overline{n-1}t-1)}{\underline{|2}}\Delta^2 u_0$$
$$+ \frac{\overline{n-1}t(\overline{n-1}t-1)(\overline{n-1}t-2)}{\underline{|3}}\Delta^3 u_0 + \&c.$$

Hence, by addition,

$$u_0+u_t+u_{2t}+\&\text{c.}+u_{(n-1)t}$$

$$=nu_0+t\{1+2+3+\&\text{c.}+(n-1)\}\Delta u_0$$

$$+\frac{t}{\underline{|2}}\{\overline{t-1}+2.\overline{2t-1}+3.\overline{3t-1}+\&\text{c.}+(n-1)(\overline{n-1}t-1)\}\Delta^2u_0$$

$$+\frac{t}{\underline{|3}}\{\overline{t-1}.\overline{t-2}+2.\overline{2t-1}.\overline{2t-2}+\&\text{c.}$$
$$+(n-1)(\overline{n-1}t-1)(\overline{n-1}t-2)\}\Delta^3u_0$$

$$+\&\text{c.}$$

The values of the coefficients of u_0, Δu_0, Δ^2u_0, &c., could easily be individually calculated by means of one or other of the formulas of summation already given in this chapter; but by observing the law which they follow, their general form can be at once determined. They are evidently the coefficients of the powers of x in the expansion of $1+(1+x)^t+(1+x)^{2t}+\&\text{c.}+(1+x)^{(n-1)t}$, which is equal to $\dfrac{(1+x)^{nt}-1}{(1+x)^t-1}$, or to $\dfrac{(1+x)-1}{(1+x)^t-1}$ since $nt=1$, or to $\dfrac{x}{(1+x)^t-1}$. But this expansion is given in Chap. xxii, formula 18; whence we have, remembering that $\dfrac{1}{t}=n$,

$$u_0+u_t+u_{2t}+\&\text{c.}+u_{(n-1)t}$$

$$=nu_0+\frac{n-1}{2}\Delta u_0-\frac{n^2-1}{12n}\Delta^2u_0+\frac{n^2-1}{24n}\Delta^3u_0-\frac{(n^2-1)(19n^2-1)}{720n^3}\Delta^4u_0$$
$$+\frac{(n^2-1)(9n^2-1)}{480n^3}\Delta^5u_0-\&\text{c.}$$

Similarly, expressing $u_{(n+1)t}$, $u_{(n+2)t}$, &c., to $u_{(2n-1)t}$, in terms of u_{nt}, or u_1, and its differences; and $u_{(2n+1)t}$, $u_{(2n+2)t}$, &c., to $u_{(3n-1)t}$, in terms of u_2 and its differences; and so on; we have,

$$u_{nt}+u_{(n+1)t}+u_{(n+2)t}+\&\text{c.}+u_{(2n-1)t}$$
$$=nu_1+\frac{n-1}{2}\Delta u_1-\frac{n^2-1}{12n}\Delta^2u_1+\frac{n^2-1}{24n}\Delta^3u_1-\&\text{c.}$$

$$u_{2nt}+u_{(2n+1)t}+u_{(2n+2)t}+\&\text{c.}+u_{(3n-1)t}$$
$$=nu_2+\frac{n-1}{2}\Delta u_2-\frac{n^2-1}{12n}\Delta^2u_2+\frac{n^2-1}{24n}\Delta^3u_2-\&\text{c.}$$

$$\&\text{c.}=\&\text{c.}$$

$$u_{(m-1)nt}+u_{(m-1)nt+t}+u_{(m-1)nt+2t}+\&\text{c.}+u_{(mn-1)t}$$
$$=nu_{m-1}+\frac{n-1}{2}\Delta u_{m-1}-\frac{n^2-1}{12n}\Delta^2u_{m-1}+\frac{n^2-1}{24n}\Delta^3u_{m-1}-\&\text{c.}$$

Adding now together all these series, which will give the sum of all the values of u (for intervals of t) from u_0 to u_{m-t}, we have

$$u_0+u_t+u_{2t}+\&c.+u_1+u_{1+t}+u_{1+2t}+\&c.+\&c.+u_{m-1}+u_{m-1+t}+\&c.+u_{m-t}$$
$$=n(u_0+u_1+u_2+\&c.+u_{m-1})$$
$$+\frac{n-1}{2}(\Delta u_0+\Delta u_1+\Delta u_2+\&c.+\Delta u_{m-1})$$
$$-\frac{n^2-1}{12n}(\Delta^2u_0+\Delta^2u_1+\Delta^2u_2+\&c.+\Delta^2u_{m-1})$$
$$+\frac{n^2-1}{24n}(\Delta^3u_0+\Delta^3u_1+\Delta^3u_2+\&c.+\Delta^3u_{m-1})$$
$$-\quad\&c.\qquad\qquad\&c.$$

But

$$\Delta u_0=u_1-u_0$$
$$\Delta u_1=u_2-u_1$$
$$\Delta u_2=u_3-u_2$$
$$\&c.=\&c.$$
$$\Delta u_{m-1}=u_m-u_{m-1}$$

Whence, by addition,

$$\Delta u_0+\Delta u_1+\Delta u_2+\&c.+\Delta u_{m-1}=u_m-u_0$$

Similarly,

$$\Delta^2u_0+\Delta^2u_1+\Delta^2u_2+\&c.+\Delta^2u_{m-1}=\Delta u_m-\Delta u_0$$
$$\&c.=\&c.$$

Thus, finally, we have

$$u_0+u_t+u_{2t}+\&c.+u_1+u_{1+t}+u_{1+2t}+\&c.+\&c.+u_{m-1}+u_{m-1+t}+\&c.+u_{m-t}$$
$$=n(u_0+u_1+u_2+\&c.+u_{m-1})$$
$$+\frac{n-1}{2}(u_m-u_0)-\frac{n^2-1}{12n}(\Delta u_m-\Delta u_0)$$
$$+\frac{n^2-1}{24n}(\Delta^2u_m-\Delta^2u_0)-\frac{(n^2-1)(19n^2-1)}{720n^3}(\Delta^3u_m-\Delta^3u_0)$$
$$+\frac{(n^2-1)(9n^2-1)}{480n^3}(\Delta^4u_m-\Delta^4u_0)-\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

[23] In order, by formula 14, to find the approximate sum of the complete series, we must know the equidistant terms u_0, u_1, &c., up to u_m, and also the differences both of u_0 and u_m; and to know the differences of u_m we must know a sufficient number of the terms u_{m+1}, u_{m+2}, &c., which follow u_m. Generally, however, in questions connected

with Life Contingencies, the formula is required for finding the values of whole-term annuities and assurances; and in such cases u_m falls beyond the mortality table, and, along with its differences, vanishes. Under these circumstances Lubbock's formula becomes

$$\begin{aligned}
&u_0+u_t+u_{2t}+\&c.\\
&\quad=n(u_0+u_1+u_2+\&c.)-\frac{n-1}{2}u_0+\frac{n^2-1}{12n}\Delta u_0\\
&\qquad-\frac{n^2-1}{24n}\Delta^2u_0+\frac{(n^2-1)(19n^2-1)}{720n^3}\Delta^3u_0-\frac{(n^2-1)(9n^2-1)}{480n^3}\Delta^4u_0+\&c.
\end{aligned} \quad \ldots\ldots \quad (15)$$

By means of formula 18 of Chap. xxii, we can supply further terms if required; but generally it will be sufficient to stop at fourth differences. Indeed, frequently second differences give an approximation which is close enough.

[24] It will very often be convenient to make t the unit of measurement instead of n; in which case u_1, u_2, &c., become respectively u_n, u_{2n}, &c.; while u_t, u_{2t}, &c., become respectively u_1, u_2, &c. Making these transformations, formula 14 becomes

$$\begin{aligned}
\Sigma_0^{mn-1}u&=n(u_0+u_n+u_{2n}+\&c.+u_{(m-1)n})\\
&\quad+\frac{n-1}{2}(u_{mn}-u_0)-\frac{n^2-1}{12n}(\Delta u_{mn}-\Delta u_0)\\
&\quad+\frac{n^2-1}{24n}(\Delta^2u_{mn}-\Delta^2u_0)-\frac{(n^2-1)(19n^2-1)}{720n^3}(\Delta^3u_{mn}-\Delta^3u_0)\\
&\quad+\frac{(n^2-1)(9n^2-1)}{480n^3}(\Delta^4u_{mn}-\Delta^4u_0)-\&c. \quad \ldots\ldots \quad (16)
\end{aligned}$$

and formula 15 becomes

$$\begin{aligned}
\Sigma_0^{\infty}u&=n(u_0+u_n+u_{2n}+\&c.)-\frac{n-1}{2}u_0+\frac{n^2-1}{12n}\Delta u_0\\
&\qquad-\frac{n^2-1}{24n}\Delta^2u_0+\frac{(n^2-1)(19n^2-1)}{720n^3}\Delta^3u_0-\frac{(n^2-1)(9n^2-1)}{480n^3}\Delta^4u_0+\&c.
\end{aligned} \quad \ldots\ldots \quad (17)$$

[25] The advantages of Lubbock's formula are easily seen. For instance, if the value of a joint-life annuity be desired, to calculate it at full length without previously prepared tables would be very laborious. It is, however, comparatively easy to calculate the limited number of

terms D_{xy}, $D_{x+n:y+n}$, &c., when n is taken equal to 10, or to some other convenient number; and the formula then by a short arithmetical process gives the value of the annuity. The chief difficulty in using the formula is to calculate the values of the coefficients of the differences, but the following table, computed by McLauchlan and given by Sprague, (*J.I.A.*, xviii, 313,) supplies these coefficients for the more usual values of n, and renders the remaining arithmetical work easy. In order to abbreviate the statement of the formulas, it is convenient to denote the coefficients of the successive orders of differences by the symbols c_1, c_2, c_3, &c., as in the headings of the columns of the table.

Table of the Numerical Values of the Coefficients required in the application of Lubbock's Formula of Summation.

n	$\frac{n^2-1}{12n}$		$\frac{n^2-1}{24n}$		$\frac{(n^2-1)(19n^2-1)}{720n^3}$	
	c_1	$\log c_1$	c_2	$\log c_2$	c_3	$\log c_3$
2	·125000	$\bar{1}\cdot096910$	·062500	$\bar{2}\cdot795880$	·039063	$\bar{2}\cdot591760$
3	·222222	$\bar{1}\cdot346787$	·111111	$\bar{1}\cdot045757$	·069959	$\bar{2}\cdot844843$
4	·312500	$\bar{1}\cdot494850$	·156250	$\bar{1}\cdot193820$	·098633	$\bar{2}\cdot994021$
5	·400000	$\bar{1}\cdot602060$	·200000	$\bar{1}\cdot301030$	·126400	$\bar{1}\cdot101747$
6	·486111	$\bar{1}\cdot687736$	·243056	$\bar{1}\cdot385706$	·153710	$\bar{1}\cdot186703$
7	·571429	$\bar{1}\cdot756962$	·285714	$\bar{1}\cdot455932$	·180758	$\bar{1}\cdot257098$
8	·656250	$\bar{1}\cdot817069$	·328125	$\bar{1}\cdot516039$	·207642	$\bar{1}\cdot317314$
9	·740741	$\bar{1}\cdot869667$	·370370	$\bar{1}\cdot568637$	·234416	$\bar{1}\cdot369986$
10	·825000	$\bar{1}\cdot916454$	·412500	$\bar{1}\cdot615424$	·261113	$\bar{1}\cdot416828$
11	·909091	$\bar{1}\cdot958607$	·454545	$\bar{1}\cdot657577$	·287754	$\bar{1}\cdot459021$

n	$\frac{(n^2-1)(9n^2-1)}{480n^3}$		$\frac{(n^2-1)(863n^4-145n^2+2)}{60480n^5}$		$\frac{(n^2-1)(275n^4-61n^2+2)}{24192n^5}$	
	c_4	$\log c_4$	c_5	$\log c_5$	c_6	$\log c_6$
2	·027344	$\bar{2}\cdot436858$	·020508	$\bar{2}\cdot311919$	·016113	$\bar{2}\cdot207184$
3	·049383	$\bar{2}\cdot693575$	·037342	$\bar{2}\cdot572196$	·029569	$\bar{2}\cdot470832$
4	·069824	$\bar{2}\cdot844006$	·052948	$\bar{2}\cdot723849$	·042038	$\bar{2}\cdot623642$
5	·089600	$\bar{2}\cdot952308$	·068032	$\bar{2}\cdot832713$	·054080	$\bar{2}\cdot733037$
6	·109037	$\bar{1}\cdot037576$	·082849	$\bar{2}\cdot918285$	·065902	$\bar{2}\cdot818896$
7	·128280	$\bar{1}\cdot108159$	·097510	$\bar{2}\cdot989051$	·077595	$\bar{2}\cdot889835$
8	·147400	$\bar{1}\cdot168497$	·112075	$\bar{1}\cdot049508$	·089208	$\bar{2}\cdot950404$
9	·166438	$\bar{1}\cdot221253$	·126574	$\bar{1}\cdot102345$	·100767	$\bar{1}\cdot003318$
10	·185419	$\bar{1}\cdot268154$	·141028	$\bar{1}\cdot149304$	·112288	$\bar{1}\cdot050332$
11	·204358	$\bar{1}\cdot310391$	·155448	$\bar{1}\cdot191584$	·123781	$\bar{1}\cdot092653$

[26] When Lubbock's formula, as given in equation 17, is applied to the calculation of Life Contingencies, it gives the values of benefits of which the first payment is due at once; that is, for instance, if the benefit be an annuity, the value brought out by the formula is that of an annuity-due. To get the value of the annuity we must deduct u_0 from each side of the equation, which then becomes

$$\Sigma_1^\infty u = n(u_0+u_n+u_{2n}+\&c.) - \frac{n+1}{2}u_0 + c_1\Delta u_0 - c_2\Delta^2 u_0 + c_3\Delta^3 u_0 - c_4\Delta^4 u_0 \&c. \quad . \quad . \quad . \quad . \quad (18)$$

[27] As an example of the use of Lubbock's formula, No. 18, let it be required to find the value, at 3 per-cent interest, of an annuity on a life aged 30. It will be convenient here to take $n=10$.

The work may be arranged advantageously in columns, as thereby, by comparing the additions, useful checks are obtained at each stage. At first the computer may find it irksome to add together the numbers when they are placed side by side instead of under each other in the more usual way, but after a little practice this difficulty is overcome. To make the work more easy, cross-ruled paper should be used, and the figures should be carefully made, and each placed in its proper position. Although in the case of such a simple benefit as an annuity on a single life the advantages of the columnar method are not so apparent, yet it will be seen from the examples of more complicated functions given in Chaps. xiii, xiv, and xv, that it is the only method practically applicable. In the case of an annuity on a single life, $u_t = \frac{v^t l_{x+t}}{l_x}$. The following is a type of the calculations:

(1) t	(2) $\log v^t$	(3) $\log l_{x+t}$	(4) (2)+(3)	(5) (4)+Colog l_x
0	0·00000	4·95272	4·95272	0·00000
10	$\bar{1}$·87163	4·91528	4·78691	$\bar{1}$·83419
20	$\bar{1}$·74326	4·86210	4·60536	$\bar{1}$·65264
30	$\bar{1}$·61488	4·76969	4·38457	$\bar{1}$·43185
40	$\bar{1}$·48651	4·57952	4·06603	$\bar{1}$·11331
50	$\bar{1}$·35814	4·14572	3·50386	$\bar{2}$·55114
60	$\bar{1}$·22977	3·10483	2·33460	$\bar{3}$·38188
70	$\bar{1}$·10139	0·60206	$\bar{1}$·70345	$\bar{6}$·75073
	$\bar{4}$·40558	31·93192	28·33750	$\overline{12}$·71574

	u_t	Δ	Δ^2	Δ^3	Δ^4
0	1·00000	−·31736	+·08413	−·03001	+·01451
10	·68264	·23323	·05412	·01550	
20	·44941	·17911	·03862		
30	·27030	·14049			
40	·12981				
50	·03558				
60	·00241				
70	·00001				
	2·57016				
Deduct	·58070				

1·98946 × 10 = 19·8946 = a_{30}

$$\frac{n+1}{2}u_0 = 5\cdot50000$$

$$c_1\Delta = \cdot26183$$

$$c_2\Delta^2 = \cdot03470$$

$$c_3\Delta^3 = \cdot00783$$

$$c_4\Delta^4 = \cdot00268$$

5·80704

[28] The correct value of the annuity is 19·8950, so that the approximation is deficient only 4 in the fourth place of decimals. Had we used only two orders of differences, the value would have come out 19·9051; or using three orders, 19·8972.

[29] As another example, let it be required to find the value of an assurance on a life aged 30. Here $u_{x+n} = \frac{v^{n+1} . d_{x+n}}{l_x}$; and as the first term must be included in the summation, formula 17 is the one to be used. We have,

(1) t	(2) $\log v^{t+1}$	(3) $\log d_{x+t}$	(4) (2)+(3)	(5) (4)+Colog l_x
0	$\bar{1}$·98716	2·83948	2·82664	$\bar{3}$·87392
10	$\bar{1}$·85879	2·91540	2·77419	$\bar{3}$·82147
20	$\bar{1}$·73042	3·05843	2·78885	$\bar{3}$·83613
30	$\bar{1}$·60205	3·24428	2·84633	$\bar{3}$·89361
40	$\bar{1}$·47367	3·38632	2·85999	$\bar{3}$·90727
50	$\bar{1}$·34530	3·30492	2·65022	$\bar{3}$·69750
60	$\bar{1}$·21693	2·60423	1·82116	$\bar{4}$·86844
70	$\bar{1}$·08856	0·47712	$\bar{1}$·56568	$\bar{6}$·61296
	$\bar{4}$·30288	21·83018	18·13306	$\overline{22}$·51130

	u_t	Δ			
0	·0074803	− ·0008510	+ ·0010786	− ·0003358	− ·0011273
10	·0066293	+ ·0002276	+ ·0007428	− ·0014631	
20	·0068569	+ ·0009704	− ·0007203		
30	·0078273	+ ·0002501			
40	·0080774				
50	·0049831				
60	·0007387				
70	·0000041				

·0425971 × n = ·425971

Deduct ·034687

A_{30} = ·391284

$\frac{n-1}{2}u_0$ = ·033661

$c_1\Delta u_0$ = ·000702

$c_2\Delta^2 u_0$ = ·000445

$c_3\Delta^3 u_0$ = ·000088

·034896

$c_4\Delta^4 u_0$ = ·000209

·034687

The correct value of the assurance is ·39141.

[30] For more detailed demonstrations and examinations of the formula the reader is referred to Lubbock's original paper in the Transactions of the Cambridge Philosophical Society, reprinted *J.I.A.*, v, 277: to Woolhouse's paper on "Summation", *J.I.A.*, xi, 301: and to Sprague's paper on "Lubbock's Formula for approximating to the Value of a Life Annuity", *J.I.A.*, xviii, 305. Both Woolhouse and Sprague give methods of eliminating Δu_m, $\Delta^2 u_m$, &c., so that formula 14 can be written in terms to involve only u_0 to u_m, without requiring u_{m+1}, &c. It is not, however, necessary to pursue the subject further here.

Woolhouse's Formulas.

[31] In *J.I.A.*, xi, 61 and 301, Woolhouse published papers on "Interpolation" and "Summation", and in *J.I.A.*, xv, 95, a paper on "An Improved Theory of Annuities and Assurances", in which he demonstrated the following summation formulas, and applied them to Life Contingencies. In speaking, therefore, on Life Contingencies, the formulas go by his name, although they were originally due to Euler.

[32] Let u_x be a function of x, and let $\frac{du_0}{dx}$, $\frac{d^2u_0}{dx^2}$, &c., $\frac{du_{\frac{1}{m}}}{dx}$, $\frac{d^2u_{\frac{1}{m}}}{dx^2}$, &c., represent the numerical values of the differential coefficients when x has the values 0, $\frac{1}{m}$, &c.

Let u_x be developed as follows:—

$$(a)\quad u_x=u_0+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\&c.$$

By successive differentiation,

$$(b)\quad \frac{du_x}{dx}=A+2Bx+3Cx^2+4Dx^3+5Ex^4+\&c.$$

$$\frac{d^2u_x}{dx^2}=2B+6Cx+12Dx^2+20Ex^3+\&c.$$

$$(c)\quad \frac{d^3u_x}{dx^3}=6C+24Dx+60Ex^2+\&c.$$

Also, writing $x=0$ in (*b*) and (*c*),

$$\frac{du_0}{dx}=A;\quad \text{and}\quad \frac{d^3u_0}{dx^3}=6C.$$

Therefore,

$$(d)\qquad u_0+u_x=2u_0+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\&c.$$

$$(e)\quad \frac{du_0}{dx}-\frac{du_x}{dx}=-2Bx-3Cx^2-4Dx^3-5Ex^4-\&c.$$

$$(f)\quad \frac{d^3u_0}{dx^3}-\frac{d^3u_x}{dx^3}=-24Dx-60Ex^2-\&c.$$

Also, by integration of (*a*),

$$(g)\qquad \int u_x dx=u_0x+A\frac{x^2}{2}+B\frac{x^3}{3}+C\frac{x^4}{4}+D\frac{x^5}{5}+E\frac{x^6}{6}+\&c.$$

The integral in (g) may be expressed in terms of the differential coefficients as follows:—

From (d),

$$\frac{x}{2}(u_0+u_x)=u_0x+\text{A}\frac{x^2}{2}+\text{B}\frac{x^3}{2}+\text{C}\frac{x^4}{2}+\text{D}\frac{x^5}{2}+\text{E}\frac{x^6}{2}+\text{\&c.}$$

From (e),

$$\frac{x^2}{12}\left\{\frac{du_0}{dx}-\frac{du_x}{dx}\right\}=-\tfrac{1}{6}\text{B}x^3-\tfrac{1}{4}\text{C}x^4-\tfrac{1}{3}\text{D}x^5-\tfrac{5}{12}\text{E}x^6+\text{\&c.}$$

From (f),

$$-\frac{x^4}{720}\left\{\frac{d^3u_0}{dx^3}-\frac{d^3u_x}{dx^3}\right\}=\tfrac{1}{30}\text{D}x^5+\tfrac{1}{12}\text{E}x^6+\text{\&c.}$$

Adding together these last three equations, the sum of the right-hand members is

$$u_0x+\text{A}\frac{x^2}{2}+\text{B}\frac{x^3}{3}+\text{C}\frac{x^4}{4}+\text{D}\frac{x^5}{5}+\text{E}\frac{x^6}{6}+\text{\&c.},$$

or, by (g), $\int u_x dx$: whence

$$\int u_x dx=\frac{x}{2}(u_0+u_x)+\frac{x^2}{12}\left\{\frac{du_0}{dx}-\frac{du_x}{dx}\right\}-\frac{x^4}{720}\left\{\frac{d^3u_0}{dx^3}-\frac{d^3u_x}{dx^3}\right\}+\text{\&c.} \quad . \ (19)$$

Now suppose the variable x to pass over successive intervals each equal to $\frac{1}{m}$, namely, from 0 to $\frac{1}{m}$, from $\frac{1}{m}$ to $\frac{2}{m}$, from $\frac{2}{m}$ to $\frac{3}{m}$, &c. By applying formula 19 to each of these intervals, we get

$$\int_0^{\frac{1}{m}} u_x dx=\frac{1}{2m}(u_0+u_{\frac{1}{m}})+\frac{1}{12m^2}\left\{\frac{du_0}{dx}-\frac{du_{\frac{1}{m}}}{dx}\right\}-\frac{1}{720m^4}\left\{\frac{d^3u_0}{dx^3}-\frac{d^3u_{\frac{1}{m}}}{dx^3}\right\}+\text{\&c.}$$

$$\int_{\frac{1}{m}}^{\frac{2}{m}} u_x dx=\frac{1}{2m}(u_{\frac{1}{m}}+u_{\frac{2}{m}})+\frac{1}{12m^2}\left\{\frac{du_{\frac{1}{m}}}{dx}-\frac{du_{\frac{2}{m}}}{dx}\right\}-\frac{1}{720m^4}\left\{\frac{d^3u_{\frac{1}{m}}}{dx^3}-\frac{d^3u_{\frac{2}{m}}}{dx^3}\right\}+\text{\&c.}$$

$$\int_{\frac{2}{m}}^{\frac{3}{m}} u_x dx=\frac{1}{2m}(u_{\frac{2}{m}}+u_{\frac{3}{m}})+\frac{1}{12m^2}\left\{\frac{du_{\frac{2}{m}}}{dx}-\frac{du_{\frac{3}{m}}}{dx}\right\}-\frac{1}{720m^4}\left\{\frac{d^3u_{\frac{2}{m}}}{dx^3}-\frac{d^3u_{\frac{3}{m}}}{dx^3}\right\}+\text{\&c.}$$

$$\text{\&c.}=\text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.}$$

By adding together the successive portions of the integral, we get the complete limits, in which ω denotes the terminal value of x,

$$\int_0^\omega u_x dx = \frac{1}{m}(u_0 + u_{\frac{1}{m}} + u_{\frac{2}{m}} + \&c. + u_\omega) - \frac{1}{2m}(u_0 + u_\omega)$$

$$+\frac{1}{12m^2}\left\{\frac{du_0}{dx} - \frac{du_\omega}{dx}\right\} - \frac{1}{720m^4}\left\{\frac{d^3u_0}{dx^3} - \frac{d^3u_\omega}{dx^3}\right\} + \&c. \quad . \quad . \quad (20)$$

Let $\Sigma^{(m)}u$ denote $\frac{1}{m}(u_0 + u_{\frac{1}{m}} + u_{\frac{2}{m}} + \&c. + u_\omega)$. Then, from formula 20,

$$\Sigma^{(m)}u = \int_0^\omega u_x dx + \frac{1}{2m}(u_0 + u_\omega) - \frac{1}{12m^2}\left\{\frac{du_0}{dx} - \frac{du_\omega}{dx}\right\}$$

$$+\frac{1}{720m^4}\left\{\frac{d^3u_0}{dx^3} - \frac{d^3u_\omega}{dx^3}\right\} + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

[33] In questions of Life Contingencies, when benefits are taken for the whole of life, the terminal values u_ω, $\frac{du_\omega}{dx}$, $\frac{d^3u_\omega}{dx^3}$, &c., disappear, and formula 21 becomes

$$\Sigma^{(m)}u = \int_0^\infty u_x dx + \frac{1}{2m}u_0 - \frac{1}{12m^2}\frac{du_0}{dx} + \frac{1}{720m^4}\frac{d^3u_0}{dx^3} - \&c. \quad . \quad . \quad (22)$$

[34] In equation 22, making $m=1$, we have

$$\Sigma^{(1)}u = \int_0^\infty u_x dx + \frac{1}{2}u_0 - \frac{1}{12}\frac{du_0}{dx} + \frac{1}{720}\frac{d^3u_0}{dx^3} - \&c. \quad . \quad . \quad . \quad (23)$$

or
$$\int_0^\infty u_x dx = \Sigma^{(1)}u - \frac{1}{2}u_0 + \frac{1}{12}\frac{du_0}{dx} - \frac{1}{720}\frac{d^3u_0}{dx^3} + \&c. \quad . \quad . \quad . \quad . \quad (24)$$

[35] By formula 24 we can get the value of a continuous benefit from that of the benefit in the more usual form at yearly intervals, as in Chap. ix, Art. 26.

[36] By combining equations 22 and 23, we obtain the relation

$$\Sigma^{(m)}u = \Sigma^{(1)}u - \frac{m-1}{2m}u_0 + \frac{m^2-1}{12m^2}\frac{du_0}{dx} - \frac{m^4-1}{720m^4}\frac{d^3u_0}{dx^3} - \&c. \quad . \quad . \quad (25)$$

which enables us to get the values of benefits at intervals $\frac{1}{m}$ from those at yearly intervals, as in Chap. ix, Art. 21.

[37] In equation 25, writing $\frac{1}{n}$ for m, we get a formula of summation similar to that of Lubbock. When $\frac{1}{m}=n$, then

$$\Sigma^{\left(\frac{1}{n}\right)}u=n(u_0+u_n+u_{2n}+\&c.)$$

whence, reverting to the notation of equation 17, where $\Sigma^{(1)}u$ is written $\Sigma_0^\infty u$,

$$\Sigma_0^\infty u=n(u_0+u_n+u_{2n}+\&c.)-\frac{n-1}{2}u_0$$
$$+\frac{n^2-1}{12}\frac{du_0}{dx}-\frac{n^4-1}{720}\frac{d^3u_0}{dx^3}+\&c. \quad . \quad . \quad . \quad . \quad (26)$$

[38] Formula 26 gives the values of benefits first payment due at once, such as annuities-due. To obtain the values of benefits first payment at the end of a year, such as annuities, we must deduct u_0 from each side of the equation. We then have

$$\Sigma_1^\infty u=n(u_n+u_{2n}+u_{3n}+\&c.)+\frac{n-1}{2}u_0$$
$$+\frac{n^2-1}{12}\frac{du_0}{dx}-\frac{n^4-1}{720}\frac{d^3u_0}{dx^3}+\&c. \quad . \quad . \quad . \quad . \quad (27)$$

[39] Formula 23 may be deduced very easily by the method of separation of symbols. We have seen, Chap. XXII, Art. 19, that $D=1+\Delta$. Also, by definition $Du_x=u_{x+1}$; whence, by Taylor's Theorem (Chap. xxii, formula 17), employed symbolically,

$$Du_x=u_x+\frac{du_x}{dx}+\frac{1}{\underline{|2}}\cdot\frac{d^2u_x}{dx^2}+\frac{1}{\underline{|3}}\cdot\frac{d^3u_x}{dx^3}+\&c.$$
$$=\left\{1+\frac{d}{dx}+\frac{1}{\underline{|2}}\cdot\frac{d^2}{dx^2}+\frac{1}{\underline{|3}}\frac{d^3}{dx^3}+\&c.\right\}u_x$$
$$=e^{\frac{d}{dx}}u_x.$$

Therefore, $D=e^{\frac{d}{dx}}$; and $\Delta=(e^{\frac{d}{dx}}-1)$. But by Art. 2 of the present chapter, $\Sigma=\Delta^{-1}$; whence $\Sigma u_x=(e^{\frac{d}{dx}}-1)^{-1}u_x$. We must now expand $(e^{\frac{d}{dx}}-1)^{-1}$ in powers of $\frac{d}{dx}$. For convenience, writing $\frac{d}{dx}=t$, we have, from the usual formula for the expansion of e^t,

$$(e^t-1)^{-1}=\frac{1}{e^t-1}=\frac{1}{t+\frac{t^2}{\underline{|2}}+\frac{t^3}{\underline{|3}}+\frac{t^4}{\underline{|4}}+\&c.}$$

$$=\frac{1}{t}-\frac{1}{2}+\frac{t}{12}-\frac{t^3}{720}+\&c.$$

by actual division. Replacing t by $\frac{d}{dx}$, and remembering that $\left(\frac{d}{dx}\right)^{-1}=\int$ we have, after restoring u_x,

$$\Sigma u_x=\int u_x dx-\frac{1}{2}u_x+\frac{1}{12}\frac{du_x}{dx}-\frac{1}{720}\frac{d^3u_x}{dx^3}+\&c.;$$

or, taking the integral between limits, we have, as before,

$$\Sigma_0^\infty u_x=\int_0^\infty u_x dx+\frac{1}{2}u_0-\frac{1}{12}\frac{du_0}{dx}+\frac{1}{720}\frac{d^3u_0}{dx^3}-\&c.$$

[40] The very intimate connection between the formulas of Lubbock and Woolhouse can be easily shown. Taking Lubbock's formula, we have, by equation 18, stopping at third differences,

$$\Sigma_1^\infty u=n(u_0+u_n+u_{2n}+\&c.)-\frac{n+1}{2}u_0$$

$$+\frac{n^2-1}{12}\cdot\frac{\Delta u_0-\frac{1}{2}\Delta^2u_0}{n}+\frac{(n^2-1)(19n^2-1)}{720}\cdot\frac{\Delta^3u_0}{n^3}.$$

But by Chap. xxiii, formula 19, $\frac{du_x}{dx}=\frac{1}{n}\{\Delta u_x-\frac{1}{2}\Delta^2u_x+\frac{1}{3}\Delta^3u_x\}$, and by Chap. xxiii, formula 22, $\frac{d^3u_x}{dx^3}=\frac{\Delta^3u_x}{n^3}$, approximately.

Therefore Lubbock's formula becomes

$$\Sigma_1^\infty u=n(u_0+u_n+u_{2n}+\&c.)-\frac{n+1}{2}u_0$$

$$+\frac{n^2-1}{12}\cdot\frac{du_0}{dx}-\frac{n^4-1}{720}\cdot\frac{d^3u_0}{dx^3},$$

which is the same as Woolhouse's (see formula 27). We have thus proved that Lubbock's formula, taken as far as third differences, is identical with Woolhouse's, taken as far as third differential coefficients.

[41] As an example of the use of Woolhouse's formula, we may find again, at 3 per-cent interest, the value of an annuity on a life aged 30.

Here, as $u_x=\frac{v^x l_x}{l}$, therefore $\frac{du_x}{dx}=-(\mu_x+\delta)$. Also $\frac{d^3u_x}{dx^3}$ may be neglected. From Table I, $\mu_{30}=00768$; also $\delta=\cdot02956$; so that $(\mu_{30}+\delta)=\cdot03724$. Taking in this case $n=7$, we have

(1) t	(2) $\log v^t$	(3) $\log l_{x+t}$	(4) (2)+(3)	(5) (4) + Colog l_x	(6) Antilog (5)
7	$\bar{1}$·91014	4·92757	4·83771	$\bar{1}$·88499	·76734
14	$\bar{1}$·82028	4·89676	4·71704	$\bar{1}$·76432	·58119
21	$\bar{1}$·73042	4·85522	4·58564	$\bar{1}$·63292	·42946
28	$\bar{1}$·64056	4·79334	4·43390	$\bar{1}$·48118	·30282
35	$\bar{1}$·55070	4·69293	4·24363	$\bar{1}$·29091	·19539
42	$\bar{1}$·46084	4·51950	3·98034	$\bar{1}$·02762	·10657
49	$\bar{1}$·37098	4·20772	3·57870	$\bar{2}$·62598	·04227
56	$\bar{1}$·28112	3·63377	2·91489	$\bar{3}$·96217	·00917
63	$\bar{1}$·19125	2·56348	1·75473	$\bar{4}$·80201	·00063
70	$\bar{1}$·10139	0·60206	$\bar{1}$·70345	$\bar{6}$·75073	·00001
	$\bar{5}$·05768	39·69235	34·75003	$\overline{15}$·22283	2·43485
					7
					17·04395
				$\frac{n-1}{2}u_0=$	3·00000
					20·04395
				$\frac{n^2-1}{12}\frac{du_0}{dx}=$	·14896
				$a_{30}=$	19·89499

As we have already seen, the correct value of the annuity is 19·8950, so that in this case Woolhouse's formula gives the annuity-value correct to at least four places of decimals.

G. F. Hardy's Formulas.

[42] In using Lubbock's formula the successive orders of differences are required, and are multiplied into coefficients which are not rapidly convergent; while in using Woolhouse's formula, recourse must be had to the differential coefficients, which with some functions are not readily calculated. To obviate these objections G. F. Hardy, in a paper (*J.I.A.*, xxiv, 95) on some "Formulas for Approximate Summation," applied

other formulas of the Integral Calculus and produced results of a very convenient description. The more useful of these are given in the following Articles. To understand the paper referred to, a good knowledge of the higher mathematics is required; but here the demonstrations have been very much simplified. These more easy developments are also for the most part due to Mr. Hardy himself, he having supplied them specially for this work.

[43] In formula 20, writing n for $\frac{1}{m}$, and tn for ω, we have

$$(a)\quad \int_0^{tn} u_x dx = n(\tfrac{1}{2}u_0 + u_n + u_{2n} + \&\text{c.} + \tfrac{1}{2}u_{tn})$$

$$-\frac{n^2}{12}\left\{\frac{du_{tn}}{dx} - \frac{du_0}{dx}\right\} + \frac{n^4}{720}\left\{\frac{d^3u_{tn}}{dx^3} - \frac{d^3u_0}{dx^3}\right\} - \&\text{c.}$$

Also, taking intervals of $\frac{n}{2}$,

$$(b)\quad \int_0^{tn} u_x dx = \frac{n}{2}(\tfrac{1}{2}u_0 + u_{\frac{n}{2}} + u_n + u_{\frac{3n}{2}} + \&\text{c.} + \tfrac{1}{2}u_{tn})$$

$$-\frac{\left(\frac{n}{2}\right)^2}{12}\left\{\frac{du_{tn}}{dx} - \frac{du_0}{dx}\right\} + \frac{\left(\frac{n}{2}\right)^4}{720}\left\{\frac{d^3u_{tn}}{dx^3} - \frac{d^3u_0}{dx^3}\right\} - \&\text{c.}$$

Doubling both sides of (b),

$$(c)\quad 2\int_0^{tn} u_x dx = n(\tfrac{1}{2}u_0 + u_{\frac{n}{2}} + u_n + u_{\frac{3n}{2}} + \&\text{c.} + \tfrac{1}{2}u_{tn})$$

$$-\frac{n^2}{24}\left\{\frac{du_{tn}}{dx} - \frac{du_0}{dx}\right\} + \frac{n^4}{5760}\left\{\frac{d^3u_{tn}}{dx^3} - \frac{d^3u_0}{dx^3}\right\} - \&\text{c.}$$

Subtracting (a) from (c)

$$\int_0^{tn} u_x dx = n(u_{\frac{n}{2}} + u_{\frac{3n}{2}} + u_{\frac{5n}{2}} + \&\text{c.} + u_{\frac{(2t-1)n}{2}})$$

$$+\frac{n^2}{24}\left\{\frac{du_{tn}}{dx} - \frac{du_0}{dx}\right\} - \frac{7n^4}{5760}\left\{\frac{d^3u_{tn}}{dx^3} - \frac{d^3u_0}{dx^3}\right\} \quad . \quad . \quad (28)$$

or, making t infinite, and assuming that the terminal values of the function and its differential coefficients then vanish,

$$\int_0^{\infty} u_x dx = n(u_{\frac{n}{2}} + u_{\frac{3n}{2}} + \&\text{c.}) - \frac{n^2}{24}\frac{du_0}{dx} + \frac{7n^4}{5760}\frac{d^3u_0}{dx^3} \quad . \quad . \quad . \quad . \quad (29)$$

or, writing $2n$ for n, in order to avoid fractional suffixes,

$$\int_0^\infty u_x dx = 2n(u_n + u_{3n} + u_{5n} + \&c.) - \frac{n^2}{6}\frac{du_0}{dx} + \frac{7n^4}{360}\frac{d^3u_0}{dx^3} \quad . \quad . \quad (30)$$

The last term in equations 29 and 30 is generally insignificant.

[44] As an example of formula 30, let it be required to find at 3 per-cent interest the value of a continuous annuity on a life aged 30. Here, as in the example of Art. 41, $\frac{du_0}{dx} = -(\mu_{30}+\delta) = -\cdot03724$. Taking $n=5$,

(1) t	(2) $\log v^t$	(3) $\log l_{x+t}$	(4) (2)+(3)	(5) (4) + Colog l_x	(6) Antilog (5)
5	$\bar{1}\cdot93581$	4·93519	4·87100	$\bar{1}\cdot91828$	·82848
15	$\bar{1}\cdot80744$	4·89164	4·69908	$\bar{1}\cdot74636$	·55765
25	$\bar{1}\cdot67907$	4·82325	4·50232	$\bar{1}\cdot54960$	·35449
35	$\bar{1}\cdot55070$	4·69293	4·24363	$\bar{1}\cdot29091$	·19539
45	$\bar{1}\cdot42232$	4·40827	3·83059	$\bar{2}\cdot87787$	·07549
55	$\bar{1}\cdot29395$	3·73902	3·03297	$\bar{2}\cdot08025$	·01203
65	$\bar{1}\cdot16558$	2·11059	1·27617	$\bar{4}\cdot32345$	·00021
	$\bar{4}\cdot85487$	29·60089	26·45576	$\bar{9}\cdot78672$	2·02374
					10
					20·2374
				$\frac{25}{6}(\mu_{30}+\delta)=$	·1552
				$\bar{a}_{30}=$	20·3926

The value of $\bar{a}_{30}$ by the formula $\bar{a}_x = a_x + \frac{1}{2} - \frac{\mu_x+\delta}{12}$ is 20·3919, so that the approximate value is only ·0007 in excess.

[45] The formulas of Art. 43 are not much less laborious to apply than Woolhouse's, and, like it, they require the differential coefficients, but they may be transformed as follows. If in equation 29 we write (by Chap. xxiii, formulas 20 and 24), $\frac{du_0}{dx} = \frac{1}{n}\left(\Delta u_{-\frac{n}{2}} - \frac{1}{24}\Delta^3 u_{-\frac{3n}{2}}\right)$, and $\frac{d^3u_0}{dx^3} = \frac{1}{n^3}\Delta^3 u_{-\frac{3n}{2}}$, we have,

$$\int_0^\infty u_x dx = n(u_{\frac{n}{2}} + u_{\frac{3n}{2}} + u_{\frac{5n}{2}} + \&c.) - \frac{n}{24}\Delta u_{-\frac{n}{2}} + \frac{17n}{5760}\Delta^3 u_{-\frac{3n}{2}} \quad . \quad (31)$$

or, omitting the last term as insignificant,

$$\int_0^\infty u_x dx = n\left(u_{\frac{n}{2}} + u_{\frac{3n}{2}} + u_{\frac{5n}{2}} + \text{\&c.} + \frac{u_{-\frac{n}{2}} - u_{\frac{n}{2}}}{24}\right) \quad . \quad . \quad . \quad (32)$$

or, writing $2n$ for n,

$$\int_0^\infty u_x dx = 2n\left(u_n + u_{3n} + u_{5n} + \text{\&c.} + \frac{u_{-n} - u_n}{24}\right) \quad . \quad . \quad (33)$$

[46] As an example, to find by formula 33 the value of $\bar{a}_{30}$ at 3 per-cent interest, taking $n=5$, we have $\dfrac{u_{-n}-u_n}{24} = \dfrac{v^{-5}l_{25} - v^5 l_{35}}{24 l_{30}} = \cdot 01559$. In Art. 44 the calculation of the other portion of the formula is given.

Whence, $\qquad \bar{a}_{30} = 10(2\cdot 02374 + \cdot 01559)$

$\qquad\qquad = 20\cdot 3933.$

or only ·0014 in excess.

[47] It will be found that the correction for the last term of equation 31 is in this case subtractive, and equal to ·0010, so that using equation 31, we have $\bar{a}_{30} = 20\cdot 3923$, or only ·0004 in excess.

[48] As showing the remarkable power of equation 33, it will be found that if we take n so large as 10, thus giving the values of u_x at intervals of 20 years, we shall have $\bar{a}_{30} = 20\cdot 4011$, or only ·0092 in excess. In this case, the only values of u used in finding the value of the annuity are, u_{20}, u_{40}, u_{60}, u_{80}, and u_{100}, with l_{30} for denominator, and the result is sufficiently accurate for most practical purposes.

[49] Another very useful formula of summation may be simply established as follows :—

By Maclaurin's Theorem, Chap. xxii, formula 17,

$$u_x = u_0 + x\frac{du_0}{dx} + \frac{x^2}{\underline{|2}}\frac{d^2u_0}{dx^2} + \frac{x^3}{\underline{|3}}\frac{d^3u_0}{dx^3} + \text{\&c.}$$

Hence, integrating,

$$\int u_x dx = xu_0 + \frac{x^2}{\underline{|2}}\frac{du_0}{dx} + \frac{x^3}{\underline{|3}}\frac{d^2u_0}{dx^2} + \frac{x^4}{\underline{|4}}\frac{d^3u_0}{dx^3} + \text{\&c.}$$

and, taking the integral between the limits $-n$ and $+n$,

$$\int_{-n}^{n} u_x dx = 2nu_0 + \frac{2n^3}{\underline{|3}}\frac{d^2u_0}{dx^2} + \frac{2n^5}{\underline{|5}}\frac{d^4u_0}{dx^4} + \text{\&c.} \quad . \quad . \quad (34)$$

Taking only the first two terms, we have,

$$(1)\qquad \int_{-n}^{n} u_x dx = 2nu_0 + \frac{2n^3}{\underline{|3}}\frac{d^2u_0}{dx^2}.$$

We may also assume that, approximately,

$$(2)\qquad \int_{-n}^{n} u_x dx = au_0 + b(u_{-n} + u_n).$$

Expanding the last expression by means of Maclaurin's Theorem, we have

$$(3)\qquad \int_{-n}^{n} u_x dx = au_0 + b\left\{2u_0 + \frac{2n^2}{\underline{|2}}\frac{d^2u_0}{dx^2} + \&c.\right\}$$

Equating the two expressions (1) and (3) for the integral, we get

$$2nu_0 + \frac{2n^3}{\underline{|3}}\frac{d^2u_0}{dx^2} = (a+2b)u_0 + bn^2\frac{d^2u_0}{dx^2}.$$

Whence, equating coefficients,

$$\left.\begin{aligned} a+2b &= 2n \\ bn^2 &= \frac{n^3}{3} \end{aligned}\right\} \text{ and } \left\{\begin{aligned} a &= \frac{4n}{3} \\ b &= \frac{n}{3} \end{aligned}\right.$$

Therefore, by (2)

$$\int_{-n}^{n} u_x dx = \frac{n}{3}(u_{-n} + 4u_0 + u_n) \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Changing the limits from $(-n, n)$ to $(0, 2n)$, we have

$$\int_{0}^{2n} u_x dx = \frac{n}{3}(u_0 + 4u_n + u_{2n}).$$

Applying the formula to successive sections of the integral, and adding the results, we have

$$\int_{0}^{\infty} u_x dx = \int_{0}^{2n} u_x dx + \int_{2n}^{4n} u_x dx + \int_{4n}^{6n} u_x dx + \&c.$$

$$= \frac{n}{3}\{(u_0 + 4u_n + u_{2n}) + (u_{2n} + 4u_{3n} + u_{4n}) + \&c.\}$$

That is,

$$\int_{0}^{\infty} u_x dx = \frac{n}{3}\{u_0 + 2(u_{2n} + u_{4n} + u_{6n} + \&c.) + 4(u_n + u_{3n} + \&c.)\} \quad . \quad (36)$$

[50] As an example, let it be required to find, as before, $\bar{a}_{30}$, taking $n=15$. We have

(1)	(2)	(3)	(4)	(5)	(6)	
t	$\log v^t$	$\log l_{x+t}$	(2)+(3)	(4) + Colog l_x	Antilog (5)	
15	$\bar{1}$·80744	4·89164	4·69908	$\bar{1}$·74636		·55765
30	$\bar{1}$·61488	4·76969	4·38457	$\bar{1}$·43185	·27030	
45	$\bar{1}$·42232	4·40827	3·83059	$\bar{2}$·87787		·07549
60	$\bar{1}$·22977	3·10483	2·33460	$\bar{3}$·38188	·00241	
	$\bar{2}$·07441	17·17443	15·24884	$\bar{5}$·43796	·27271	·63314
					2	4
					·54542	2·53256
						·54542
						3·07798
					$u_0=$	1·00000
						4·07798
						5
					$\bar{a}_{30}=$	20·38990

To obtain the value of the annuity, it has been necessary to employ only four values of u.

[51] Other formulas may be obtained without difficulty by taking in more terms of equation 34. By including four terms we really retain the differential coefficient of the seventh order, although only that of the sixth order appears, because the seventh order is eliminated in taking the integral between the limits. Recurring therefore to equation 34 and reducing the arithmetical coefficients, we have

$$\int_{-n}^{n} u_x dx = 2nu_0 + \frac{n^3}{3}\frac{d^2u_0}{dx^2} + \frac{n^5}{60}\frac{d^4u_0}{dx^4} + \frac{n^7}{2520}\frac{d^6u_0}{dx^6}$$

In Art. 49 we assumed three equidistant values of u for the purpose of obtaining an expansion of $\int_{-n}^{n} u_x dx$. If we now assume five values, we may write

$$\int_{-n}^{n} u_x dx = au_0 + b(u_{-n} + u_n) + c(u_{-h} + u_h)$$

where we have four unknown quantities, a, b, c, and h. Expanding by Maclaurin's Theorem,

$$\int_{-n}^{n} u_x dx = au_0$$

$$+b\left\{2u_0 + \frac{2n^2}{\lfloor 2}\frac{d^2u_0}{dx^2} + \frac{2n^4}{\lfloor 4}\frac{d^4u_0}{dx^4} + \frac{2n^6}{\lfloor 6}\frac{d^6u_0}{dx^6}\right\}$$

$$+c\left\{2u_0 + \frac{2h^2}{\lfloor 2}\frac{d^2u_0}{dx^2} + \frac{2h^4}{\lfloor 4}\frac{d^4u_0}{dx^4} + \frac{2h^6}{\lfloor 6}\frac{d^6u_0}{dx^6}\right\}$$

Equating the coefficients of u_0 and its differential coefficients in the two expansions of the integral, we have

$$a+2b+2c = 2n$$

$$bn^2+ch^2 = \frac{1}{3}n^3$$

$$bn^4+ch^4 = \frac{1}{5}n^5$$

$$bn^6+ch^6 = \frac{1}{7}n^7$$

which may be written

$$(1) \qquad a+2b+2c = 2n$$

$$(2) \qquad b+c\frac{h^2}{n^2} = \frac{1}{3}n$$

$$(3) \qquad b+c\frac{h^4}{n^4} = \frac{1}{5}n$$

$$(4) \qquad b+c\frac{h^6}{n^6} = \frac{1}{7}n$$

From (3) and (4), $c\left(\frac{h^4}{n^4} - \frac{h^6}{n^6}\right) = \frac{2}{35}n$

From (2) and (3), $c\left(\frac{h^2}{n^2} - \frac{h^4}{n^4}\right) = \frac{2}{15}n$

Whence $\frac{h^2}{n^2} = \frac{3}{7}$, and $h = n\sqrt{\frac{3}{7}}$.

We might take this value for h, and from it find the values of a, b, and c; but the resulting formula would be very troublesome to use on account of the fractions involved. It will be much more convenient, and sufficiently accurate for practical purposes, if we reject the surd, and take the approximate value $h=\frac{2}{3}n$. Substituting this value in (1), (2), and (3) above, we obtain

$$a=\frac{110}{150}n \qquad c=\frac{81}{150}n$$

$$b=\frac{14}{150}n \qquad h=\frac{2}{3}n.$$

Hence, substituting these values in the assumed expression for the integral,

$$\int_{-n}^{n} u_x dx=\frac{n}{150}\left\{110u_0+14(u_{-n}+u_n)+81(u_{-\frac{2}{3}n}+u_{\frac{2}{3}n})\right\}$$

or, changing the limits,

$$\int_{0}^{2n} u_x dx=\frac{n}{150}\left\{14(u_0+u_{2n})+81(u_{\frac{1}{3}n}+u_{\frac{5}{3}n})+110u_n\right\}$$

or writing $3n$ for n, and turning the numerical coefficients into decimals,

$$\int_{0}^{6n} u_x dx=n\{\cdot 28(u_0+u_{6n})+1\cdot 62(u_n+u_{5n})+2\cdot 2u_{3n}\} \quad . \quad . \quad (37)$$

[52] In using formula 37 the value of n must be taken to suit the limits of summation; but it will be found that if the summation be a long one, that is, if the benefit sought depend upon a comparatively young life, the formula is not to be implicitly relied on. Thus, taking the same example as before, to find the value of $\bar{a}_{30}$ we have 101 the oldest age in the table. Hence, $6n=101-30=71$, and approximately $n=12$. Therefore,

$$\bar{a}_{30}=\frac{12}{l_{30}}\{\cdot 28l_{30}+1\cdot 62(v^{12}l_{42}+v^{60}l_{90})+2\cdot 2v^{36}l_{66}\}$$

and performing the calculations $\bar{a}_{30}=20\cdot 4533$. In this case the formula requires only three values of the function in addition to the initial value, but the result cannot be considered as satisfactory, as it is ·0614 in excess. The formula may, however, be regarded as exceedingly useful where a very rapid but somewhat rough calculation is required.

[53] By summation of the series in sections a very accurate formula may be derived from No. 37. We have

$$\int_0^\infty u_x dx=\int_0^{6n} u_x dx+\int_{6n}^{12n} u_x dx+\int_{12n}^{18n} u_x dx+\&c.$$

That is, after collecting together the terms,

$$\begin{aligned}\int_0^\infty u_x dx=n\{&{\cdot}28u_0+{\cdot}56(u_{6n}+u_{12n}+u_{18n}+\&c.)\\&+1{\cdot}62(u_n+u_{5n}+u_{7n}+u_{11n}+\&c.)\\&+2{\cdot}2(u_{3n}+u_{9n}+u_{15n}+\&c.)\}\quad .\ .\ .\ .\quad (38)\end{aligned}$$

[54] Adopting the same example as before, and taking $n=5$, we have $\bar{a}_{30}=20{\cdot}3921$. To obtain this nine values of the function are used besides the initial value, but the result is only ·0002 in excess. Had we made $n=7$, we should have had $\bar{a}_{30}=20{\cdot}3976$. Here only six values of the function are used besides the initial value, and the result is only ·0057 in excess. Taking $n=10$, we have $\bar{a}_{30}=20{\cdot}3954$, or only ·0035 in excess. Here only five values of the function are used besides the initial value, but by dropping u_{7n}, which is insignificant, only four values would have been included, and accuracy would not have been lost. This leads us to the very convenient formula,

$$\int_0^\omega u_x dx=n\{{\cdot}28u_0+{\cdot}56u_{6n}+1{\cdot}62(u_n+u_{5n}+u_{7n})+2{\cdot}2u_{3n}\}\quad .\quad (39)$$

$$=n\{{\cdot}28u_0+1{\cdot}62u_n+2{\cdot}2u_{3n}+1{\cdot}62u_{5n}+{\cdot}56u_{6n}+1{\cdot}62u_{7n}\}\quad .\quad (39a)$$

where ω is the limiting age of the table, and n is so taken that u_{7n} falls just within or just beyond the table.

The second form is given as being generally more convenient in numerical calculations, because in it the terms follow each other in their natural order. In using the formula the logarithms of the coefficients are required, and they are therefore here supplied for reference.

Term.	Coefficient.	Logarithm.
u_0	·28	$\bar{1}$·44716
u_n	1·62	0·20952
u_{3n}	2·2	0·34242
u_{5n}	1·62	0·20952
u_{6n}	·56	$\bar{1}$·74819
u_{7n}	1·62	0·20952

To find the value of $\bar{a}_{30}$ by formula 39a, n must be made equal to 10, because $7n=102-x=72$. The following is the work.

(1) t	(2) $\log v^t$	(3) $\log l_{x+t}$	(4) log Coeff.	(5) (2)+(3)+(4)	(6) (5) + Colog l_x	(7) Antilog (6)
0	0·00000	4·95272	$\bar{1}$·44716	4·39988	$\bar{1}$·44716	·2800
10	$\bar{1}$·87163	4·91528	0·20952	4·99643	0·04371	1·1059
30	$\bar{1}$·61488	4·76969	0·34242	4·72699	$\bar{1}$·77427	·5947
50	$\bar{1}$·35814	4·14572	0·20952	3·71338	$\bar{2}$·76066	·0576
60	$\bar{1}$·22977	3·10483	$\bar{1}$·74819	2·08279	$\bar{3}$·13007	·0013
70	$\bar{1}$·10139	0·60206	0·20952	$\bar{1}$·91297	$\bar{6}$·96025	·0000
	$\bar{3}$·17581	22·49030	0·16633	19·83244	$\overline{10}$·11612	2·0395
						10
						$\bar{a}_{30}=$20·3950

[55] If we return to equation 34, and take in the term including $\dfrac{d^8u_0}{dx^8}$, we may, by a process very similar to that adopted in deducing formula 37, obtain another very convenient formula of summation. We have

$$\int_{-n}^{n} u_x dx = 2nu_0 + \frac{2n^3}{\lfloor\underline{3}}\,\frac{d^2u_0}{dx^2} + \frac{2n^5}{\lfloor\underline{5}}\,\frac{d^4u_0}{dx^4} + \frac{2n^7}{\lfloor\underline{7}}\,\frac{d^6u_0}{dx^6} + \frac{2n^9}{\lfloor\underline{9}}\,\frac{d^8u_0}{dx^8}$$

Also, assuming

$$\int_{-n}^{n} u_x dx = a(u_{-n}+u_n) + b(a_{-h}+a_h) + c(u_{-k}+u_k),$$

expanding by Maclaurin's Theorem, and equating coefficients with those in the other expression for the integral, and solving for the five unknown quantities a, b, c, h, and k, we shall find $h^2=n^2\dfrac{162}{283}$; whence $h=n\dfrac{13}{17}$ nearly. By adopting this approximate value for h, we shall obtain $k^2=n^2\dfrac{316}{3892}$; or $k=n\dfrac{5}{17}$ nearly. Whence

$$c=n\frac{958}{1700};\quad b=n\frac{622}{1700};\quad a=\frac{120}{1700}.$$

Adopting these approximate values for the assumed coefficients, changing the limits of summation from $(-n, n)$ to $(0, 2n)$, and writing $8{\cdot}5\times n$ for n, we finally have

$$\int_0^{17n} u_x dx = n\{{\cdot}60(u_0+u_{17n})+3{\cdot}11(u_{2n}+u_{15n})+4{\cdot}79(u_{6n}+u_{11n})\} \quad . \quad (40)$$

[56] In using this last formula n must be so taken as to suit the limits of summation in question; that is, $17n$ must be as nearly as possible equal to the difference between the age at which the benefit is required, and the limiting age in the table. Thus, to find $\bar{a}_{30}$ we must have $17n=102-30=72$, as nearly as possible: therefore we must assume $n=4$. Hence,

$$\bar{a}_{30}=\frac{4}{l_{30}}\{{\cdot}60(l_{30}+v^{68}l_{98})+3{\cdot}11(v^{8}l_{38}+v^{60}l_{90})+4{\cdot}79(v^{24}l_{54}+v^{44}l_{74})\}$$

$$=20{\cdot}3833 \text{ or } {\cdot}0086 \text{ too small.}$$

Had we taken $n=5$, we should have had

$$\bar{a}_{30}=\frac{5}{l_{30}}\{{\cdot}60l_{30}+3{\cdot}11v^{10}l_{40}+4{\cdot}79(v^{30}l_{60}+v^{55}l_{85})\}$$

$$=20{\cdot}3830.$$

Here we have employed only three values of the function besides the initial value, and yet the result is only ·0089 too small.

[57] All of Hardy's formulas of summation produce, as result, the *continuous* benefit; that is, in the case of annuities they give us the values of annuities payable momently, and in the case of assurances, assurances payable at the moment of death. In practice these continuous values are often those most convenient, but if annuities payable yearly or assurances payable at the end of the year of death be desired, their values can usually be very readily obtained from the continuous values.

[58] Throughout this chapter the formulas have been illustrated by finding the value of an annuity on a single life. In the preceding chapters, however, there are numerous examples of their application to other benefits.

[59] For convenience of reference, the following tabular summary of the formulas of summation is appended

Reference Table of Formulas of Approximate Summation.

No.	Page.	Formula.
17	470	$\Sigma_0^\infty u = n(u_0 + u_n + u_{2n} + \text{\&c.}) - \frac{n-1}{2} u_0 + c_1 \Delta u_0 - c_2 \Delta^2 u_0 + \text{\&c.}$
18	472	$\Sigma_1^\infty u = n(u_0 + u_n + u_{2n} + \text{\&c.}) - \frac{n+1}{2} u_0 + c_1 \Delta u_0 - c_2 \Delta^2 u_0 + \text{\&c.}$
		For values of coefficients c_1, c_2, &c., see Table, p. 471.
26	478	$\Sigma_0^\infty u = n(u_0 + u_n + u_{2n} + \text{\&c.}) - \frac{n-1}{2} u_0 + \frac{n^2-1}{12} \cdot \frac{du_0}{dx} - \frac{n^4-1}{720} \cdot \frac{d^3u_0}{dx^3} + \text{\&c.}$
27	478	$\Sigma_1^\infty u = n(u_n + u_{2n} + u_{3n} + \text{\&c.}) + \frac{n-1}{2} u_0 + \frac{n^2-1}{12} \cdot \frac{du_0}{dx} - \frac{n^4-1}{720} \cdot \frac{d^3u_0}{dx^3} + \text{\&c.}$
30	482	$\int_0^\infty u_x dx = 2n(u_n + u_{3n} + u_{5n} + \text{\&c.}) - \frac{n^2}{6} \cdot \frac{du_0}{dx} + \frac{7n^4}{360} \cdot \frac{d^3u_0}{dx^3} - \text{\&c.}$
33	483	$\int_0^\infty u_x dx = 2n\left(u_n + u_{3n} + u_{5n} + \text{\&c.} + \frac{u_{-n} - u_n}{24}\right)$
36	484	$\int_0^\infty u_x dx = \frac{n}{3}\left\{u_0 + 2(u_{2n} + u_{4n} + u_{6n} + \text{\&c.}) + 4(u_n + u_{3n} + u_{5n} + \text{\&c.})\right\}$
37	487	$\int_0^{6n} u_x dx = n\left\{{\cdot}28(u_0 + u_{6n}) + 1{\cdot}62(u_n + u_{5n}) + 2{\cdot}2u_{3n}\right\}$
38	488	$\int_0^\infty u_x dx = n\left\{{\cdot}28u_0 + {\cdot}56(u_{6n} + u_{12n} + u_{18n} + \text{\&c.})\right.$ $+ 1{\cdot}62(u_n + u_{5n} + u_{7n} + u_{11n} + u_{13n} + \text{\&c.})$ $\left.+ 2{\cdot}2(u_{3n} + u_{9n} + u_{15n} + \text{\&c.})\right\}$
39	488	$\int_0^\omega u_x dx = n\left\{{\cdot}28u_0 + {\cdot}56u_{6n} + 1{\cdot}62(u_n + u_{5n} + u_{7n}) + 2{\cdot}2u_{3n}\right\}$
39*a*	488	$= n\left\{{\cdot}28u_0 + 1{\cdot}62u_n + 2{\cdot}2u_{3n} + 1{\cdot}62u_{5n} + {\cdot}56u_{6n} + 1{\cdot}62u_{7n}\right\}$
		Where ω is limiting age of the table, and u_{7n} falls just within or just beyond the table. For logarithms of coefficients, see page 488.
40	490	$\int_0^{17n} u_x dx = n\left\{{\cdot}60(u_0 + u_{17n}) + 3{\cdot}11(u_{2n} + u_{15n}) + 4{\cdot}79(u_{6n} + u_{11n})\right\}$

TABLES.

MORTALITY ONLY.

Table No. I.—The Life Table.
„ „ II.—Elementary Values.
„ „ III.—Expectations—Joint Lives.

Constants.

Constant.	Common Logarithm.	Napierian Logarithm.
c	0·03965686	0·0913133
g	$\bar{1}$·9995432	$\bar{1}$·9989482
s	$\bar{1}$·997310673	$\bar{1}$·9938077
k	4·0404723	9·3035310
γ	$\bar{1}$·54698	$\bar{2}$·9568829
σ	0·06820	0·1570363
κ	3·9103	9·0037985

Modulus of Common Logarithms, $\cdot434294482 = \frac{1}{2\cdot30258509}$.

Base of Napierian Logarithms, 2·718281828.

TABLE NO. I.

The Life Table.

x	l_x	d_x	μ_x	$\Delta\mu_x$	$p_x = (1-q_x)$	p_x^{-1}	x
0	127 283	14 358	·15920	$\bar{1}$·91981	·88720	1·1271	0
1	112 925	3 962	·07901	·94465	·96492	1·0364	1
2	108 963	2 375	·02366	·99421	·97821	1·0223	2
3	106 588	1 646	·01787	·99592	·98456	1·0157	3
4	104 942	1 325	·01379	·99763	·98737	1·0128	4
5	103 617	1 061	·01142	·99783	·98976	1·0103	5
6	102 556	852	·00925	·99823	·99170	1·0084	6
7	101 704	683	·00748	·99859	·99328	1·0068	7
8	101 021	557	·00607	·99895	·99449	1·0055	8
9	100 464	464	·00502	·99926	·99538	1·0046	9
10	100 000	408	·00428	·99960	·99591	1·0041	10
1	99 592	369	·00388	·99971	·99630	1·0037	1
2	99 223	346	·00359	·99983	·99653	1·0035	2
3	98 877	337	·00342	·99998	·99658	1·0034	3
4	98 540	337	·00340	0·00013	·99658	1·0034	4
15	98 203	360	·00353	·00025	·99635	1·0037	15
6	97 843	384	·00378	·00036	·99607	1·0039	6
7	97 459	425	·00414	·00044	·99563	1·0044	7
8	97 034	465	·00458	·00046	·99522	1·0048	8
9	96 569	508	·00504	·00046	·99474	1·0053	9
20	96 061	548	·00550	·00042	·99428	1·0058	20
1	95 513	582	·00592	·00037	·99392	1·0061	1
2	94 931	609	·00629	·00030	·99357	1·0065	2
3	94 322	631	·00659	·00023	·99332	1·0067	3
4	93 691	647	·00682	·00019	·99309	1·0070	4
25	93 044	658	·00701	·00015	·99293	1·0071	25
6	92 386	664	·00716	·00013	·99280	1·0073	6
7	91 722	673	·00729	·00013	·99268	1·0074	7
8	91 049	678	·00742	·00013	·99254	1·0075	8
9	90 371	686	·00755	·00013	·99241	1·0076	9
30	89 685	691	·00768	·00014	·99229	1·0078	30
1	88 994	700	·00782	·00016	·99213	1·0079	1
2	88 294	709	·00798	·00017	·99197	1·0081	2
3	87 585	719	·00815	·00018	·99179	1·0083	3
4	86 866	729	·00833	·00021	·99161	1·0085	4
35	86 137	742	·00854	·00022	·99138	1·0087	35
6	85 395	756	·00876	·00025	·99115	1·0089	6
7	84 639	770	·00901	·00027	·99090	1·0092	7
8	83 869	786	·00928	·00029	·99063	1·0095	8
9	83 083	806	·00957	·00033	·99031	1·0098	9
40	82 277	823	·00990	·00035	·98999	1·0101	40
1	81 454	846	·01025	·00039	·98962	1·0105	1
2	80 608	871	·01064	·00042	·98919	1·0109	2
3	79 737	895	·01106	·00047	·98878	1·0113	3
4	78 842	924	·01153	·00051	·98828	1·0119	4
45	77 918	954	·01204	·00056	·98776	1·0124	45
6	76 964	986	·01260	·00061	·98719	1·0130	6
7	75 978	1 021	·01321	·00067	·98655	1·0136	7
8	74 957	1 061	·01388	·00074	·98585	1·0144	8
9	73 896	1 101	·01462	·00080	·98510	1·0151	9

for n years

$n|\ell_x = \frac{p_{x+n}}{\ell_x}$

$= |_n e_x = \frac{N'_x}{\ell_x}$

~~$\ell_x - {}_np_x$~~

$n/\mathring{e}_x - {}_np_x \, \mathring{e}_{x+n}$

$|_n \mathring{e}_x = \mathring{\ell}_x - {}_np_x \, \mathring{\ell}_{x+n}$

TABLE No. I.

The Life Table.

x	q_x $=(1-p_x)$	L_x $=\frac{1}{2}(l_x+l_{x+1})$	N'_x $=\Sigma l_{x+1}$	T_x $=\Sigma L_x$	$\mathring{e}_x$ $=\frac{1}{2}+e_x$ $= T_x/l_x$	x
0	·11280	120 104	6 018 388	6 082 031	47·784	0
1	·03508	110 944	5 905 463	5 961 927	52·796	1
2	·02179	107 776	5 796 500	5 850 983	53·697	2
3	·01544	105 765	5 689 912	5 743 207	53·881	3
4	·01263	104 279	5 584 970	5 637 442	53·719	4
5	·01024	103 087	5 481 353	5 533 163	53·401	5
6	·00830	102 130	5 378 797	5 430 076	52·948	6
7	·00672	101 362	5 277 093	5 327 946	52·387	7
8	·00551	100 743	5 176 072	5 226 584	51·738	8
9	·00462	100 232	5 075 608	5 125 841	51·022	9
10	·00409	99 796	4 975 608	5 025 609	50·257	10
1	·00370	99 408	4 876 016	4 925 813	49·460	1
2	·00347	99 050	4 776 793	4 826 405	48·643	2
3	·00342	98 708	4 677 916	4 727 355	47·810	3
4	·00342	98 372	4 579 376	4 628 647	46·973	4
15	·00365	98 023	4 481 173	4 530 275	46·132	15
6	·00393	97 651	4 383 330	4 432 252	45·299	6
7	·00437	97 246	4 285 871	4 334 601	44·476	7
8	·00478	96 802	4 188 837	4 237 355	43·669	8
9	·00526	96 315	4 092 268	4 140 553	42·877	9
20	·00572	95 787	3 996 207	4 044 238	42·101	20
1	·00608	95 222	3 900 694	3 948 451	41·339	1
2	·00643	94 626	3 805 763	3 853 229	40·590	2
3	·00668	94 007	3 711 441	3 758 603	39·849	3
4	·00691	93 367	3 617 750	3 664 596	39·114	4
25	·00707	92 715	3 524 706	3 571 229	38·382	25
6	·00720	92 054	3 432 320	3 478 514	37·652	6
7	·00732	91 386	3 340 598	3 386 460	36·921	7
8	·00746	90 710	3 249 549	3 295 074	36·189	8
9	·00759	90 028	3 159 178	3 204 364	35·458	9
30	·00771	89 339	3 069 493	3 114 336	34·726	30
1	·00787	88 644	2 980 499	3 024 997	33·991	1
2	·00803	87 940	2 892 205	2 936 353	33·257	2
3	·00821	87 225	2 804 620	2 848 413	32·521	3
4	·00839	86 502	2 717 754	2 761 188	31·787	4
35	·00862	85 766	2 631 617	2 674 686	31·051	35
6	·00885	85 017	2 546 222	2 588 920	30·317	6
7	·00910	84 254	2 461 583	2 503 903	29·584	7
8	·00937	83 476	2 377 714	2 419 649	28·850	8
9	·00969	82 680	2 294 631	2 336 173	28·118	9
40	·01001	81 865	2 212 354	2 253 493	27·389	40
1	·01038	81 031	2 130 900	2 171 628	26·661	1
2	·01081	80 173	2 050 292	2 090 597	25·935	2
3	·01122	79 289	1 970 555	2 010 424	25·214	3
4	·01172	78 380	1 891 713	1 931 135	24·493	4
45	·01224	77 441	1 813 795	1 852 755	23·778	45
6	·01281	76 471	1 736 831	1 775 314	23·066	6
7	·01345	75 468	1 660 853	1 698 843	22·360	7
8	·01415	74 426	1 585 896	1 623 375	21·658	8
9	·01490	73 346	1 512 000	1 548 949	20·961	9

Table No. I.

The Life Table—(continued).

x	l_x	d_x	μ_x	$\Delta\mu_x$	p_x $=(1-q_x)$	p_x^{-1}	x
50	72 795	1 144	·01542	·00089	·98428	1·0160	50
1	71 651	1 193	·01631	·00096	·98335	1·0169	1
2	70 458	1 243	·01727	·00106	·98236	1·0180	2
3	69 215	1 296	·01833	·00117	·98127	1·0191	3
4	67 919	1 353	·01950	·00127	·98008	1·0203	4
55	66 566	1 414	·02077	·00139	·97877	1·0217	55
6	65 152	1 475	·02216	·00153	·97735	1·0232	6
7	63 677	1 541	·02369	·00167	·97580	1·0248	7
8	62 136	1 612	·02536	·00183	·97407	1·0266	8
9	60 524	1 682	·02719	·00201	·97221	1·0286	9
60	58 842	1 755	·02920	·00220	·97017	1·0307	60
1	57 087	1 830	·03140	·00241	·96794	1·0331	1
2	55 257	1 906	·03381	·00264	·96549	1·0357	2
3	53 351	1 983	·03645	·00289	·96283	1·0386	3
4	51 368	2 059	·03934	·00317	·95993	1·0417	4
65	49 309	2 133	·04251	·00348	·95673	1·0452	65
6	47 176	2 204	·04599	·00380	·95328	1·0490	6
7	44 972	2 273	·04979	·00417	·94947	1·0532	7
8	42 699	2 334	·05396	·00457	·94534	1·0578	8
9	40 365	2 388	·05853	·00500	·94083	1·0629	9
70	37 977	2 434	·06353	·00548	·93590	1·0685	70
1	35 543	2 468	·06901	·00601	·93057	1·0746	1
2	33 075	2 490	·07502	·00658	·92472	1·0814	2
3	30 585	2 496	·08160	·00721	·91840	1·0889	3
4	28 089	2 487	·08881	·00790	·91144	1·0972	4
75	25 602	2 459	·09671	·00865	·90396	1·1062	75
6	23 143	2 412	·10536	·00949	·89578	1·1163	6
7	20 731	2 343	·11485	·01038	·88697	1·1274	7
8	18 388	2 255	·12523	·01139	·87738	1·1398	8
9	16 133	2 146	·13662	·01247	·86696	1·1535	9
80	13 987	2 018	·14909	·01366	·85574	1·1686	80
1	11 969	1 873	·16275	·01497	·84351	1·1855	1
2	10 096	1 712	·17772	·01640	·83042	1·2042	2
3	8 384	1 540	·19412	·01797	·81632	1·2250	3
4	6 844	1 361	·21209	·01968	·80114	1·2482	4
85	5 483	1 180	·23177	·02157	·78478	1·2742	85
6	4 303	1 002	·25343	·02363	·76715	1·3035	6
7	3 301	830	·27697	·02589	·74855	1·3359	7
8	2 471	671	·30286	·02837	·72845	1·3728	8
9	1 800	527	·33123	·03107	·70723	1·4140	9
90	1 273	402	·36230	·03405	·68421	1·4615	90
1	871	296	·39635	·03731	·66016	1·5148	1
2	575	209	·43366	·04087	·63652	1·5710	2
3	366	144	·47453	·04477	·60655	1·6487	3
4	222	93	·51930	·04906	·58109	1·7209	4
95	129	58	·56836	·05375	·55039	1·8169	95
6	71	34	·62211	·05889	·52112	1·9189	6
7	37	18	·68100	·06452	·51351	1·9474	7
8	19	10	·74552	·07069	·47369	2·1111	8
9	9	5	·81621	·07745	·44445	2·2500	9
100	4	3	·89366	·08485	·25000	4·0000	100
1	1	1	·97851	∞	·00000	∞	1
2	0	...	∞	...	...	...	2

Table No. I.

The Life Table—(continued).

x	$q_x = (1-p_x)$	$L_x = \frac{1}{2}(l_x+l_{x+1})$	$N'_x = \Sigma l_{x+1}$	$T_x = \Sigma L_x$	$\overset{\circ}{e}_x = \frac{1}{2}+e_x$	x
50	·01572	72 223	1 439 205	1 475 603	20·271	50
1	·01665	71 054	1 367 554	1 403 380	19·587	1
2	·01764	69 837	1 297 096	1 332 326	18·909	2
3	·01873	68 567	1 227 881	1 262 489	18·240	3
4	·01992	67 242	1 159 962	1 193 922	17·579	4
55	·02123	65 859	1 093 396	1 126 680	16·926	55
6	·02265	64 415	1 028 244	1 060 821	16·282	6
7	·02420	62 906	964 567	996 406	15·648	7
8	·02593	61 330	902 431	933 500	15·023	8
9	·02779	59 683	841 907	872 170	14·410	9
60	·02983	57 965	783 065	812 487	13·808	60
1	·03206	56 172	725 978	754 522	13·217	1
2	·03451	54 304	670 721	698 350	12·638	2
3	·03717	52 359	617 370	644 046	12·072	3
4	·04007	50 339	566 002	591 687	11·519	4
65	·04327	48 242	516 693	541 348	10·979	65
6	·04672	46 074	469 517	493 106	10·452	6
7	·05053	43 836	424 545	447 032	9·940	7
8	·05466	41 532	381 846	403 196	9·443	8
9	·05917	39 171	341 481	361 664	8·960	9
70	·06410	36 760	303 504	322 493	8·492	70
1	·06943	34 309	267 961	285 733	8·039	1
2	·07528	31 830	234 886	251 424	7·602	2
3	·08160	29 337	204 301	219 594	7·180	3
4	·08856	26 845	176 212	190 257	6·773	4
75	·09604	24 373	150 610	163 412	6·383	75
6	·10422	21 937	127 467	139 039	6·008	6
7	·11303	19 559	106 736	117 102	5·649	7
8	·12262	17 261	88 348	97 543	5·305	8
9	·13304	15 060	72 215	80 282	4·976	9
80	·14426	12 978	58 228	65 222	4·663	80
1	·15649	11 033	46 259	52 244	4·365	1
2	·16958	9 240	36 163	41 211	4·082	2
3	·18368	7 614	27 779	31 971	3·813	3
4	·19886	6 164	20 935	24 357	3·559	4
85	·21522	4 893	15 452	18 193	3·318	85
6	·23285	3 802	11 149	13 300	3·091	6
7	·25145	2 886	7 848	9 498	2·877	7
8	·27155	2 135	5 377	6 612	2·676	8
9	·29277	1 537	3 577	4 477	2·487	9
90	·31579	1 072	2 304	2 940	2·310	90
1	·33984	723	1 433	1 868	2·145	1
2	·36348	470	858	1 145	1·992	2
3	·39345	294	492	675	1·844	3
4	·41891	176	270	381	1·716	4
95	·44961	100	141	205	1·593	95
6	·47888	54	70	105	1·486	6
7	·48649	28	33	51	1·392	7
8	·52631	14	14	23	1·237	8
9	·55555	6	5	9	1·056	9
100	·75000	3	1	3	0·750	100
1	1·00000	0	...	...	0·500	1
2	...	...	...	...	0·000	2

TABLE No. II.

Elementary Values derived from the Life Table.

x	$\log l_x$	$\Delta\log l_x$ $=\log p_x$	$\Delta\log p_x$	$\text{Colog}\, l_x$	$\Delta\text{Colog}\, l_x$ $=\text{Colog}\, p_x$	$\Delta\text{Colog}\, p_x$	$\log \mu_x$	$\log L_x$	x
0	5·10477	$\bar{1}$·94802	0·03647	$\bar{6}$·89523	0·05198	$\bar{1}$·96353	$\bar{1}$·20194	5·07954	0
1	·05279	·98449	·00594	·94721	·01551	·99406	$\bar{2}$·89768	·04509	1
2	·03728	·99043	·00281	·96272	·00957	·99719	·37401	·03254	2
3	·02771	·99324	·00124	·97229	·00676	·99876	·25212	·02436	3
4	·02095	·99448	·00105	·97905	·00552	·99895	·13956	·01820	4
5	·01543	·99553	·00085	·98457	·00447	·99915	·05767	·01322	5
6	·01096	·99638	·00069	·98904	·00362	·99931	$\bar{3}$·96614	·00915	6
7	·00734	·99707	·00053	·99266	·00293	·99947	·87390	·00587	7
8	·00441	·99760	·00039	·99559	·00240	·99961	·78319	·00320	8
9	·00201	·99799	·00023	·99799	·00201	·99977	·70070	·00100	9
10	·00000	·99822	·00017	$\bar{5}$·00000	·00178	·99983	·63144	4·99911	10
1	4·99822	·99839	·00010	·00178	·00161	·99990	·58883	·99742	1
2	·99661	·99849	·00002	·00339	·00151	·99998	·55509	·99585	2
3	·99510	·99851	·00000	·00490	·00149	0·00000	·53403	·99435	3
4	·99361	·99851	$\bar{1}$·99990	·00639	·00149	·00010	·53148	·99287	4
15	·99212	·99841	·99988	·00788	·00159	·00012	·54777	·99133	15
6	·99053	·99829	·99981	·00947	·00171	·00019	·57749	·98968	6
7	·98882	·99810	·99982	·01118	·00190	·00018	·61700	·98787	7
8	·98692	·99792	·99979	·01308	·00208	·00021	·66087	·98588	8
9	·98484	·99771	·99980	·01516	·00229	·00020	·70243	·98369	9
20	·98255	·99751	·99984	·01745	·00249	·00016	·74036	·98131	20
1	·98006	·99735	·99985	·01994	·00265	·00015	·77232	·97874	1
2	·97741	·99720	·99989	·02259	·00280	·00011	·79865	·97601	2
3	·97461	·99709	·99990	·02539	·00291	·00010	·81889	·97316	3
4	·97170	·99699	·99993	·02830	·00301	·00007	·83378	·97019	4
25	·96869	·99692	·99994	·03131	·00308	·00006	·84572	·96715	25
6	·96561	·99686	·99995	·03439	·00314	·00005	·85491	·96404	6
7	·96247	·99681	·99994	·03753	·00319	·00006	·86273	·96088	7
8	·95928	·99675	·99994	·04072	·00325	·00006	·87040	·95766	8
9	·95603	·99669	·99995	·04397	·00331	·00005	·87795	·95438	9
30	·95272	·99664	·99993	·04728	·00336	·00007	·88536	·95104	30
1	·94936	·99657	·99993	·05064	·00343	·00007	·89321	·94765	1
2	·94593	·99650	·99992	·05407	·00350	·00008	·90200	·94419	2
3	·94243	·99642	·99992	·05757	·00358	·00008	·91116	·94064	3
4	·93885	·99634	·99990	·06115	·00366	·00010	·92065	·93703	4
35	·93519	·99624	·99990	·06481	·00376	·00010	·93146	·93332	35
6	·93143	·99614	·99989	·06857	·00386	·00011	·94250	·92951	6
7	·92757	·99603	·99988	·07243	·00397	·00012	·95472	·92559	7
8	·92360	·99591	·99986	·07640	·00409	·00014	·96755	·92156	8
9	·91951	·99577	·99986	·08049	·00423	·00014	·98091	·91740	9
40	·91528	·99563	·99984	·08472	·00437	·00016	·99564	·91310	40
1	·91091	·99547	·99981	·08909	·00453	·00019	$\bar{2}$·01072	·90865	1
2	·90638	·99528	·99982	·09362	·00472	·00018	·02694	·90403	2
3	·90166	·99510	·99978	·09834	·00490	·00022	·04376	·89921	3
4	·89676	·99488	·99977	·10324	·00512	·00023	·06183	·89421	4
45	·89164	·99465	·99975	·10836	·00535	·00025	·08063	·88897	45
6	·88629	·99440	·99972	·11371	·00560	·00028	·10037	·88350	6
7	·88069	·99412	·99969	·11931	·00588	·00031	·12090	·87776	7
8	·87481	·99381	·99967	·12519	·00619	·00033	·14239	·87172	8
9	·86862	·99348	·99964	·13138	·00652	·00036	·16495	·86538	9

TABLE No. II.

Elementary Values derived from the Life Table.

x	log d_x	Δlog d_x	$-\Delta$log d_x	log $\frac{1}{2}(p_x^{-1}+1)$	Δlog $\frac{1}{2}(p_x^{-1}+1)$	log $(p_x^{-1}-1)$	Δlog $(p_x^{-1}-1)$	x
0	4·15709	$\bar{1}$·44082	0·55918	0·02675	$\bar{1}$·98106	$\bar{1}$·10430	$\bar{1}$·45633	0
1	3·59791	·77775	·22225	·00781	·99702	$\bar{2}$·56063	·78732	1
2	·37566	·84077	·15923	·00483	·99858	·34795	·84753	2
3	·21643	·90579	·09421	·00341	·99936	·19548	·91131	3
4	·12222	·90350	·09650	·00277	·99949	·10679	·90797	4
5	·02572	·90472	·09528	·00226	·99955	·01476	·90834	5
6	2·93044	·90398	·09602	·00181	·99965	$\bar{3}$·92310	·90691	6
7	·83442	·91144	·08856	·00146	·99973	·83001	·91384	7
8	·74586	·92066	·07934	·00119	·99981	·74385	·92267	8
9	·66652	·94414	·05586	·00100	·99989	·66652	·94592	9
10	·61066	·95637	·04363	·00089	·99992	·61244	·95798	10
1	·56703	·97205	·02795	·00081	·99994	·57042	·97356	1
2	·53908	·98855	·01145	·00075	·99999	·54398	·99004	2
3	·52763	0·00000	·00000	·00074	0·00001	·53402	0·00149	3
4	·52763	·02867	$\bar{1}$·97133	·00075	·00005	·53551	·03026	4
15	·55630	·02803	·97197	·00080	·00006	·56577	·02974	15
6	·58433	·04406	·95594	·00086	·00009	·59551	·04596	6
7	·62839	·03906	·96094	·00095	·00009	·64147	·04114	7
8	·66745	·03841	·96159	·00104	·00010	·68261	·04070	8
9	·70586	·03292	·96708	·00114	·00011	·72331	·03541	9
20	·73878	·02614	·97386	·00125	·00008	·75872	·02879	20
1	·76492	·01970	·98030	·00133	·00007	·78751	·02250	1
2	·78462	·01541	·98459	·00140	·00006	·81001	·01832	2
3	·80003	·01087	·98913	·00146	·00004	·82833	·01388	3
4	·81090	·00733	·99267	·00150	·00004	·84221	·01041	4
25	·81823	·00394	·99606	·00154	·00003	·85262	·00708	25
6	·82217	·00585	·99415	·00157	·00003	·85970	·00904	6
7	·82802	·00321	·99679	·00160	·00003	·86874	·00646	7
8	·83123	·00509	·99491	·00163	·00003	·87520	·00840	8
9	·83632	·00316	·99684	·00166	·00002	·88360	·00652	9
30	·83948	·00562	·99438	·00168	·00004	·89012	·00905	30
1	·84510	·00555	·99445	·00172	·00004	·89917	·00905	1
2	·85065	·00608	·99392	·00176	·00003	·90822	·00966	2
3	·85673	·00600	·99400	·00179	·00005	·91788	·00966	3
4	·86273	·00767	·99233	·00184	·00005	·92754	·01143	4
35	·87040	·00812	·99188	·00189	·00005	·93897	·01198	35
6	·87852	·00797	·99203	·00194	·00005	·95095	·01194	6
7	·88649	·00893	·99107	·00199	·00006	·96289	·01302	7
8	·89542	·01092	·98908	·00205	·00007	·97591	·01515	8
9	·90634	·00906	·99094	·00212	·00007	·99106	·01343	9
40	·91540	·01197	·98803	·00219	·00008	$\bar{2}$·00449	·01650	40
1	·92737	·01265	·98735	·00227	·00010	·02099	·01737	1
2	·94002	·01180	·98820	·00237	·00008	·03836	·01670	2
3	·95182	·01385	·98615	·00245	·00012	·05506	·01897	3
4	·96567	·01388	·98612	·00257	·00011	·07403	·01923	4
45	·97955	·01433	·98567	·00268	·00013	·09326	·01993	45
6	·99388	·01515	·98485	·00281	·00014	·11319	·02103	6
7	3·00903	·01669	·98331	·00295	·00015	·13422	·02288	7
8	·02572	·01607	·98393	·00310	·00017	·15710	·02259	8
9	·04179	·01664	·98336	·00327	·00019	·17969	·02352	9

Table No. II.

Elementary Values derived from the Life Table—(continued).

x	log l_x	Δlog l_x = log p_x	Δlog p_x	Colog l_x	ΔColog l_x = Colog p_x	ΔColog p_x	log μ_x	log L_x	x
50	4·86210	$\bar{1}$·99312	$\bar{1}$·99959	$\bar{5}$·13790	0·00688	0·00041	$\bar{2}$·18808	4·85868	50
1	·85522	·99271	·99956	·14478	·00729	·00044	·21245	·85159	1
2	·84793	·99227	·99952	·15207	·00773	·00048	·23729	·84409	2
3	·84020	·99179	·99947	·15980	·00821	·00053	·26316	·83612	3
4	·83199	·99126	·99942	·16801	·00874	·00058	·29003	·82764	4
55	·82325	·99068	·99937	·17675	·00932	·00063	·31744	·81862	55
6	·81393	·99005	·99931	·18607	·00995	·00069	·34557	·80899	6
7	·80398	·98936	·99923	·19602	·01064	·00077	·37457	·79869	7
8	·79334	·98859	·99917	·20666	·01141	·00083	·40415	·78767	8
9	·78193	·98776	·99909	·21807	·01224	·00091	·43441	·77585	9
60	·76969	·98685	·99900	·23031	·01315	·00100	·46538	·76317	60
1	·75654	·98585	·99890	·24346	·01415	·00110	·49693	·74952	1
2	·74239	·98475	·99880	·25761	·01525	·00120	·52905	·73483	2
3	·72714	·98355	·99869	·27286	·01645	·00131	·56170	·71899	3
4	·71069	·98224	·99855	·28931	·01776	·00145	·59483	·70190	4
65	·69293	·98079	·99843	·30707	·01921	·00157	·62849	·68343	65
6	·67372	·97922	·99826	·32628	·02078	·00174	·66266	·66346	6
7	·65294	·97748	·99811	·34706	·02252	·00189	·69714	·64183	7
8	·63042	·97559	·99792	·36958	·02441	·00208	·73207	·61838	8
9	·60601	·97351	·99772	·39399	·02649	·00228	·76738	·59296	9
70	·57952	·97123	·99752	·42048	·02877	·00248	·80298	·56538	70
1	·55075	·96875	·99726	·44925	·03125	·00274	·83891	·53541	1
2	·51950	·96601	·99702	·48050	·03399	·00298	·87518	·50284	2
3	·48551	·96303	·99670	·51449	·03697	·00330	·91169	·46742	3
4	·44854	·95973	·99642	·55146	·04027	·00358	·94846	·42886	4
75	·40827	·95615	·99605	·59173	·04385	·00395	·98547	·38691	75
6	·36442	·95220	·99571	·63558	·04780	·00429	$\bar{1}$·02268	·34118	6
7	·31662	·94791	·99528	·68338	·05209	·00472	·06013	·29135	7
8	·26453	·94319	·99481	·73547	·05681	·00519	·09771	·23707	8
9	·20772	·93800	·99434	·79228	·06200	·00566	·13551	·17783	9
80	·14572	·93234	·99375	·85428	·06766	·00625	·17345	·11321	80
1	·07806	·92609	·99321	·92194	·07391	·00679	·21152	·04265	1
2	·00415	·91930	·99256	·99585	·08070	·00744	·24974	3·96567	2
3	3·92345	·91186	·99185	$\bar{4}$·07655	·08814	·00815	·28807	·88161	3
4	·83531	·90371	·99104	·16469	·09629	·00896	·32652	·78986	4
85	·73902	·89475	·99013	·26098	·10525	·00987	·36506	·68958	85
6	·63377	·88488	·98934	·36623	·11512	·01066	·40370	·58001	6
7	·51865	·87422	·98818	·48135	·12578	·01182	·44243	·46030	7
8	·39287	·86240	·98716	·60713	·13760	·01284	·48124	·32940	8
9	·25527	·84956	·98563	·74473	·15044	·01437	·52013	·18667	9
90	·10483	·83519	·98446	·89517	·16481	·01554	·55907	·03019	90
1	2·94002	·81965	·98416	$\bar{3}$·05998	·18035	·01584	·59808	2·85914	1
2	·75967	·80381	·97906	·24033	·19619	·02094	·63715	·67210	2
3	·56348	·78287	·98137	·43652	·21713	·01863	·67626	·46835	3
4	·34635	·76424	·97643	·65365	·23576	·02357	·71542	·24551	4
95	·11059	·74067	·97627	·88941	·25933	·02373	·75462	·00000	95
6	1·85126	·71694	·99361	$\bar{2}$·14874	·28306	·00639	·79387	1·73239	6
7	·56820	·71055	·96494	·43180	·28945	·03506	·83315	·44716	7
8	·27875	·67549	·97233	·72125	·32451	·02767	·87246	·14613	8
9	0·95424	·64782	·75012	$\bar{1}$·04576	·35218	·24988	·91180	0·77815	9
100	·60206	·39794	...	·39794	·60206	...	·95117	·47712	100
1	·00000	...	...	0·00000	...	...	·99057	...	1

TABLE No. II.

Elementary Values derived from the Life Table—(continued).

x	log d_x	Δlog d_x	$-\Delta$log d_x	log $\frac{1}{2}(p_x^{-1}+1)$	Δlog $\frac{1}{2}(p_x^{-1}+1)$	log $(p_x^{-1}-1)$	Δlog $(p_x^{-1}-1)$	x
50	3·05843	0·01821	$\bar{1}$·98179	0·00346	0·00020	$\bar{2}$·20321	0·02550	50
1	·07664	·01783	·98217	·00366	·00023	·22871	·02556	1
2	·09447	·01814	·98186	·00389	·00024	·25427	·02635	2
3	·11261	·01869	·98131	·00413	·00026	·28062	·02743	3
4	·13130	·01915	·98085	·00439	·00030	·30805	·02847	4
55	·15045	·01834	·98166	·00469	·00032	·33652	·02829	55
6	·16879	·01901	·98099	·00501	·00034	·36481	·02965	6
7	·18780	·01957	·98043	·00535	·00039	·39446	·03098	7
8	·20737	·01846	·98154	·00574	·00042	·42544	·03070	8
9	·22583	·01845	·98155	·00616	·00047	·45614	·03160	9
60	·24428	·01817	·98183	·00663	·00050	·48774	·03232	60
1	·26245	·01767	·98233	·00713	·00056	·52006	·03292	1
2	·28012	·01720	·98280	·00769	·00061	·55298	·03365	2
3	·29732	·01634	·98366	·00830	·00067	·58663	·03410	3
4	·31366	·01533	·98467	·00897	·00074	·62073	·03454	4
65	·32899	·01422	·98578	·00971	·00081	·65527	·03500	65
6	·34321	·01339	·98661	·01052	·00089	·69027	·03591	6
7	·35660	·01150	·98850	·01141	·00096	·72618	·03591	7
8	·36810	·00993	·99007	·01237	·00107	·76209	·03642	8
9	·37803	·00829	·99171	·01344	·00119	·79851	·03706	9
70	·38632	·00603	·99397	·01463	·00128	·83557	·03728	70
1	·39235	·00385	·99615	·01591	·00142	·87285	·03784	1
2	·39620	·00104	·99896	·01733	·00155	·91069	·03801	2
3	·39724	$\bar{1}$·99844	0·00156	·01888	·00171	·94870	·03871	3
4	·39568	·99508	·00492	·02059	·00190	·98741	·03893	4
75	·39076	·99162	·00838	·02249	·00207	$\bar{1}$·02634	·03942	75
6	·38238	·98739	·01261	·02456	·00226	·06576	·03948	6
7	·36977	·98338	·01662	·02682	·00253	·10524	·04019	7
8	·35315	·97848	·02152	·02935	·00276	·14543	·04048	8
9	·33163	·97329	·02671	·03211	·00304	·18591	·04095	9
80	·30492	·96762	·03238	·03515	·00335	·22686	·04153	80
1	·27254	·96096	·03904	·03850	·00372	·26839	·04166	1
2	·23350	·95402	·04598	·04222	·00408	·31005	·04216	2
3	·18752	·94634	·05366	·04630	·00454	·35221	·04263	3
4	·13386	·93802	·06198	·05084	·00497	·39484	·04327	4
85	·07188	·92899	·07101	·05581	·00555	·43811	·04411	85
6	·00087	·91821	·08179	·06136	·00607	·48222	·04399	6
7	2·91908	·90764	·09236	·06743	·00670	·52621	·04524	7
8	·82672	·89509	·10491	·07413	·00771	·57145	·04553	8
9	·72181	·88242	·11758	·08184	·00833	·61698	·04723	9
90	·60423	·86706	·13294	·09017	·00930	·66421	·04741	90
1	·47129	·84886	·15114	·09947	·00915	·71162	·04505	1
2	·32015	·83821	·16179	·10862	·01338	·75667	·05534	2
3	·15836	·81012	·18988	·12200	·01292	·81201	·04588	3
4	1·96848	·79495	·20505	·13492	·01382	·85789	·05428	4
95	·76343	·76805	·23195	·14874	·01545	·91217	·05111	95
6	·53148	·72379	·27621	·16419	·00422	·96328	·01324	6
7	·25527	·74473	·25527	·16841	·02348	·97652	·06924	7
8	·00000	·69897	·30103	·19189	·01896	0·04576	·05115	8
9	0·69897	·77815	·22185	·21085	·18709	·09691	·38021	9
100	·47712	·52288	·47712	·39794	·60206	·47712	·52288	100
1	·00000	...	...	...	...	...	...	1

TABLE NO. III.

Curtate Expectations of Life. Joint Lives. Equal Ages.

Common Age x	Two Lives e_{xx}	Three Lives e_{xxx}	Four Lives e_{xxxx}	Common Age x	Two Lives e_{xx}	Three Lives e_{xxx}	Four Lives e_{xxxx}
0	30·716	21·291	15·270	50	13·874	10·891	9·014
1	38·023	29·488	23·646	1	13·321	10·421	8·604
2	39·839	31·823	26·278	2	12·776	9·959	8·201
3	40·634	32·997	27·699	3	12·239	9·505	7·806
4	40·919	33·576	28·479	4	11·710	9·060	7·419
5	40·972	33·881	28·964	55	11·191	8·623	7·042
6	40·824	33·944	29·182	6	10·682	8·197	6·673
7	40·512	33·803	29·171	7	10·183	7·780	6·313
8	40·061	33·494	28·968	8	9·694	7·374	5·963
9	39·507	33·054	28·616	9	9·217	6·978	5·623
10	38·874	32·516	28·151	60	8·752	6·594	5·295
1	38·194	31·918	27·616	1	8·298	6·221	4·976
2	37·477	31·275	27·029	2	7·857	5·860	4·669
3	36·740	30·603	26·407	3	7·429	5·511	4·373
4	35·992	29·920	25·772	4	7·013	5·174	4·089
15	35·240	29·229	25·128	65	6·611	4·849	3·815
6	34·499	28·552	24·499	6	6·222	4·537	3·554
7	33·771	27·891	23·887	7	5·847	4·238	3·303
8	33·069	27·260	23·309	8	5·486	3·951	3·064
9	32·387	26·654	22·760	9	5·139	3·677	2·837
20	31·730	26·080	22·246	70	4·806	3·415	2·621
1	31·096	25·531	21·761	1	4·486	3·166	2·416
2	30·478	25·003	21·299	2	4·181	2·929	2·222
3	29·874	24·492	20·856	3	3·889	2·704	2·038
4	29·277	23·989	20·422	4	3·611	2·491	1·865
25	28·686	23·493	19·996	75	3·347	2·289	1·702
6	28·095	22·998	19·571	6	3·095	2·099	1·550
7	27·504	22·503	19·147	7	2·858	1·921	1·407
8	26·912	22·004	18·716	8	2·632	1·752	1·273
9	26·317	21·504	18·285	9	2·419	1·594	1·147
30	25·722	21·000	17·851	80	2·219	1·447	1·031
1	25·123	20·493	17·412	1	2·030	1·308	·923
2	24·522	19·985	16·971	2	1·853	1·180	·823
3	23·921	19·474	16·528	3	1·687	1·060	·731
4	23·319	18·962	16·082	4	1·532	0·949	·646
35	22·715	18·448	15·633	85	1·386	·847	·567
6	22·111	17·933	15·184	6	1·251	·751	·495
7	21·508	17·418	14·733	7	1·125	·664	·430
8	20·906	16·902	14·282	8	1·008	·584	·371
9	20·302	16·386	13·830	9	0·900	·510	·317
40	19·702	15·872	13·380	90	·800	·443	·269
1	19·102	15·358	12·929	1	·708	·382	·226
2	18·505	14·846	12·480	2	·625	·329	·189
3	17·911	14·338	12·035	3	·543	·275	·152
4	17·321	13·832	11·590	4	·477	·234	·125
45	16·734	13·330	11·150	95	·413	·194	·099
6	16·151	12·832	10·713	6	·363	·163	·079
7	15·573	12·338	10·280	7	·335	·151	·073
8	15·001	11·850	9·852	8	·271	·116	·052
9	14·434	11·367	9·430	9	·210	·089	·039
				100	·063	·016	·004
				1	·000	·000	·000

TABLES.

TWO-AND-A-HALF PER CENT.

ONE LIFE.

TABLE NO. IV.—D, N, S, C, M, R.
„ „ V.—LOG D, LOG N, LOG M.
„ „ VI.—*a*, A, P.

TWO, THREE, AND FOUR LIVES.

TABLE NO. VII.—LOG D, LOG N.
„ „ VIII.—*a*.

Constants.

Constant.	Number.	Logarithm.
i	·025	$\bar{2}{\cdot}3979400$
$(1+i)$	1·025	0·0107239
$(1+i)^{\frac{1}{2}}$	1·0124228	0·0053619
$(1+i)^{\frac{1}{4}}$	1·0061922	0·0026810
v	0·9756098	$\bar{1}{\cdot}9892761$
$v^{\frac{1}{2}}$	0·9877296	$\bar{1}{\cdot}9946381$
$v^{\frac{1}{4}}$	0·9938459	$\bar{1}{\cdot}9973190$
d	0·0243902	$\bar{2}{\cdot}3872161$
δ	0·0246926	$\bar{2}{\cdot}3925670$
$j_{(2)}$	0·0251563	$\bar{2}{\cdot}4006459$
$i_{(4)}$	0·0252354	$\bar{2}{\cdot}4020094$

Table No. IV.

Commutation Table.

2½ per-cent.

x	D_x	N_x	S_x	C_x	M_x	R_x
0	127 280	3 001 994	70 539 692	14 008·	50 959·	1 332 478·
1	110 170	2 891 824	67 537 698	3 771·0	36 951·2	1 281 519·4
2	103 710	2 788 114	64 645 874	2 205·4	33 180·2	1 244 568·2
3	98 978	2 689 136	61 857 760	1 491·2	30 974·8	1 211 388·0
4	95 071	2 594 065	59 168 624	1 171·1	29 483·6	1 180 413·2
5	91 582	2 502 483	56 574 559	914·91	28 312·46	1 150 929·60
6	88 434	2 414 049	54 072 076	716·75	27 397·55	1 122 617·14
7	85 560	2 328 489	51 658 027	560·57	26 680·80	1 095 219·59
8	82 913	2 245 576	49 329 538	446·02	26 120·23	1 068 538·79
9	80 445	2 165 131	47 083 962	362·48	25 674·21	1 042 418·56
10	78 120	2 087 011	44 918 831	310·96	25 311·73	1 016 744·35
1	75 903	2 011 108	42 831 820	274·37	25 000·77	991 432·62
2	73 777	1 937 331	40 820 712	251·00	24 726·40	966 431·85
3	71 728	1 865 603	38 883 381	238·51	24 475·40	941 705·45
4	69 740	1 795 863	37 017 778	232·69	24 236·89	917 230·05
15	67 805	1 728 058	35 221 915	242·50	24 004·20	892 993·16
6	65 910	1 662 148	33 493 857	252·36	23 761·70	868 988·96
7	64 049	1 598 099	31 831 709	272·50	23 509·34	845 227·26
8	62 214	1 535 885	30 233 610	290·87	23 236·84	821 717·92
9	60 407	1 475 478	28 697 725	310·01	22 945·97	798 481·08
20	58 623	1 416 855	27 222 247	326·27	22 635·96	775 535·11
1	56 867	1 359 988	25 805 392	338·06	22 309·69	752 899·15
2	55 142	1 304 846	24 445 404	345·12	21 971·63	730 589·46
3	53 452	1 251 394	23 140 558	348·87	21 626·51	708 617·83
4	51 800	1 199 594	21 889 164	348·98	21 277·64	686 991·32
25	50 187	1 149 407	20 689 570	346·27	20 928·66	665 713·68
6	48 617	1 100 790	19 540 163	340·90	20 582·39	644 785·02
7	47 090	1 053 700	18 439 373	337·09	20 241·49	624 202·63
8	45 605	1 008 095	17 385 673	331·31	19 904·40	603 961·14
9	44 161	963 934	16 377 578	327·04	19 573·09	584 056·74
30	42 756	921 178	15 413 644	321·40	19 246·05	564 483·65
1	41 392	879 786	14 492 466	317·64	18 924·65	545 237·60
2	40 065	839 721	13 612 680	313·88	18 607·01	526 312·95
3	38 774	800 947	12 772 959	310·54	18 293·13	507 705·94
4	37 518	763 429	11 972 012	307·18	17 982·59	489 412·81
35	36 295	727 134	11 208 583	305·03	17 675·41	471 430·22
6	35 105	692 029	10 481 449	303·21	17 370·38	453 754·81
7	33 946	658 083	9 789 420	301·29	17 067·17	436 384·43
8	32 816	625 267	9 131 337	300·05	16 765·88	419 317·26
9	31 716	593 551	8 506 070	300·19	16 465·83	402 551·38
40	30 643	562 908	7 912 519	299·03	16 165·64	386 085·55
1	29 596	533 312	7 349 611	299·90	15 866·61	369 919·91
2	28 575	504 737	6 816 299	301·22	15 566·71	354 053·30
3	27 576	477 161	6 311 562	301·97	15 265·49	338 486·59
4	26 602	450 559	5 834 401	304·16	14 963·52	323 221·10
45	25 649	424 910	5 383 842	306·37	14 659·36	308 257·58
6	24 717	400 193	4 958 932	308·93	14 352·99	293 598·22
7	23 805	376 388	4 558 739	312·09	14 044·06	279 245·23
8	22 912	353 476	4 182 351	316·41	13 731·97	265 201·17
9	22 037	331 439	3 828 875	320·33	13 415·56	251 469·20

TABLE NO. IV.

Commutation Table.

2½ per-cent.

x	D_x	N_x	S_x	C_x	M_x	R_x
50	21 179·	310 260·	3 497 436·	324·72	13 095·23	238 053·64
1	20 338·	289 922·	3 187 176·	330·37	12 770·51	224 958·41
2	19 511·	270 411·	2 897 254·	335·82	12 440·14	212 187·90
3	18 700·	251 711·	2 626 843·	341·60	12 104·32	199 747·76
4	17 902·	233 809·	2 375 132·	347·93	11 762·72	187 643·44
55	17 117·	216 692·	2 141 323·	354·74	11 414·79	175 880·72
6	16 345·	200 347·	1 924 631·	361·02	11 060·05	164 465·93
7	15 585·	184 762·	1 724 284·	367·98	10 699·03	153 405·88
8	14 837·	169 925·	1 539 522·	375·54	10 331·05	142 706·85
9	14 100·	155 825·	1 369 597·	382·30	9 955·51	132 375·80
60	13 374·	142 451·	1 213 772·	389·15	9 573·21	122 420·29
1	12 658·	129 793·	1 071 321·	395·89	9 184·06	112 847·08
2	11 954·	117 839·	941 528·	402·27	8 788·17	103 663·02
3	11 260·	106 579·	823 689·	408·31	8 385·90	94 874·85
4	10 577·	96 002·0	717 110·5	413·63	7 977·59	86 488·95
65	9 905·6	86 096·4	621 108·5	418·03	7 563·96	78 511·36
6	9 245·7	76 850·7	535 012·1	421·42	7 145·93	70 947·40
7	8 598·8	68 251·9	458 161·4	424·01	6 724·51	63 801·47
8	7 965·3	60 286·6	389 909·5	424·77	6 300·50	57 076·96
9	7 346·2	52 940·4	329 622·9	423·99	5 875·73	50 776·46
70	6 743·0	46 197·4	276 682·5	421·63	5 451·74	44 900·73
1	6 156·9	40 040·5	230 485·1	417·09	5 030·11	39 448·99
2	5 589·6	34 450·9	190 444·6	410·54	4 613·02	34 418·88
3	5 042·8	29 408·1	155 993·7	401·49	4 202·48	29 805·86
4	4 518·2	24 889·9	126 585·6	390·29	3 800·99	25 603·38
75	4 017·7	20 872·2	101 695·7	376·49	3 410·70	21 802·39
6	3 543·3	17 328·9	80 823·5	360·28	3 034·21	18 391·69
7	3 096·6	14 232·3	63 494·6	341·44	2 673·93	15 357·48
8	2 679·6	11 552·7	49 262·3	320·60	2 332·49	12 683·55
9	2 293·7	9 259·0	37 709·6	297·66	2 011·89	10 351·06
80	1 940·0	7 319·0	28 450·6	273·08	1 714·23	8 339·17
1	1 619·7	5 699·3	21 131·6	247·27	1 441·15	6 624·94
2	1 332·9	4 366·4	15 432·3	220·51	1 193·88	5 183·79
3	1 079·9	3 286·47	11 065·86	193·52	973·37	3 989·91
4	860·02	2 426·45	7 779·39	166·85	779·85	3 016·54
85	672·19	1 754·26	5 352·94	141·13	613·00	2 236·69
6	514·66	1 239·60	3 598·68	116·92	471·87	1 623·69
7	385·19	854·41	2 359·08	94·489	354·952	1 151·821
8	281·30	573·11	1 504·67	74·525	260·463	796·869
9	199·92	373·19	931·56	57·103	185·938	536·406
90	137·94	235·252	558·368	42·497	128·835	350·468
1	92·077	143·175	323·116	30·527	86·338	221·633
2	59·302	83·873	179·941	21·030	55·811	135·295
3	36·826	47·047	96·068	14·136	34·781	79·484
4	21·793	25·254	49·021	8·9066	20·6452	44·7029
95	12·354	12·900	23·767	5·4193	11·7386	24·0577
6	6·634	6·266	10·867	3·0993	6·3193	12·3191
7	3·373	2·893	4·601	1·6008	3·2200	5·9998
8	1·690	1·203	1·708	·8676	1·6192	2·7798
9	·781	·422	·505	·4232	·7516	1·1606
100	·339	·083	·083	·2478	·3284	·4090
1	·083	...	...	·0806	·0806	·0806

Table No. V.

Logarithms of D_x, N_x, *and* M_x.

$2\frac{1}{2}$ *per-cent.*

x	log D_x	log N_x	log M_x	x	log D_x	log N_x	log M_x
0	5·10477	6·47741	4·70722	50	4·32591	5·49173	4·11711
1	·04207	·46117	·56763	1	·30830	·46228	·10622
2	·01583	·44531	·52088	2	·29029	·43202	·09482
3	4·99554	·42961	·49101	3	·27184	·40090	·08293
4	·97805	·41399	·46959	4	·25290	·36886	·07052
5	·96181	·39837	·45197	55	·23344	·33584	·05748
6	·94662	·38274	·43772	6	·21339	·30179	·04376
7	·93227	·36708	·42620	7	·19272	·26661	·02934
8	·91862	·35133	·41697	8	·17136	·23027	·01414
9	·90550	·33548	·40949	9	·14922	·19265	3·99806
10	·89276	·31952	·40333	60	·12626	·15366	·98106
1	·88026	·30343	·39796	1	·10238	·11324	·96304
2	·86792	·28720	·39315	2	·07751	·07129	·94390
3	·85569	·27082	·38872	3	·05154	·02768	·92355
4	·84348	·25428	·38448	4	·02436	4·98228	·90187
15	·83126	·23757	·38028	65	3·99588	·93498	·87875
6	·81895	·22066	·37588	6	·96594	·88565	·85406
7	·80651	·20360	·37123	7	·93444	·83412	·82766
8	·79389	·18636	·36618	8	·90120	·78022	·79938
9	·78109	·16894	·36071	9	·86606	·72378	·76906
20	·76807	·15134	·35480	70	·82885	·66461	·73653
1	·75486	·13354	·34850	1	·78936	·60250	·70158
2	·74148	·11554	·34187	2	·74738	·53720	·66398
3	·72796	·09740	·33500	3	·70267	·46847	·62351
4	·71433	·07904	·32793	4	·65497	·39602	·57990
25	·70059	·06047	·32075	75	·60398	·31956	·53284
6	·68679	·04171	·31349	6	·54941	·23877	·48204
7	·67293	·02272	·30623	7	·49088	·15327	·42715
8	·65901	·00350	·29894	8	·42807	·06269	·36782
9	·64504	5·98405	·29166	9	·36053	3·96656	·30361
30	·63100	·96434	·28434	80	·28781	·86445	·23406
1	·61692	·94438	·27704	1	·20943	·75582	·15872
2	·60277	·92413	·26968	2	·12479	·64012	·07697
3	·58854	·90361	·26228	3	·03337	·51673	2·98828
4	·57424	·88277	·25486	4	2·93451	·38498	·89201
35	·55985	·86161	·24736	85	·82749	·24410	·78746
6	·54537	·84012	·23980	6	·71152	·09328	·67382
7	·53079	·81828	·23216	7	·58567	2·93167	·55017
8	·51609	·79607	·22443	8	·44917	·75824	·41574
9	·50128	·77346	·21659	9	·30085	·57193	·26937
40	·48633	·75044	·20860	90	·13968	·37153	·11005
1	·47123	·72698	·20049	1	1·96415	·15588	1·93620
2	·45598	·70307	·19220	2	·77307	1·92362	·74672
3	·44053	·67866	·18370	3	·56616	·67253	·54134
4	·42491	·65375	·17505	4	·33831	·40233	·31481
45	·40907	·62830	·16610	95	·09182	·11059	·06963
6	·39299	·60227	·15694	6	0·82177	0·79696	0·80067
7	·37667	·57564	·14749	7	·52799	·46130	·50786
8	·36006	·54836	·13773	8	·22781	·08027	·20930
9	·34315	·52040	·12762	9	$\bar{1}$·89258	$\bar{1}$·62446	$\bar{1}$·87596
				100	·52967	$\bar{2}$·91689	·51630
				1	$\bar{2}$·91689	...	$\bar{2}$·90617

TABLE NO. VI.

Value of Annuities, and Single and Annual Premiums for Assurance of a Unit.

2½ per-cent.

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	23·585	·40036	·01628	50	14·649	·61830	·03951
1	26·248	·33540	·01231	1	14·255	·62794	·04116
2	26·883	·31993	·01147	2	13·859	·63757	·04291
3	27·169	·31295	·01111	3	13·460	·64728	·04476
4	27·286	·31013	·01096	4	13·061	·65708	·04673
5	27·325	·30914	·01091	55	12·659	·66687	·04882
6	27·297	·30981	·01095	6	12·257	·67666	·05104
7	27·215	·31184	·01105	7	11·855	·68647	·05340
8	27·084	·31503	·01122	8	11·453	·69627	·05592
9	26·914	·31915	·01143	9	11·052	·70606	·05859
10	26·715	·32402	·01169	60	10·651	·71581	·06143
1	26·495	·32938	·01198	1	10·253	·72554	·06447
2	26·259	·33514	·01230	2	9·858	·73517	·06771
3	26·009	·34122	·01263	3	9·465	·74475	·07116
4	25·751	·34754	·01299	4	9·077	·75424	·07485
15	25·486	·35401	·01337	65	8·692	·76361	·07879
6	25·218	·36052	·01375	6	8·312	·77289	·08300
7	24·951	·36705	·01414	7	7·937	·78202	·08750
8	24·687	·37350	·01454	8	7·569	·79101	·09231
9	24·426	·37986	·01494	9	7·206	·79983	·09746
20	24·170	·38613	·01534	70	6·851	·80850	·10298
1	23·916	·39232	·01575	1	6·503	·81700	·10889
2	23·662	·39846	·01616	2	6·163	·82528	·11521
3	23·412	·40461	·01658	3	5·832	·83337	·12199
4	23·158	·41077	·01700	4	5·509	·84126	·12925
25	22·902	·41702	·01745	75	5·195	·84891	·13703
6	22·642	·42335	·01791	6	4·891	·85631	·14537
7	22·376	·42983	·01839	7	4·596	·86352	·15431
8	22·105	·43645	·01889	8	4·311	·87046	·16389
9	21·828	·44322	·01942	9	4·037	·87716	·17415
30	21·545	·45013	·01997	80	3·773	·88359	·18514
1	21·255	·45721	·02055	1	3·519	·88980	·19691
2	20·958	·46442	·02115	2	3·276	·89574	·20948
3	20·657	·47178	·02179	3	3·043	·90138	·22293
4	20·348	·47931	·02245	4	2·821	·90678	·23729
35	20·034	·48698	·02315	85	2·610	·91195	·25263
6	19·713	·49480	·02389	6	2·409	·91685	·26898
7	19·386	·50277	·02466	7	2·218	·92151	·28635
8	19·054	·51090	·02548	8	2·037	·92591	·30484
9	18·715	·51917	·02633	9	1·867	·93008	·32444
40	18·370	·52756	·02724	90	1·706	·93405	·34524
1	18·020	·53612	·02819	1	1·555	·93767	·36700
2	17·664	·54478	·02919	2	1·414	·94113	·38980
3	17·303	·55357	·03024	3	1·278	·94445	·41469
4	16·937	·56252	·03136	4	1·159	·94733	·43881
45	16·566	·57152	·03254	95	1·044	·95019	·46484
6	16·191	·58070	·03378	6	·944	·95258	·48987
7	15·811	·58996	·03509	7	·858	·95471	·51393
8	15·428	·59934	·03648	8	·712	·95827	·55976
9	15·040	·60879	·03795	9	·539	·96245	·62473
				100	·244	·96968	·77954
				1	...	·97561	·97561

TABLE No. VII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.* 2½ *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
0	10·20954	11·43908	15·31431	16·41387	20·41908	21·39651	0
1	·09486	·41896	·14765	·38968	·20044	·36795	1
2	·05311	·39985	·09039	·36732	·12767	·34223	2
3	·02325	·38121	·05096	·34582	·07867	·31788	3
4	9·99900	·36280	·01995	·32482	·04090	·29431	4
5	·97724	·34455	14·99267	·30412	·00810	·27124	5
6	·95758	·32636	·96854	·28360	19·97950	·24846	6
7	·93961	·30816	·94695	·26312	·95429	·22583	7
8	·92303	·28988	·92744	·24259	·93185	·20317	8
9	·90751	·27149	·90952	·22194	·91153	·18038	9
10	·89276	·25295	·89276	·20110	·89276	·15740	10
1	·87848	·23421	·87670	·18001	·87492	·13411	1
2	·86453	·21527	·86114	·15863	·85775	·11052	2
3	·85079	·19609	·84589	·13697	·84099	·08650	3
4	·83709	·17667	·83070	·11498	·82431	·06209	4
15	·82338	·15697	·81550	·09262	·80762	·03723	15
6	·80948	·13701	·80001	·06989	·79054	·01191	6
7	·79533	·11677	·78415	·04681	·77297	20·98609	7
8	·78081	·09625	·76773	·02333	·75465	·95982	8
9	·76593	·07547	·75077	15·99950	·73561	·93310	9
20	·75062	·05438	·73317	·97533	·71572	·90594	20
1	·73492	·03306	·71498	·95080	·69504	·87836	1
2	·71889	·01149	·69630	·92593	·67371	·85036	2
3	·70257	10·98961	·67718	·90072	·65179	·82197	3
4	·68603	·96747	·65773	·87516	·62943	·79315	4
25	·66928	·94504	·63797	·84925	·60666	·76392	25
6	·65240	·92232	·61801	·82297	·58362	·73426	6
7	·63540	·89929	·59787	·79630	·56034	·70414	7
8	·61829	·87593	·57757	·76923	·53685	·67354	8
9	·60107	·85224	·55710	·74173	·51313	·64243	9
30	·58372	·82818	·53644	·71378	·48916	·61080	30
1	·56628	·80374	·51564	·68535	·46500	·57858	1
2	·54870	·77890	·49463	·65641	·44056	·54578	2
3	·53097	·75364	·47340	·62695	·41583	·51232	3
4	·51309	·72794	·45194	·59690	·39079	·47818	4
35	·49504	·70177	·43023	·56626	·36542	·44333	35
6	·47680	·67509	·40823	·53498	·33966	·40770	6
7	·45836	·64789	·38593	·50301	·31350	·37123	7
8	·43969	·62014	·36329	·47034	·28689	·33391	8
9	·42079	·59178	·34030	·43688	·25981	·29568	9
40	·40161	·56280	·31689	·40262	·23217	·25643	40
1	·38214	·53317	·29305	·36749	·20396	·21611	1
2	·36236	·50282	·26874	·33145	·17512	·17470	2
3	·34219	·47173	·24385	·29440	·14551	·13207	3
4	·32167	·43985	·21843	·25633	·11519	·08814	4
45	·30071	·40714	·19235	·21714	·08399	·04285	45
6	·27928	·37350	·16557	·17673	·05186	19·99606	6
7	·25736	·33893	·13805	·13507	·01874	·94771	7
8	·23487	·30333	·10968	·09205	18·98449	·89767	8
9	·21177	·26666	·08039	·04759	·94901	·84582	9

TABLE No. VII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*
$2\frac{1}{2}$ *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
50	9·18801	10·22881	14·05011	15·00152	18·91221	19·79201	50
1	·16352	·18974	·01874	14·95382	·87396	·73612	1
2	·13822	·14934	13·98615	·90432	·83408	·67797	2
3	·11204	·10755	·95224	·85289	·79244	·61742	3
4	·08489	·06423	·91688	·79940	·74887	·55426	4
55	·05669	·01932	·87994	·74370	·70319	·48830	55
6	·02732	9·97265	·84125	·68562	·65518	·41933	6
7	8·99670	·92413	·80068	·62496	·60466	·34710	7
8	·96470	·87363	·75804	·56155	·55138	·27133	8
9	·93115	·82098	·71308	·49517	·49501	·19179	9
60	·89595	·76603	·66564	·42557	·43533	·10813	60
1	·85892	·70862	·61546	·35251	·37200	·02003	1
2	·81990	·64853	·56229	·27570	·30468	18·92713	2
3	·77868	·58557	·50582	·19485	·23296	·82901	3
4	·73505	·51953	·44574	·10961	·15643	·72525	4
65	·68881	·45013	·38174	·01966	·07467	·61535	65
6	·63966	·37714	·31338	13·92453	17·98710	·49881	6
7	·58738	·30025	·24032	·82385	·89326	·37502	7
8	·53162	·21914	·16204	·71713	·79246	·24339	8
9	·47207	·13344	·07808	·60385	·68409	·10319	9
70	·40837	·04281	12·98789	·48344	·56741	17·95370	70
1	·34011	8·94685	·89086	·35530	·44161	·79409	1
2	·26688	·84505	·78638	·21872	·30588	·62344	2
3	·18818	·73695	·67369	·07298	·15920	·44075	3
4	·10351	·62200	·55205	12·91725	·00059	·24492	4
75	·01225	·49959	·42052	·75065	16·82879	·03479	75
6	7·91383	·36909	·27825	·57217	·64267	16·80902	6
7	·80750	·22976	·12412	·38079	·44074	·56621	7
8	·69260	·08084	11·95713	·17528	·22166	·30475	8
9	·56825	7·92145	·77597	11·95435	15·98369	·02288	9
80	·43353	·75066	·57925	·71659	·72497	15·71869	80
1	·28749	·56743	·36555	·46040	·44361	·39007	1
2	·12894	·37064	·13309	·18415	·13724	·03483	2
3	6·95682	·15906	10·88027	10·88594	14·80372	14·65043	3
4	·76982	6·93131	·60513	·56371	·44044	·23409	4
85	·56651	·68588	·30553	·21514	·04455	13·78267	85
6	·34529	·42109	9·97906	9·83771	13·61283	·29279	6
7	·10432	·13526	·62297	·42888	·14162	12·76108	7
8	5·84204	5·82626	·23491	8·98541	12·62778	·18318	8
9	·55612	·49196	8·81139	·50412	·06666	11·55479	9
90	·24451	·12982	·34934	7·98097	11·45417	10·87035	90
1	4·90417	4·73755	7·84419	·41259	10·78421	·12538	1
2	·53274	·31273	·29241	6·79598	·05208	9·31660	2
3	·12964	3·84930	6·69312	·12001	9·25660	8·42734	3
4	3·68466	·34832	·03101	5·38798	8·37736	7·46356	4
95	·20241	2·80371	5·31300	4·58814	7·42359	6·40780	95
6	2·67303	·21843	4·52429	3·72395	6·37555	5·26250	6
7	·09619	1·60819	3·66439	2·83169	5·23259	4·08565	7
8	1·50656	0·92761	2·78531	1·83727	4·06406	2·77159	8
9	0·84682	·15743	1·80106	0·74036	2·75530	1·33752	9
100	·13173	$\bar{2}$·91689	0·73379	$\bar{2}$·91689	1·33585	$\bar{2}$·91689	100
1	$\bar{2}$·91689	...	$\bar{2}$·91689	...	$\bar{2}$·91689	...	1

Table No. VIII.

Values of Annuities. Joint Lives. Equal Ages.
$2\frac{1}{2}$ *per-cent.*

x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}	x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}
0	16·964	12·577	9·494	50	10·985	8·942	7·582
1	21·091	17·459	14·707	1	10·622	8·612	7·281
2	22·220	18·920	16·389	2	10·259	8·283	6·981
3	22·801	19·718	17·346	3	9·897	7·955	6·683
4	23·110	20·178	17·923	4	9·535	7·630	6·388
5	23·298	20·486	18·329	55	9·176	7·307	6·097
6	23·377	20·657	18·576	6	8·817	6·988	5·810
7	23·364	20·710	18·687	7	8·461	6·672	5·526
8	23·273	20·661	18·678	8	8·108	6·361	5·248
9	23·120	20·531	18·572	9	7·759	6·055	4·975
10	22·919	20·339	18·392	60	7·415	5·754	4·708
1	22·685	20·105	18·163	1	7·075	5·458	4·447
2	22·425	19·838	17·897	2	6·740	5·169	4·192
3	22·146	19·547	17·600	3	6·411	4·887	3·945
4	21·856	19·243	17·289	4	6·088	4·612	3·705
15	21·557	18·929	16·967	65	5·772	4·344	3·473
6	21·258	18·616	16·648	6	5·464	4·085	3·249
7	20·962	18·309	16·335	7	5·163	3·833	3·032
8	20·675	18·014	16·039	8	4·870	3·590	2·824
9	20·396	17·731	15·758	9	4·585	3·356	2·625
20	20·126	17·465	15·496	70	4·310	3·130	2·434
1	19·867	17·212	15·252	1	4·043	2·914	2·252
2	19·616	16·968	15·019	2	3·786	2·706	2·078
3	19·366	16·732	14·797	3	3·538	2·508	1·912
4	19·118	16·498	14·579	4	3·300	2·319	1·755
25	18·869	16·266	14·363	75	3·071	2·139	1·607
6	18·617	16·031	14·146	6	2·853	1·968	1·467
7	18·361	15·792	13·925	7	2·644	1·806	1·335
8	18·098	15·547	13·699	8	2·445	1·653	1·211
9	17·831	15·298	13·468	9	2·255	1·508	1·094
30	17·557	15·043	13·232	80	2·076	1·372	·986
1	17·277	14·781	12·989	1	1·905	1·244	·884
2	16·990	14·514	12·741	2	1·745	1·125	·790
3	16·698	14·241	12·488	3	1·593	1·013	·703
4	16·400	13·962	12·229	4	1·450	·909	·622
35	16·096	13·678	11·965	85	1·316	·812	·547
6	15·787	13·389	11·696	6	1·191	·722	·479
7	15·471	13·094	11·422	7	1·074	·640	·416
8	15·151	12·795	11·143	8	·964	·563	·359
9	14·825	12·491	10·861	9	·863	·493	·308
40	14·494	12·182	10·575	90	·768	·428	·261
1	14·159	11·870	10·284	1	·681	·370	·219
2	13·818	11·553	9·990	2	·603	·319	·184
3	13·475	11·234	9·695	3	·524	·267	·148
4	13·127	10·912	9·396	4	·461	·227	·122
45	12·777	10·587	9·096	95	·399	·188	·096
6	12·423	10·260	8·794	6	·351	·158	·077
7	12·066	9·932	8·491	7	·325	·147	·071
8	11·707	9·602	8·188	8	·264	·113	·051
9	11·347	9·273	7·885	9	·204	·087	·038
				100	·061	·015	·004

TABLES.

THREE PER CENT.

ONE LIFE.

Table No. IX.—D, N, S, C, M, R.
" " X.—Log D, Log N, Log M.
" " XI.—*a*, A, P.

TWO, THREE, AND FOUR LIVES.

Table No. XII.—Log D, Log N.
" " XIII.—*a*.

Constants.

Constant.	Number.	Logarithm.
i	·03	$\bar{2}$·4771213
$(1+i)$	1·03	0·0128372
$(1+i)^{\frac{1}{2}}$	1·0148892	0·0064186
$(1+i)^{\frac{1}{4}}$	1·0074171	0·0032093
v	0·9708738	$\bar{1}$·9871628
$v^{\frac{1}{2}}$	0·9853293	$\bar{1}$·9935814
$v^{\frac{1}{4}}$	0·9926375	$\bar{1}$·9967907
d	0·0291262	$\bar{2}$·4642840
δ	0·0295588	$\bar{2}$·4706868
$j_{(2)}$	0·0297783	$\bar{2}$·4738999
$j_{(4)}$	0·0296683	$\bar{2}$·4722927

Table No. IX.

Commutation Table.

3 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
0	127 283	2 688 021	58 943 170	13 940·	45 284·	1 016 517·
1	109 636	2 578 385	56 255 149	3 734·6	31 344·2	971 233·2
2	102 708	2 475 677	53 676 764	2 173·5	27 609·6	939 889·0
3	97 544	2 378 133	51 201 087	1 462·4	25 436·1	912 279·4
4	93 240	2 284 893	48 822 954	1 143·0	23 973·7	886 843·3
5	89 380	2 195 513	46 538 061	888·59	22 830·68	862 869·57
6	85 889	2 109 624	44 342 548	692·76	21 942·09	840 038·89
7	82 695	2 026 929	42 232 924	539·16	21 249·33	818 096·80
8	79 746	1 947 183	40 205 995	426·89	20 710·17	796 847·47
9	76 996	1 870 187	38 258 812	345·26	20 283·28	776 137·30
10	74 410	1 795 777	36 388 625	294·75	19 938·02	755 854·02
1	71 947	1 723 830	34 592 848	258·81	19 643·27	735 916·00
2	69 592	1 654 238	32 869 018	235·61	19 384·46	716 272·73
3	67 332	1 586 906	31 214 780	222·80	19 148·85	696 888·27
4	65 146	1 521 760	29 627 874	216·31	18 926·05	677 739·42
15	63 032	1 458 728	28 106 114	224·34	18 709·74	658 813·37
6	60 972	1 397 756	26 647 386	232·33	18 485·40	640 103·63
7	58 964	1 338 792	25 249 630	249·64	18 253·07	621 618·23
8	56 997	1 281 795	23 910 838	265·18	18 003·43	603 365·16
9	55 072	1 226 723	22 629 043	281·27	17 738·25	585 361·73
20	53 188	1 173 535	21 402 320	294·58	17 456·98	567 623·48
1	51 343	1 122 192	20 228 785	303·74	17 162·40	550 166·50
2	49 544	1 072 648	19 106 593	308·57	16 858·66	533 004·10
3	47 791	1 024 857	18 033 945	310·41	16 550·09	516 145·44
4	46 090	978 767	17 009 088	309·01	16 239·68	499 595·35
25	44 439	934 328	16 030 321	305·11	15 930·67	483 355·67
6	42 839	891 489	15 095 993	298·92	15 625·56	467 425·00
7	41 291	850 198	14 204 504	294·16	15 326·64	451 799·44
8	39 796	810 402	13 354 306	287·71	15 032·48	436 472·80
9	38 349	772 053	12 543 904	282·62	14 744·77	421 440·32
30	36 949	735 104	11 771 851	276·39	14 462·15	406 695·55
1	35 597	699 507	11 036 747	271·84	14 185·76	392 233·40
2	34 288	665 219	10 337 240	267·31	13 913·92	378 047·64
3	33 022	632 197	9 672 021	263·18	13 646·61	364 133·72
4	31 797	600 400	9 039 824	259·08	13 383·43	350 487·11
35	30 612	569 788	8 439 424	256·01	13 124·35	337 103·68
6	29 464	540 324	7 869 636	253·24	12 868·34	323 979·33
7	28 352	511 972	7 329 312	250·43	12 615·10	311 110·99
8	27 277	484 695	6 817 340	248·18	12 364·67	298 495·89
9	26 234	458 461	6 332 645	247·09	12 116·49	286 131·22
40	25 223	433 238	5 874 184	244·95	11 869·40	274 014·73
1	24 243	408 995	5 440 946	244·46	11 624·45	262 145·33
2	23 293	385 702	5 031 951	244·35	11 379·99	250 520·88
3	22 370	363 332	4 646 249	243·77	11 135·64	239 140·89
4	21 474	341 858	4 282 917	244·34	10 891·87	228 005·25
45	20 604	321 254	3 941 059	244·93	10 647·53	217 113·38
6	19 760	301 494	3 619 805	245·77	10 402·60	206 465·85
7	18 938	282 556	3 318 311	247·08	10 156·83	196 063·25
8	18 139	264 417	3 035 755	249·29	9 909·75	185 906·42
9	17 362	247 055	2 771 338	251·15	9 660·46	175 996·67

TABLE No. IX.

Commutation Table.

3 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
50	16 605·	230 450·	2 524 283·	253·36	9 409·31	166 336·21
1	15 868·	214 582·	2 293 833·	256·51	9 155·95	156 926·90
2	15 149·	199 433·	2 079 251·	259·48	8 899·44	147 770·95
3	14 449·	184 984·	1 879 818·	262·66	8 639·96	138 871·51
4	13 765·	171 219·	1 694 834·	266·23	8 377·30	130 231·55
55	13 098·	158 121·	1 523 615·	270·13	8 111·07	121 854·25
6	12 447·	145 674·	1 365 494·	273·57	7 840·94	113 743·18
7	11 810·	133 864·	1 219 820·	277·49	7 567·37	105 902·24
8	11 189·	122 675·	1 085 956·	281·82	7 289·88	98 334·87
9	10 581·	112 093·8	963 280·6	285·50	7 008·06	91 044·99
60	9 987·6	1 02 106·2	851 186·8	289·21	6 722·56	84 036·93
1	9 407·4	92 698·8	749 080·6	292·78	6 433·35	77 314·37
2	8 840·6	83 858·2	656 381·8	296·05	6 140·57	70 881·02
3	8 286·9	75 571·3	572 523·6	299·05	5 844·52	64 740·45
4	7 746·6	67 824·7	496 952·3	301·47	5 545·47	58 895·93
65	7 219·5	60 605·2	429 127·6	303·20	5 244·00	53 350·46
6	6 705·9	53 899·3	368 522·4	304·17	4 940·80	48 106·46
7	6 206·5	47 692·8	314 623·1	304·56	4 636·63	43 165·66
8	5 721·2	41 971·6	266 930·3	303·62	4 332·07	38 529·03
9	5 251·0	36 720·6	224 958·7	301·59	4 028·45	34 196·96
70	4 796·3	31 924·3	188 238·1	298·46	3 726·86	30 168·51
1	4 358·2	27 566·1	156 313·8	293·81	3 428·40	26 441·65
2	3 937·5	23 628·6	128 747·7	287·79	3 134·59	23 013·25
3	3 535·0	20 093·6	105 119·1	280·09	2 846·80	19 878·66
4	3 152·0	16 941·6	85 025·5	270·95	2 566·71	17 031·86
75	2 789·2	14 152·4	68 083·9	260·09	2 295·76	14 465·15
6	2 447·9	11 704·5	53 931·5	247·69	2 035·67	12 169·39
7	2 128·9	9 575·6	42 227·0	233·60	1 787·98	10 133·72
8	1 833·3	7 742·3	32 651·4	218·28	1 554·38	8 345·74
9	1 561·6	6 180·7	24 909·1	201·67	1 336·10	6 791·36
80	1 314·4	4 866·3	18 728·4	184·12	1 134·43	5 455·26
1	1 092·0	3 774·27	13 862·06	165·92	950·31	4 320·83
2	894·33	2 879·94	10 087·79	147·23	784·39	3 370·52
3	721·04	2 158·90	7 207·85	128·58	637·16	2 586·13
4	571·45	1 587·45	5 048·95	110·33	508·58	1 948·97
85	444·49	1 142·96	3 461·50	92·871	398·246	1 440·392
6	338·66	804·30	2 318·54	76·565	305·375	1 042·146
7	252·24	552·06	1 514·24	61·574	228·810	736·771
8	183·31	368·75	962·18	48·329	167·236	507·961
9	129·65	239·100	593·425	36·852	118·907	340·725
90	89·018	150·082	354·325	27·292	82·055	221·818
1	59·133	90·949	204·243	19·511	54·763	139·763
2	37·901	53·048	113·294	13·375	35·252	85·000
3	23·422	29·626	60·246	8·9466	21·8765	49·7476
4	13·793	15·833	30·620	5·6097	12·9299	27·8711
95	7·781	8·052	14·787	3·3967	7·3202	14·9412
6	4·158	3·894	6·735	1·9332	3·9235	7·6210
7	2·104	1·790	2·841	·9936	1·9903	3·6975
8	1·049	·741	1·051	·5359	·9967	1·7072
9	·482	·259	·310	·2602	·4608	·7105
100	·208	·051	·051	·1515	·2006	·2497
1	·051	...	...	·0491	·0491	·0491

TABLE No. X.

Logarithms of D_x, N_x, *and* M_x.

3 *per-cent.*

x	log D_x	log N_x	log M_x	x	log D_x	log N_x	log M_x
0	5·10477	6·42943	4·65594	50	4·22024	5·36258	3·97356
1	·03995	·41135	·49615	1	·20052	·33159	·96171
2	·01161	·39370	·44107	2	·18039	·29979	·94936
3	4·98920	·37623	·40545	3	·15983	·26712	·93651
4	·96960	·35887	·37974	4	·13878	·23355	·92310
5	·95124	·34153	·35852	55	·11720	·19899	·90908
6	·93394	·32420	·34128	6	·09505	·16337	·89437
7	·91748	·30683	·32734	7	·07226	·12665	·87895
8	·90171	·28941	·31618	8	·04878.	·08877	·86272
9	·88647	·27189	·30713	9	·02453	·04957	·84560
10	·87163	·25426	·29968	60	3·99946	·00907	·82754
1	·85701	·23649	·29321	1	·97347	4·96708	·80844
2	·84256	·21859	·28747	2	·94648	·92354	·78821
3	·82822	·20055	·28215	3	·91839	·87836	·76675
4	·81389	·18236	·27706	4	·88911	·83139	·74394
15	·79956	·16397	·27207	65	·85851	·78251	·71966
6	·78513	·14545	·26682	6	·82646	·73158	·69380
7	·77059	·12672	·26133	7	·79285	·67845	·66620
8	·75585	·10782	·25534	8	·75749	·62296	·63670
9	·74093	·08874	·24890	9	·72024	·56491	·60514
20	·72581	·06948	·24197	70	·68091	·50412	·57135
1	·71048	·05007	·23457	1	·63931	·44037	·53509
2	·69499	·03044	·22683	2	·59522	·37345	·49618
3	·67935	·01068	·21880	3	·54839	·30307	·45436
4	·66361	5·99068	·21059	4	·49859	·22896	·40938
25	·64776	·97050	·20224	75	·44548	·15082	·36093
6	·63184	·95012	·19385	6	·38879	·06837	·30871
7	·61586	·92952	·18546	7	·32815	3·98117	·25237
8	·59984	·90870	·17702	8	·26323	·88887	·19156
9	·58375	·88765	·16864	9	·19358	·79104	·12584
30	·56760	·86635	·16023	80	·11874	·68720	·05477
1	·55141	·84479	·15186	1	·03824	·57684	2·97787
2	·53514	·82297	·14345	2	2·95150	·45938	·89453
3	·51880	·80085	·13504	3	·85796	·33423	·80425
4	·50238	·77844	·12655	4	·75698	·20071	·70636
35	·48589	·75571	·11807	85	·64786	·05805	·60016
6	·46929	·73265	·10951	6	·52977	2·90542	·48483
7	·45259	·70924	·10089	7	·40181	·74199	·35948
8	·43579	·68547	·09219	8	·26319	·56673	·22334
9	·41886	·66130	·08336	9	·11276	·37858	·07522
40	·40179	·63673	·07441	90	1·94948	·17632	1·91411
1	·38458	·61172	·06536	1	·77183	1·95880	·73849
2	·36722	·58625	·05614	2	·57865	·72467	·54718
3	·34966	·56030	·04673	3	·36962	·47167	·33999
4	·33192	·53385	·03711	4	·13965	·19956	·11160
45	·31396	·50684	·02727	95	0·89105	0·90590	0·86452
6	·29578	·47927	·01716	6	·61889	·59040	·59367
7	·27734	·45111	·00677	7	·32299	·25285	·29892
8	·25862	·42229	3·99606	8	·02070	$\bar{1}$·86982	$\bar{1}$·99856
9	·23960	·39280	·98500	9	$\bar{1}$·68335	·41330	·66351
				100	·31834	$\bar{2}$·70344	·30233
				1	$\bar{2}$·70344	...	$\bar{2}$·69064

TABLE NO. XI.

Values of Annuities, and Single and Annual Premiums for Assurance of a Unit.

3 *per-cent.*

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	21·11846	·35577	·01609	50	13·87828	·56666	·03809
1	23·51770	·28589	·01166	1	13·52286	·57702	·03973
2	24·10401	·26882	·01071	2	13·16439	·58745	·04147
3	24·38034	·26077	·01027	3	12·80283	·59797	·04332
4	24·50562	·25712	·01008	4	12·43853	·60858	·04529
5	24·56355	·25543	·00999	55	12·07210	·61927	·04737
6	24·56220	·25547	·00999	6	11·70413	·62997	·04959
7	24·51100	·25696	·01007	7	11·33450	·64075	·05195
8	24·41703	·25970	·01022	8	10·96407	·65154	·05446
9	24·28897	·26343	·01042	9	10·59376	·66232	·05713
10	24·13373	·26795	·01066	60	10·22349	·67310	·05997
1	23·95956	·27302	·01094	1	9·85391	·68386	·06300
2	23·77013	·27855	·01125	2	9·48566	·69459	·06624
3	23·56891	·28440	·01158	3	9·11929	·70528	·06970
4	23·35900	·29052	·01193	4	8·75546	·71586	·07338
15	23·14235	·29683	·01229	65	8·39470	·72636	·07732
6	22·92431	·30317	·01267	6	8·03748	·73678	·08153
7	22·70507	·30956	·01306	7	7·68433	·74705	·08602
8	22·48867	·31586	·01345	8	7·33619	·75720	·09083
9	22·27487	·32208	·01384	9	6·99319	·76718	·09598
20	22·06443	·32822	·01423	70	6·65591	·77703	·10149
1	21·85675	·33426	·01463	1	6·32507	·78665	·10739
2	21·65046	·34028	·01502	2	6·00094	·79609	·11371
3	21·44397	·34630	·01543	3	5·68418	·80532	·12048
4	21·23604	·35235	·01585	4	5·37495	·81431	·12774
25	21·02522	·35849	·01628	75	5·07399	·82310	·13551
6	20·81021	·36476	·01672	6	4·78151	·83161	·14384
7	20·58970	·37119	·01719	7	4·49796	·83989	·15276
8	20·36415	·37773	·01768	8	4·22323	·84787	·16233
9	20·13244	·38449	·01819	9	3·95794	·85558	·17257
30	19·89502	·39141	·01873	80	3·70215	·86304	·18354
1	19·65098	·39852	·01930	1	3·45613	·87022	·19529
2	19·40098	·40580	·01989	2	3·22023	·87706	·20782
3	19·14477	·41328	·02052	3	2·99413	·88367	·22124
4	18·88233	·42089	·02117	4	2·77789	·88998	·23558
35	18·61340	·42873	·02186	85	2·57145	·89598	·25087
6	18·33839	·43674	·02258	6	2·37490	·90170	·26717
7	18·05725	·44494	·02335	7	2·18866	·90713	·28449
8	17·76973	·45331	·02415	8	2·01154	·91233	·30294
9	17·47598	·46185	·02500	9	1·84424	·91719	·32247
40	17·17658	·47057	·02589	90	1·68596	·92179	·34319
1	16·87063	·47949	·02683	1	1·53802	·92610	·36490
2	16·55913	·48856	·02782	2	1·39966	·93010	·38760
3	16·24222	·49782	·02887	3	1·26488	·93405	·41240
4	15·91939	·50721	·02998	4	1·14791	·93745	·43645
45	15·59142	·51679	·03115	95	1·03473	·94074	·46234
6	15·25822	·52648	·03238	6	·93640	·94358	·48727
7	14·91992	·53633	·03369	7	·85078	·94608	·51112
8	14·57684	·54631	·03507	8	·70649	·95030	·55681
9	14·22971	·55642	·03654	9	·53623	·95534	·62186
				100	·24272	·96381	·77452
				1	...	·97087	·97087

TABLE No. XII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.* 3 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
0	10·20954	11·39724	15·31431	16·37630	20·41908	21·36229	0
1	·09274	·37515	·14553	·35000	·19832	·33145	1
2	·04889	·35417	·08617	·32566	·12345	·30367	2
3	·01691	·33371	·04462	·30231	·07233	·27740	3
4	9·99055	·31355	·01150	·27948	·03245	·25195	4
5	·96667	·29356	14·98210	·25703	19·99753	·22709	5
6	·94490	·27365	·95586	·23475	·96682	·20254	6
7	·92482	·25375	·93216	·21256	·93950	·17818	7
8	·90612	·23378	·91053	·19033	·91494	·15381	8
9	·88848	·21373	·89049	·16800	·89250	·12937	9
10	·87163	·19351	·87163	·14548	·87163	·10469	10
1	·85523	·17310	·85345	·12271	·85167	·07972	1
2	·83917	·15250	·83578	·09968	·83239	·05442	2
3	·82332	·13165	·81842	·07635	·81352	·02873	3
4	·80750	·11056	·80111	·05266	·79472	·00264	4
15	·79168	·08920	·78380	·02861	·77592	20·97607	15
6	·77566	·06755	·76619	·00424	·75672	·94904	6
7	·75941	·04568	·74823	15·97944	·73705	·92153	7
8	·74277	·02350	·72969	·95429	·71661	·89355	8
9	·72577	·00104	·71061	·92877	·69545	·86512	9
20	·70836	10·97831	·69091	·90291	·67346	·83625	20
1	·69054	·95532	·67060	·87670	·65066	·80695	1
2	·67240	·93207	·64981	·85014	·62722	·77725	2
3	·65396	·90855	·62857	·82325	·60318	·74714	3
4	·63531	·88477	·60701	·79602	·57871	·71663	4
25	·61645	·86070	·58514	·76844	·55383	·68570	25
6	·59745	·83634	·56306	·74049	·52867	·65434	6
7	·57833	·81168	·54080	·71216	·50327	·62254	7
8	·55912	·78669	·51840	·68343	·47768	·59025	8
9	·53978	·76137	·49581	·65428	·45184	·55747	9
30	·52032	·73569	·47304	·62469	·42576	·52415	30
1	·50077	·70964	·45013	·59461	·39949	·49027	1
2	·48107	·68320	·42700	·56404	·37293	·45579	2
3	·46123	·65634	·40366	·53293	·34609	·42068	3
4	·44123	·62903	·38008	·50126	·31893	·38487	4
35	·42108	·60126	·35627	·46900	·29146	·34836	35
6	·40072	·57299	·33215	·43608	·26358	·31108	6
7	·38016	·54419	·30773	·40250	·23530	·27298	7
8	·35939	·51484	·28299	·36819	·20659	·23401	8
9	·33837	·48491	·25788	·33314	·17739	·19410	9
40	·31707	·45436	·23235	·29726	·14763	·15324	40
1	·29549	·42313	·20640	·26052	·11731	·11126	1
2	·27360	·39120	·17998	·22285	·08636	·06819	2
3	·25132	·35854	·15298	·18421	·05464	·02391	3
4	·22868	·32508	·12544	·14451	·02220	19·97836	4
45	·20560	·29077	·09724	·10370	18·98888	·93142	45
6	·18207	·25556	·06836	·06168	·95465	·88300	6
7	·15803	·21940	·03872	·01841	·91941	·83301	7
8	·13343	·18224	·00824	14·97377	·88305	·78131	8
9	·10822	·14398	13·97684	·92767	·84546	·72781	9

TABLE No. XII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.* 3 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
50	9·08234	10·10456	13·94444	14·88001	18·80654	19·67235	50
1	·05574	·06390	·91096	·83067	·76618	·61479	1
2	·02832	·02193	·87625	·77954	·72418	·55500	2
3	·00003	9·97852	·84023	·72648	·68043	·49277	3
4	8·97077	·93361	·80276	·67136	·63475	·42794	4
55	·94045	·88709	·76370	·61402	·58695	·36031	55
6	·90898	·83883	·72291	·55430	·53684	·28963	6
7	·87624	·78871	·68022	·49199	·48420	·21572	7
8	·84212	·73660	·63546	·42692	·42880	·13827	8
9	·80646	·68233	·58839	·35887	·37032	·05702	9
60	·76915	·62577	·53884	·28760	·30853	18·97166	60
1	·73001	·56672	·48655	·21285	·24309	·88184	1
2	·68887	·50499	·43126	·13437	·17365	·78721	2
3	·64553	·44039	·37267	·05181	·09981	·68735	3
4	·59980	·37269	·31049	13·96487	·02118	·58184	4
65	·55144	·30164	·24437	·87318	17·93730	·47018	65
6	·50018	·22699	·17390	·77634	·84762	·35185	6
7	·44579	·14839	·09873	·67392	·75167	·22629	7
8	·38791	·06562	·01833	·56546	·64875	·09286	8
9	·32625	8·97822	12·93226	·45042	·53827	17·95088	9
70	·26043	·88589	·83995	·32826	·41947	·79957	70
1	·19006	·78820	·74081	·19833	·29156	·63814	1
2	·11472	·68467	·63422	·05998	·15372	·46565	2
3	·03390	·57483	·51941	12·91243	·00492	·28112	3
4	7·94713	·45811	·39567	·75488	16·84421	·08343	4
75	·85375	·33393	·26202	·58644	·67029	16·87143	75
6	·75321	·20164	·11763	·40613	·48205	·64379	6
7	·64477	·06055	11·96139	·21288	·27801	·39909	7
8	·52776	7·90981	·79229	·00552	·05682	·13571	8
9	·40130	·74861	·60902	11·78271	15·81674	15·85191	9
80	·26446	·57598	·41018	·54305	·55590	·54580	80
1	·11630	·39090	·19436	·28497	·27242	·21524	1
2	6·95565	·19226	10·95980	·00681	14·96395	14·85805	2
3	·78141	6·97882	·70486	10·70667	·62831	·47169	3
4	·59229	·74920	·42760	·38250	·26291	·05339	4
85	·38688	·50187	·12590	·03197	13·86492	13·59997	85
6	·16354	·23517	9·79731	9·65259	·43108	·10809	6
7	5·92046	5·94742	·43911	·24180	12·95776	12·57438	7
8	·65606	·63650	·04893	8·79635	·44180	11·99446	8
9	·36803	·30027	8·62330	·31307	11·87857	·36406	9
90	·05431	4·93618	·15914	7·78791	·26397	10·67758	90
1	4·71185	·54194	7·65187	·21754	10·59189	9·93059	1
2	·33832	·11514	·09799	6·59890	9·85766	·11975	2
3	3·93310	3·64972	6·49658	5·92091	·06006	8·22843	3
4	·48600	·14675	5·83235	·18682	8·17870	7·26259	4
95	·00164	2·60011	·11223	4·38493	7·22282	6·20474	95
6	2·47015	·01280	4·32141	3·51866	6·17267	5·05736	6
7	1·89119	1·40056	3·45939	2·62435	5·02759	3·87845	7
8	·29945	0·71800	2·57820	1·62788	3·85695	2·56228	8
9	0·63759	$\bar{1}$·94599	1·59183	0·52899	2·54607	1·12616	9
100	$\bar{1}$·92040	$\bar{2}$·70344	0·52246	$\bar{2}$·70344	1·12452	$\bar{2}$·70344	100
1	$\bar{2}$·70344	...	$\bar{2}$·70344	...	$\bar{2}$·70344	...	1

Table No. XIII.

Values of Annuities. Joint Lives. Equal Ages.
3 *per-cent.*

x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}	x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}
0	15·406	11·534	8·774	50	10·525	8·621	7·342
1	19·161	16·013	13·587	1	10·190	8·312	7·057
2	20·197	17·358	15·143	2	9·854	8·004	6·774
3	20·740	18·100	16·035	3	9·517	7·696	6·491
4	21·038	18·534	16·577	4	9·180	7·389	6·211
5	21·227	18·833	16·965	55	8·844	7·085	5·934
6	21·318	19·006	17·208	6	8·509	6·783	5·660
7	21·327	19·072	17·325	7	8·175	6·483	5·389
8	21·265	19·046	17·333	8	7·843	6·187	5·122
9	21·147	18·946	17·253	9	7·514	5·895	4·861
10	20·984	18·787	17·103	60	7·188	5·607	4·604
1	20·791	18·589	16·906	1	6·866	5·325	4·353
2	20·575	18·361	16·674	2	6·548	5·048	4·107
3	20·339	18·110	16·414	3	6·235	4·777	3·869
4	20·094	17·846	16·141	4	5·928	4·512	3·636
15	19·839	17·572	15·854	65	5·626	4·254	3·411
6	19·583	17·300	15·571	6	5·331	4·004	3·193
7	19·332	17·030	15·293	7	5·042	3·760	2·983
8	19·087	16·773	15·029	8	4·761	3·525	2·780
9	18·848	16·526	14·780	9	4·487	3·297	2·586
20	18·619	16·293	14·548	70	4·221	3·078	2·399
1	18·398	16·073	14·331	1	3·964	2·868	2·221
2	18·183	15·861	14·126	2	3·715	2·665	2·051
3	17·972	15·656	13·930	3	3·475	2·472	1·889
4	17·761	15·453	13·738	4	3·243	2·287	1·735
25	17·549	15·251	13·548	75	3·021	2·111	1·589
6	17·334	15·046	13·356	6	2·808	1·943	1·451
7	17·114	14·837	13·160	7	2·605	1·784	1·322
8	16·888	14·623	12·959	8	2·410	1·634	1·199
9	16·657	14·404	12·754	9	2·225	1·492	1·084
30	16·420	14·179	12·543	80	2·049	1·358	·977
1	16·176	13·947	12·325	1	1·882	1·232	·877
2	15·927	13·710	12·102	2	1·724	1·114	·784
3	15·671	13·467	11·874	3	1·576	1·004	·697
4	15·410	13·218	11·640	4	1·435	·901	·617
35	15·142	12·964	11·400	85	1·303	·806	·543
6	14·869	12·704	11·156	6	1·179	·717	·475
7	14·589	12·439	10·906	7	1·064	·635	·414
8	14·304	12·167	10·652	8	·956	·559	·357
9	14·013	11·892	10·392	9	·856	·490	·306
40	13·718	11·612	10·130	90	·762	·425	·259
1	13·417	11·327	9·862	1	·676	·368	·218
2	13·110	11·037	9·590	2	·598	·317	·183
3	12·800	10·746	9·317	3	·521	·266	·147
4	12·485	10·449	9·040	4	·458	·226	·121
45	12·167	10·150	8·761	95	·397	·187	·096
6	11·844	9·847	8·479	6	·349	·157	·077
7	11·518	9·543	8·196	7	·323	·146	·071
8	11·189	9·237	7·912	8	·262	·112	·051
9	10·858	8·930	7·627	9	·203	·087	·038
				100	·061	·015	·004

TABLES.

THREE-AND-A-HALF PER CENT.

ONE LIFE.

Table No. XIV.—D, N, S, C, M, R.
„ „ XV.—Log D, Log N, Log M.
„ „ XVI.—*a*, A, P.

TWO, THREE, AND FOUR LIVES.

Table No. XVII.—Log D, Log N.
„ „ XVIII.—*a*.

Constants.

Constant.	Number.	Logarithm.
i	·035	$\bar{2}$·5440680
$(1+i)$	1·035	0·0149403
$(1+i)^{\frac{1}{2}}$	1·0173495	0·0074702
$(1+i)^{\frac{1}{4}}$	1·0086374	0·0037351
v	·9661836	$\bar{1}$·9850597
$v^{\frac{1}{2}}$	·9829464	$\bar{1}$·9925298
$v^{\frac{1}{4}}$	·9914365	$\bar{1}$·9962649
d	·0338164	$\bar{2}$·5291277
δ	·0344014	$\bar{2}$·5365765
$j_{(2)}$	·0346990	$\bar{2}$·5403170
$j_{(4)}$	·0345498	$\bar{2}$·5384454

Table No. XIV.

Commutation Table.

3½ per-cent.

x	D_x	N_x	S_x	C_x	M_x	R_x
0	127 280	2 425 772	49 703 849	13 872·	40 948·	785 908·
1	109 110	2 316 662	47 278 077	3 698·5	27 075·5	744 959·6
2	101 720	2 214 942	44 961 415	2 142·1	23 377·0	717 884·1
3	96 137	2 118 805	42 746 473	1 434·4	21 234·9	694 507·1
4	91 451	2 027 354	40 627 668	1 115·6	19 800·5	673 272·2
5	87 243	1 940 111	38 600 314	863·14	18 684·94	653 471·69
6	83 430	1 856 681	36 660 203	669·67	17 821·80	634 786·75
7	79 939	1 776 742	34 803 522	518·68	17 152·13	616 964·95
8	76 717	1 700 025	33 026 780	408·70	16 633·45	599 812·82
9	73 714	1 626 311	31 326 755	328·94	16 224·75	583 179·37
10	70 892	1 555 419	29 700 444	279·46	15 895·81	566 954·62
1	68 215	1 487 204	28 145 025	244·20	15 616·35	551 058·81
2	65 664	1 421 540	26 657 821	221·24	15 372·15	535 442·46
3	63 224	1 358 316	25 236 281	208·19	15 150·91	520 070·31
4	60 877	1 297 439	23 877 965	201·15	14 942·72	504 919·40
15	58 615	1 238 824	22 580 526	207·61	14 741·57	489 976·68
6	56 426	1 182 398	21 341 702	213·96	14 533·96	475 235·11
7	54 304	1 128 094	20 159 304	228·80	14 320·00	460 701·15
8	52 238	1 075 856	19 031 210	241·87	14 091·20	446 381·15
9	50 231	1 025 625	17 955 354	255·30	13 849·33	432 289·95
20	48 277	977 348	16 929 729	266·09	13 594·03	418 440·62
1	46 378	930 970	15 952 381	273·04	13 327·94	404 846·59
2	44 537	886 433	15 021 411	276·05	13 054·90	391 518·65
3	42 754	843 679	14 134 978	276·35	12 778·85	378 463·75
4	41 033	802 646	13 291 299	273·77	12 502·50	365 684·90
25	39 371	763 275	12 488 653	269·02	12 228·73	353 182·40
6	37 771	725 504	11 725 378	262·29	11 959·71	340 953·67
7	36 231	689 273	10 999 874	256·86	11 697·42	328 993·96
8	34 750	654 523	10 310 601	250·01	11 440·56	317 296·54
9	33 324	621 199	9 656 078	244·41	11 190·55	305 855·98
30	31 953	589 246	9 034 879	237·86	10 946·14	294 665·43
1	30 634	558 612	8 445 633	232·81	10 708·28	283 719·29
2	29 366	529 246	7 887 021	227·83	10 475·47	273 011·01
3	28 145	501 101	7 357 775	223·23	10 247·64	262 535·54
4	26 970	474 131	6 856 674	218·69	10 024·41	252 287·90
35	25 839	448 292	6 382 543	215·06	9 805·72	242 263·49
6	24 750	423 542	5 934 251	211·70	9 590·66	232 457·77
7	23 702	399 840	5 510 709	208·33	9 378·96	222 867·11
8	22 692	377 148	5 110 869	205·47	9 170·63	213 488·15
9	21 719	355 429	4 733 721	203·58	8 965·16	204 317·52
40	20 781	334 648	4 378 292	200·84	8 761·58	195 352·36
1	19 877	314 771	4 043 644	199·47	8 560·74	186 590·78
2	19 006	295 765	3 728 873	198·42	8 361·27	178 030·04
3	18 165	277 600	3 433 108	196·99	8 162·85	169 668·77
4	17 353	260 247	3 155 508	196·49	7 965·86	161 505·92
45	16 570	243 677	2 895 261	196·02	7 769·37	153 540·06
6	15 814	227 863	2 651 584	195·74	7 573·35	145 770·69
7	15 083	212 780	2 423 721	195·83	7 377·61	138 197·34
8	14 377	198 403	2 210 941	196·63	7 181·78	130 819·73
9	13 694	184 709	2 012 538	197·14	6 985·15	123 637·95

TABLE NO. XIV.

Commutation Table.

$3\frac{1}{2}$ *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
50	13 034·	171 675·	1 827 829·	197·91	6 788·01	116 652·80
1	12 395·	159 280·	1 656 154·	199·41	6 590·10	109 864·79
2	11 777·	147 503·	1 496 874·	200·74	6 390·69	103 274·69
3	11 178·	136 325·	1 349 371·	202·22	6 189·95	96 884·00
4	10 598·	125 727·	1 213 046·	203·98	5 987·73	90 694·05
55	10 035·	115 691·8	1 087 319·3	205·96	5 783·75	84 706·32
6	9 490·1	106 201·7	971 627·5	207·58	5 577·79	78 922·57
7	8 961·5	97 240·2	865 425·8	209·54	5 370·21	73 344·78
8	8 448·9	88 791·3	768 185·6	211·78	5 160·67	67 974·57
9	7 951·5	80 839·8	679 394·3	213·51	4 948·89	62 813·90
60	7 469·1	73 370·7	598 554·5	215·24	4 735·38	57 865·01
1	7 001·3	66 369·4	525 183·8	216·85	4 520·14	53 129·63
2	6 547·7	59 821·7	458 814·4	218·21	4 303·29	48 609·49
3	6 108·0	53 713·7	398 992·7	219·35	4 085·08	44 306·20
4	5 682·1	48 031·6	345 279·0	220·06	3 865·73	40 221·12
65	5 270·0	42 761·6	297 247·4	220·26	3 645·67	36 355·39
6	4 871·5	37 890·1	254 485·8	219·89	3 425·41	32 709·72
7	4 486·8	33 403·3	216 595·7	219·11	3 205·52	29 284·31
8	4 116·0	29 287·3	183 192·4	217·38	2 986·41	26 078·79
9	3 759·5	25 527·8	153 905·1	214·89	2 769·03	23 092·38
70	3 417·4	22 110·4	128 377·3	211·62	2 554·14	20 323·35
1	3 090·2	19 020·2	106 266·9	207·32	2 342·52	17 769·21
2	2 778·4	16 241·8	87 246·7	202·09	2 135·20	15 426·69
3	2 482·3	13 759·5	71 004·9	195·73	1 933·11	13 291·49
4	2 202·7	11 556·8	57 245·4	188·43	1 737·38	11 358·38
75	1 939·7	9 617·1	45 688·6	180·01	1 548·95	9 621·00
6	1 694·1	7 923·0	36 071·5	170·60	1 368·94	8 072·05
7	1 466·3	6 456·7	28 148·5	160·11	1 198·34	6 703·11
8	1 256·6	5 200·1	21 691·8	148·89	1 038·23	5 504·77
9	1 065·2	4 134·89	16 491·72	136·90	889·34	4 466·54
80	892·26	3 242·63	12 356·83	124·38	752·44	3 577·20
1	737·72	2 504·91	9 114·20	111·54	628·06	2 824·76
2	601·23	1 903·68	6 609·29	98·503	516·521	2 196·695
3	482·39	1 421·29	4 705·61	85·611	418·018	1 680·174
4	380·47	1 040·82	3 284·32	73·102	332·407	1 262·156
85	294·50	746·32	2 243·50	61·236	259·305	929·749
6	223·31	523·01	1 497·18	50·241	198·069	670·444
7	165·52	357·49	974·17	40·210	147·828	472·375
8	119·71	237·784	616·680	31·407	107·618	324·547
9	84·252	153·532	378·896	23·833	76·211	216·929
90	57·571	95·961	225·364	17·565	52·378	140·718
1	38·058	57·903	129·403	12·496	34·813	88·340
2	24·275	33·628	71·500	8·5251	22·3170	53·5272
3	14·929	18·699	37·872	5·6751	13·7919	31·2102
4	8·749	9·950	19·173	3·5412	8·1168	17·4183
95	4·912	5·038	9·223	2·1338	4·5756	9·3015
6	2·612	2·426	4·185	1·2086	2·4418	4·7259
7	1·315	1·111	1·759	·6183	1·2332	2·2841
8	·653	·458	·648	·3317	·6149	1·0509
9	·299	·159	·190	·1603	·2832	·4360
100	·128	·031	·031	·0930	·1229	·1528
1	·031	...	...	·0299	·0299	·0299

TABLE NO. XV.

Logarithms of D_x, N_x, *and* M_x.
$3\frac{1}{2}$ *per-cent.*

x	log D_x	log N_x	log M_x	x	log D_x	log N_x	log M_x
0	5·10477	6·38485	4·61223	50	4·11508	5·23472	3·83174
1	·03785	·36487	·43258	1	·09326	·20216	·81889
2	·00740	·34535	·36879	2	·07103	·16879	·80555
3	4·98289	·32609	·32705	3	·04836	·13459	·79169
4	·96119	·30694	·29669	4	·02521	·09944	·77726
5	·94073	·28782	·27149	55	·00153	·06330	·76221
6	·92132	·26874	·25096	6	3·97727	·02612	·74646
7	·90276	·24961	·23431	7	·95238	4·98784	·72999
8	·88489	·23045	·22097	8	·92680	·94837	·71271
9	·86755	·21120	·21018	9	·90045	·90763	·69451
10	·85060	·19184	·20129	60	·87327	·86552	·67536
1	·83388	·17237	·19357	1	·84518	·82197	·65515
2	·81733	·15275	·18673	2	·81609	·77686	·63380
3	·80088	·13300	·18044	3	·78590	·73009	·61120
4	·78445	·11307	·17444	4	·75451	·68153	·58723
15	·76801	·09300	·16856	65	·72181	·63106	·56178
6	·75148	·07276	·16239	6	·68766	·57852	·53471
7	·73483	·05235	·15594	7	·65194	·52379	·50590
8	·71799	·03177	·14894	8	·61448	·46667	·47515
9	·70097	·01098	·14142	9	·57513	·40702	·44232
20	·68374	5·99005	·13335	70	·53370	·34459	·40724
1	·66631	·96894	·12476	1	·48999	·27921	·36968
2	·64872	·94764	·11578	2	·44379	·21064	·32944
3	·63098	·92618	·10650	3	·39486	·13862	·28625
4	·61313	·90453	·09701	4	·34295	·06285	·23990
25	·59518	·88268	·08739	75	·28774	3·98304	·19005
6	·57716	·86064	·07773	6	·22895	·89889	·13637
7	·55908	·83839	·06807	7	·16621	·81001	·07857
8	·54095	·81592	·05846	8	·09918	·71601	·01628
9	·52276	·79323	·04887	9	·02743	·61647	2·94907
30	·50451	·77030	·03926	80	2·95049	·51089	·87647
1	·48621	·74711	·02971	1	·86789	·39879	·79800
2	·46784	·72366	·02015	2	·77904	·27960	·71309
3	·44940	·69992	·01064	3	·68340	·15269	·62120
4	·43088	·67590	·00104	4	·58032	·01737	·52167
35	·41228	·65156	3·99148	85	·46909	2·87293	·41382
6	·39358	·62689	·98185	6	·34890	·71851	·29682
7	·37478	·60189	·97216	7	·21884	·55326	·16976
8	·35587	·57651	·96240	8	·07812	·37618	·03189
9	·33684	·55075	·95256	9	1·92558	·18619	1·88202
40	·31767	·52459	·94258	90	·76020	1·98209	·71915
1	·29836	·49799	·93251	1	·58045	·76270	·54174
2	·27889	·47095	·92227	2	·38516	·52670	·34864
3	·25923	·44342	·91184	3	·17403	·27182	·13963
4	·23938	·41539	·90123	4	0·94196	0·99781	0·90938
45	·21932	·38682	·89039	95	·69126	·70223	·66045
6	·19903	·35767	·87929	6	·41699	·38482	·38771
7	·17849	·32793	·86792	7	·11899	·04548	·09103
8	·15767	·29754	·85623	8	$\bar{1}$·81460	$\bar{1}$·66073	$\bar{1}$·78887
9	·13654	·26649	·84418	9	·47515	·20200	·45203
				100	·10803	$\bar{2}$·49103	·08941
				1	$\bar{2}$·49103	...	$\bar{2}$·47609

TABLE No. XVI.

Values of Annuities, and Single and Annual Premiums for Assurance of a Unit.

3½ *per-cent.*

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	19·058	·32171	·01604	50	13·172	·52079	·03675
1	21·233	·24816	·01116	1	12·850	·53166	·03839
2	21·775	·22982	·01009	2	12·524	·54265	·04012
3	22·039	·22088	·00959	3	12·196	·55377	·04197
4	22·169	·21652	·00935	4	11·864	·56500	·04392
5	22·238	·21417	·00922	55	11·528	·57634	·04600
6	22·255	·21362	·00919	6	11·191	·58775	·04821
7	22·225	·21456	·00924	7	10·851	·59925	·05057
8	22·160	·21681	·00936	8	10·509	·61082	·05307
9	22·062	·22011	·00954	9	10·167	·62239	·05574
10	21·940	·22423	·00977	60	9·823	·63400	·05858
1	21·802	·22892	·01004	1	9·480	·64561	·06161
2	21·648	·23410	·01034	2	9·136	·65722	·06484
3	21·484	·23964	·01066	3	8·794	·66881	·06829
4	21·312	·24547	·01100	4	8·453	·68033	·07197
15	21·134	·25151	·01136	65	8·114	·69178	·07590
6	20·955	·25758	·01173	6	7·778	·70315	·08010
7	20·774	·26370	·01211	7	7·445	·71443	·08460
8	20·596	·26974	·01249	8	7·115	·72555	·08940
9	20·418	·27571	·01287	9	6·790	·73653	·09455
20	20·245	·28159	·01325	70	6·470	·74738	·10005
1	20·074	·28738	·01364	1	6·155	·75804	·10595
2	19·903	·29313	·01402	2	5·846	·76851	·11226
3	19·733	·29890	·01442	3	5·543	·77874	·11902
4	19·561	·30471	·01482	4	5·247	·78877	·12626
25	19·387	·31061	·01524	75	4·958	·79856	·13403
6	19·208	·31664	·01567	6	4·677	·80802	·14234
7	19·024	·32284	·01612	7	4·404	·81726	·15124
8	18·835	·32924	·01660	8	4·138	·82623	·16079
9	18·641	·33582	·01710	9	3·882	·83491	·17103
30	18·441	·34257	·01762	80	3·634	·84330	·18197
1	18·235	·34954	·01817	1	3·395	·85135	·19369
2	18·023	·35671	·01875	2	3·166	·85911	·20621
3	17·804	·36412	·01936	3	2·946	·86656	·21958
4	17·580	·37167	·02000	4	2·736	·87368	·23387
35	17·349	·37949	·02068	85	2·534	·88050	·24914
6	17·112	·38750	·02139	6	2·342	·88699	·26539
7	16·870	·39571	·02214	7	2·160	·89314	·28265
8	16·620	·40414	·02294	8	1·986	·89902	·30104
9	16·365	·41278	·02377	9	1·822	·90457	·32051
40	16·103	·42161	·02465	90	1·667	·90981	·34116
1	15·835	·43068	·02558	1	1·521	·91472	·36279
2	15·562	·43993	·02656	2	1·385	·91935	·38543
3	15·282	·44938	·02760	3	1·253	·92385	·41014
4	14·997	·45904	·02870	4	1·137	·92773	·43407
45	14·706	·46889	·02985	95	1·026	·93152	·45988
6	14·409	·47892	·03108	6	·929	·93480	·48471
7	14·107	·48914	·03238	7	·844	·93765	·50841
8	13·800	·49953	·03375	8	·702	·94248	·55385
9	13·488	·51008	·03521	9	·533	·94816	·61844
				100	·242	·95803	·77163
				1	...	·96618	·96618

Table No. XVII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*
3½ *per-cent.*

x	Two Lives.		Three Lives.		Four Lives.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
0	10·20954	11·35811	15·31431	16·34098	20·41908	21·33001	0
1	·09064	·33399	·14343	·31252	·19622	·29686	1
2	·04468	·31108	·08196	·28619	·11924	·26701	2
3	·01060	·28876	·03831	·26093	·06602	·23877	3
4	9·98214	·26680	·00309	·23626	·02404	·21144	4
5	·95616	·24502	14·97159	·21197	19·98702	·18472	5
6	·93228	·22334	·94324	·18794	·95420	·15839	6
7	·91010	·20170	·91744	·16400	·92478	·13226	7
8	·88930	·18001	·89371	·14004	·89812	·10616	8
9	·86956	·15824	·87157	·11598	·87358	·07998	9
10	·85060	·13631	·85060	·09174	·85060	·05358	10
1	·83210	·11418	·83032	·06729	·82854	·02690	1
2	·81394	·09188	·81055	·04254	·80716	20·99990	2
3	·79598	·06934	·79108	·01749	·78618	·97251	3
4	·77806	·04653	·77167	15·99210	·76528	·94469	4
15	·76013	·02346	·75225	·96636	·74437	·91640	15
6	·74201	·00013	·73254	·94024	·72307	·88765	6
7	·72365	10·97652	·71247	·91374	·70129	·85840	7
8	·70491	·95264	·69183	·88687	·67875	·82870	8
9	·68581	·92849	·67065	·85964	·65549	·79853	9
20	·66629	·90407	·64884	·83205	·63139	·76792	20
1	·64637	·87939	·62643	·80412	·60649	·73689	1
2	·62613	·85445	·60354	·77585	·58095	·70544	2
3	·60559	·82925	·58020	·74726	·55481	·67361	3
4	·58483	·80377	·55653	·71832	·52823	·64137	4
25	·56387	·77803	·53256	·68903	·50125	·60872	25
6	·54277	·75199	·50838	·65940	·47399	·57564	6
7	·52155	·72567	·48402	·62939	·44649	·54213	7
8	·50023	·69902	·45951	·59898	·41879	·50814	8
9	·47879	·67205	·43482	·56814	·39085	·47365	9
30	·45723	·64472	·40995	·53687	·36267	·43864	30
1	·43557	·61703	·38493	·50514	·33429	·40307	1
2	·41377	·58894	·35970	·47290	·30563	·36691	2
3	·39183	·56044	·33426	·44014	·27669	·33011	3
4	·36973	·53150	·30858	·40680	·24743	·29263	4
35	·34747	·50210	·28266	·37288	·21785	·25443	35
6	·32501	·47222	·25644	·33832	·18787	·21548	6
7	·30235	·44181	·22992	·30311	·15749	·17571	7
8	·27947	·41084	·20307	·26715	·12667	·13507	8
9	·25635	·37929	·17586	·23045	·09537	·09353	9
40	·23295	·34712	·14823	·19293	·06351	·05096	40
1	·20927	·31429	·12018	·15458	·03109	·00736	1
2	·18527	·28076	·09165	·11528	18·99803	19·96263	2
3	·16089	·24650	·06255	·07500	·96421	·91670	3
4	·13614	·21144	·03290	·03367	·92966	·86948	4
45	·11096	·17554	·00260	14·99123	·89424	·82088	45
6	·08532	·13871	13·97161	·94760	·85790	·77081	6
7	·05918	·10096	·93987	·90269	·82056	·71915	7
8	·03248	·06221	·90729	·85641	·78210	·66580	8
9	·00516	·02235	·87378	·80868	·74240	·61063	9

TABLE No. XVII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*

3½ *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
50	8·97718	9·98132	13·83928	14·75939	18·70138	19·55352	50
1	·94848	·93905	·80370	·70842	·65892	·49429	1
2	·91896	·89547	·76689	·65565	·61482	·43283	2
3	·88856	·85048	·72876	·60095	·56896	·36892	3
4	·85720	·80397	·68919	·54419	·52118	·30242	4
55	·82478	·75584	·64803	·48520	·47128	·23310	55
6	·79120	·70597	·60513	·42382	·41906	·16074	6
7	·75636	·65422	·56034	·35985	·36432	·08515	7
8	·72014	·60049	·51348	·29312	·30682	·00600	8
9	·68238	·54460	·46431	·22339	·24624	18·92303	9
60	·64296	·48640	·41265	·15045	·18234	·83595	60
1	·60172	·42571	·35826	·07401	·11480	·74441	1
2	·55848	·36235	·30087	13·99384	·04326	·64805	2
3	·51304	·29609	·24018	·90959	17·96732	·54646	3
4	·46520	·22673	·17589	·82094	·88658	·43919	4
65	·41474	·15403	·10767	·72753	·80060	·32578	65
6	·36138	·07770	·03510	·62897	·70882	·20569	6
7	·30488	8·99742	12·95782	·52481	·61076	·07835	7
8	·24490	·91293	·87532	·41461	·50574	17·94313	8
9	·18114	·82385	·78715	·29780	·39316	·79934	9
70	·11322	·72981	·69274	·17386	·27226	·64623	70
1	·04074	·63040	·59149	·04218	·14224	·48297	1
2	7·96329	·52514	·48279	12·90201	·00229	·30865	2
3	·88037	·41355	·36588	·75267	16·85139	·12228	3
4	·79149	·29508	·24003	·59331	·68857	16·92275	4
75	·69601	·16915	·10428	·42305	·51255	·70889	75
6	·59337	·03507	11·95779	·24090	·32221	·47936	6
7	·48283	7·89218	·79945	·04583	·11607	·23277	7
8	·36371	·73966	·62824	11·83659	15·89277	15·96750	8
9	·23515	·57664	·44287	·61190	·65059	·68179	9
80	·09621	·40219	·24193	·37037	·38765	·37376	80
1	6·94595	·21529	·02401	·11039	·10207	·04127	1
2	·78319	·01481	10·78734	10·83032	14·79149	14·68214	2
3	·60685	6·79949	·53030	·52827	·45375	·29383	3
4	·41563	·56799	·25094	·20219	·08625	13·87354	4
85	·20811	·31877	9·94713	9·84972	13·68615	·41817	85
6	5·98267	·05019	·61644	·46838	·25021	12·92431	6
7	·73749	5·76052	·25614	·05561	12·77479	·38860	7
8	·47099	·44769	8·86386	8·60819	·25673	11·80667	8
9	·18085	·10954	·43612	·12294	11·69139	·17426	9
90	4·86503	4·74349	7·96986	7·59579	·07469	10·48575	90
1	·52047	·34729	·46049	·02342	10·40051	9·73673	1
2	·14483	3·91854	6·90450	6·40278	9·66417	8·92387	2
3	3·73751	·45113	·30099	5·72276	8·86447	·03048	3
4	·28831	2·94616	5·63466	4·98664	7·98101	7·06262	4
95	2·80185	·39751	4·91244	·18270	·02303	6·00268	95
6	·26825	1·80816	·11951	3·31435	5·97077	4·85321	6
7	1·68719	·19396	3·25539	2·41802	4·82359	3·67225	7
8	·09335	0·50944	2·37210	1·41951	3·65085	2·35401	8
9	0·42939	$\bar{1}$·73556	1·38363	0·31867	2·33787	0·91585	9
100	$\bar{1}$·71009	$\bar{2}$·49102	0·31215	$\bar{2}$·49102	0·91421	$\bar{2}$·49102	100
1	$\bar{2}$·49102	...	$\bar{2}$·49102	...	$\bar{2}$·49102	...	1

Table No. XVIII.

Values of Annuities. Joint Lives. Equal Ages.

3½ *per-cent.*

x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}	x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}
0	14·079	10·633	8·146	50	10·096	8·320	7·114
1	17·513	14·760	12·608	1	9·785	8·030	6·845
2	18·467	16·004	14·053	2	9·474	7·740	6·577
3	18·974	16·696	14·885	3	9·161	7·451	6·309
4	19·260	17·107	15·396	4	8·847	7·161	6·043
5	19·447	17·393	15·765	55	8·532	6·873	5·779
6	19·546	17·567	16·003	6	8·218	6·587	5·517
7	19·570	17·642	16·124	7	7·904	6·303	5·258
8	19·530	17·633	16·145	8	7·592	6·021	5·002
9	19·439	17·555	16·084	9	7·282	5·742	4·751
10	19·307	17·424	15·958	60	6·973	5·468	4·504
1	19·146	17·257	15·789	1	6·668	5·197	4·262
2	18·964	17·060	15·586	2	6·366	4·931	4·025
3	18·765	16·843	15·358	3	6·068	4·671	3·794
4	18·555	16·612	15·115	4	5·775	4·416	3·570
15	18·337	16·372	14·860	65	5·486	4·167	3·351
6	18·118	16·132	14·608	6	5·204	3·925	3·140
7	17·901	15·895	14·359	7	4·927	3·690	2·935
8	17·690	15·669	14·124	8	4·656	3·462	2·738
9	17·486	15·452	13·901	9	4·393	3·241	2·548
20	17·289	15·248	13·694	70	4·136	3·028	2·366
1	17·101	15·055	13·502	1	3·887	2·823	2·191
2	16·917	14·870	13·320	2	3·646	2·626	2·025
3	16·736	14·691	13·146	3	3·413	2·437	1·866
4	16·555	14·514	12·976	4	3·189	2·256	1·715
25	16·374	14·337	12·808	75	2·973	2·083	1·572
6	16·189	14·159	12·637	6	2·765	1·919	1·436
7	16·000	13·976	12·464	7	2·567	1·764	1·308
8	15·805	13·787	12·284	8	2·377	1·616	1·188
9	15·605	13·593	12·100	9	2·195	1·476	1·075
30	15·399	13·394	11·912	80	2·023	1·344	·969
1	15·187	13·189	11·716	1	1·859	1·220	·869
2	14·968	12·978	11·515	2	1·705	1·104	·777
3	14·744	12·761	11·309	3	1·558	·995	·692
4	14·513	12·538	11·097	4	1·420	·894	·613
35	14·277	12·309	10·879	85	1·290	·799	·540
6	14·035	12·075	10·656	6	1·168	·711	·472
7	13·787	11·836	10·428	7	1·055	·630	·411
8	13·532	11·590	10·195	8	·948	·555	·355
9	13·272	11·339	9·958	9	·849	·486	·304
40	13·007	11·084	9·715	90	·756	·423	·258
1	12·736	10·824	9·468	1	·671	·366	·217
2	12·459	10·559	9·217	2	·594	·315	·182
3	12·179	10·291	8·964	3	·517	·264	·147
4	11·893	10·018	8·706	4	·455	·225	·121
45	11·603	9·742	8·446	95	·394	·186	·095
6	11·308	9·462	8·183	6	·347	·157	·076
7	11·010	9·180	7·918	7	·321	·145	·071
8	10·709	8·895	7·651	8	·261	·112	·050
9	10·404	8·608	7·383	9	·202	·086	·038
				100	·060	·015	·004

TABLES.

FOUR PER CENT.

ONE LIFE.

Table No. XIX.—D, N, S, C, M, R.
" " XX.—Log D, Log N, Log M.
" " XXI.—*a*, A, P.

TWO, THREE, AND FOUR LIVES.

Table No. XXII.—Log D, Log N.
" " XXIII.—*a*.

Constants.

Constant.	Number.	Logarithm.
i	·04	$\bar{2}$·6020600
$(1+i)$	1·04	0·0170333
$(1+i)^{\frac{1}{2}}$	1·0198039	0·0085167
$(1+i)^{\frac{1}{4}}$	1·0098534	0·0042583
v	·9615385	$\bar{1}$·9829667
$v^{\frac{1}{2}}$	·9805807	$\bar{1}$·9914833
$v^{\frac{1}{4}}$	·9902427	$\bar{1}$·9957417
d	·0384615	$\bar{2}$·5850267
δ	·0392207	$\bar{2}$·5935155
$j_{(2)}$	·0396078	$\bar{2}$·5977807
$j_{(4)}$	·0394136	$\bar{2}$·5956464

TABLE NO. XIX.

Commutation Table.

4 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
0	127 280	2 204 737	42 280 578	13 806·	37 590·	616 157·
1	108 580	2 096 157	40 075 841	3 663·0	23 784·0	578 567·3
2	100 740	1 995 417	37 979 684	2 111·3	20 121·0	554 783·3
3	94 757	1 900 660	35 984 267	1 407·0	18 009·7	534 662·3
4	89 706	1 810 954	34 083 607	1 089·1	16 602·7	516 652·6
5	85 165	1 725 789	32 272 653	838·53	15 513·57	500 049·89
6	81 051	1 644 738	30 546 864	647·46	14 675·04	484 536·32
7	77 288	1 567 450	28 902 126	499·06	14 027·58	469 861·28
8	73 814	1 493 636	27 334 676	391·35	13 528·52	455 833·70
9	70 585	1 423 051	25 841 040	313·47	13 137·17	442 305·18
10	67 557	1 355 494	24 417 989	265·03	12 823·70	429 168·01
1	64 692	1 290 802	23 062 495	230·48	12 558·67	416 344·31
2	61 974	1 228 828	21 771 693	207·80	12 328·19	403 785·64
3	59 384	1 169 444	20 542 865	194·61	12 120·39	391 457·45
4	56 904	1 112 540	19 373 421	187·12	11 925·78	379 337·06
15	54 528	1 058 012	18 260 881	192·21	11 738·66	367 411·28
6	52 240	1 005 772	17 202 869	197·13	11 546·45	355 672·62
7	50 032	955 740	16 197 097	209·79	11 349·32	344 126·17
8	47 898	907 842	15 241 357	220·71	11 139·53	332 776·85
9	45 836	862 006	14 333 515	231·84	10 918·82	321 637·32
20	43 841	818 165	13 471 509	240·48	10 686·98	310 718·50
1	41 914	776 251	12 653 344	245·58	10 446·50	300 031·52
2	40 057	736 194	11 877 093	247·09	10 200·92	289 585·02
3	38 268	697 926	11 140 899	246·17	9 953·83	279 384·10
4	36 551	661 375	10 442 973	242·70	9 707·66	269 430·27
25	34 903	626 472	9 781 598	237·33	9 464·96	259 722·61
6	33 323	593 149	9 155 126	230·29	9 227·63	250 257·65
7	31 810	561 339	8 561 977	224·43	8 997·34	241 030·02
8	30 363	530 976	8 000 638	217·40	8 772·91	232 032·68
9	28 977	501 999	7 469 662	211·50	8 555·51	223 259·77
30	27 652	474 347	6 967 663	204·86	8 344·01	214 704·26
1	26 383	447 964	6 493 316	199·54	8 139·15	206 360·25
2	25 169	422 795	6 045 352	194·33	7 939·61	198 221·10
3	24 007	398 788	5 622 557	189·50	7 745·28	190 281·49
4	22 894	375 894	5 223 769	184·74	7 555·78	182 536·21
35	21 828	354 066	4 847 875	180·80	7 371·04	174 980·43
6	20 808	333 258	4 493 809	177·13	7 190·24	167 609·39
7	19 831	313 427	4 160 551	173·47	7 013·11	160 419·15
8	18 894	294 533	3 847 124	170·26	6 839·64	153 406·04
9	17 997	276 536	3 552 591	167·88	6 669·38	146 566·40
40	17 138	259 398	3 276 055	164·83	6 501·50	139 897·02
1	16 313	243 085	3 016 657	162·92	6 336·67	133 395·52
2	15 523	227 562	2 773 572	161·28	6 173·75	127 058·85
3	14 765	212 797	2 546 010	159·35	6 012·47	120 885·10
4	14 038	198 759	2 333 213	158·19	5 853·12	114 872·63
45	13 340	185 419	2 134 454	157·04	5 694·93	109 019·51
6	12 670	172 749	1 949 035	156·07	5 537·89	103 324·58
7	12 026	160 723	1 776 286	155·39	5 381·82	97 786·69
8	11 408	149 315	1 615 563	155·27	5 226·43	92 404·87
9	10 814	138 501	1 466 248	154·92	5 071·16	87 178·44

TABLE NO. XIX.

Commutation Table.

4 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
50	10 243·	128 258·0	1 327 747·3	154·79	4 916·24	82 107·28
1	9 694·4	118 563·6	1 199 489·3	155·21	4 761·45	77 191·04
2	9 166·4	109 397·2	1 080 925·7	155·49	4 606·24	72 429·59
3	8 658·2	100 739·0	971 528·5	155·89	4 450·75	67 823·35
4	8 169·4	92 569·6	870 789·5	156·48	4 294·86	63 372·60
55	7 698·7	84 870·9	778 219·9	157·25	4 138·38	59 077·74
6	7 245·4	77 625·5	693 349·0	157·72	3 981·13	54 939·36
7	6 808·9	70 816·6	615 723·5	158·44	3 823·41	50 958·23
8	6 388·7	64 427·9	544 906·9	159·37	3 664·97	47 134·82
9	5 983·6	58 444·3	480 479·0	159·89	3 505·60	43 469·85
60	5 593·6	52 850·7	422 034·7	160·42	3 345·71	39 964·25
1	5 218·1	47 632·6	369 184·0	160·83	3 185·29	36 618·54
2	4 856·5	42 776·1	321 551·4	161·07	3 024·46	33 433·25
3	4 508·6	38 267·5	278 775·3	161·14	2 863·39	30 408·79
4	4 174·1	34 093·4	240 507·8	160·88	2 702·25	27 545·40
65	3 852·7	30 240·7	206 414·4	160·25	2 541·37	24 843·15
6	3 544·2	26 696·5	176 173·7	159·21	2 381·12	22 301·78
7	3 248·7	23 447·8	149 477·2	157·88	2 221·91	19 920·66
8	2 965·9	20 481·9	126 029·4	155·88	2 064·03	17 698·75
9	2 695·9	17 786·0	105 547·5	153·36	1 908·15	15 634·72
70	2 438·9	15 347·1	87 761·5	150·30	1 754·79	13 726·57
1	2 194·7	13 152·4	72 414·4	146·54	1 604·49	11 971·78
2	1 963·8	11 188·6	59 262·0	142·16	1 457·95	10 367·29
3	1 746·1	9 442·5	48 073·4	137·02	1 315·79	8 909·34
4	1 541·9	7 900·6	38 630·9	131·27	1 178·77	7 593·55
75	1 351·4	6 549·2	30 730·3	124·80	1 047·50	6 414·78
6	1 174·6	5 374·6	24 181·1	117·71	922·70	5 367·28
7	1 011·7	4 362·91	18 806·47	109·94	804·99	4 444·58
8	862·84	3 500·07	14 443·56	101·75	695·05	3 639·59
9	727·93	2 772·14	10 943·49	93·102	593·299	2 944·536
80	606·81	2 165·33	8 171·35	84·182	500·197	2 351·237
1	499·30	1 666·03	6 006·02	75·129	416·015	1 851·040
2	404·97	1 261·06	4 339·99	66·028	340·886	1 435·025
3	323·36	937·70	3 078·93	57·111	274·858	1 094·139
4	253·81	683·89	2 141·23	48·532	217·747	819·281
85	195·52	488·37	1 457·34	40·459	169·215	601·534
6	147·54	340·83	968·97	33·035	128·756	432·319
7	108·83	232·001	628·136	26·312	95·721	303·563
8	78·332	153·669	396·135	20·453	69·409	207·842
9	54·866	98·803	242·466	15·446	48·956	138·433
90	37·310	61·493	143·663	11·329	33·510	89·477
1	24·547	36·946	82·170	8·0208	22·1811	55·9668
2	15·581	21·365	45·224	5·4457	14·1603	33·7857
3	9·536	11·829	23·859	3·6077	8·7146	19·6254
4	5·562	6·267	12·030	2·2403	5·1069	10·9108
95	3·108	3·159	5·763	1·3435	2·8666	5·8039
6	1·645	1·514	2·604	·7573	1·5231	2·9373
7	·824	·690	1·090	·3855	·7658	1·4142
8	·407	·283	·400	·2059	·3803	·6484
9	·185	·098	·117	·0990	·1744	·2681
100	·079	·019	·019	·0571	·0754	·0937
1	·019	...	...	·0183	·0183	·0183

TABLE NO. XX.

Logarithms of D_x, N_x, *and* M_x.
4 *per-cent.*

x	log D_x	log N_x	log M_x	x	log D_x	log N_x	log M_x
0	5·10477	6·34335	4·57507	50	4·01043	5·10809	3·69163
1	·03576	·32143	·37628	1	3·98652	·07394	·67774
2	·00321	·30003	·30365	2	·96220	·03902	·66334
3	4·97661	·27891	·25551	3	·93743	·00320	·64844
4	·95282	·25792	·22019	4	·91219	4·96647	·63295
5	·93026	·23699	·19072	55	·88642	·92876	·61683
6	·90876	·21609	·16658	6	·86006	·89001	·60000
7	·88811	·19521	·14700	7	·83308	·85014	·58245
8	·86814	·17423	·13127	8	·80541	·80907	·56407
9	·84871	·15324	·11850	9	·77696	·76674	·54476
10	·82967	·13210	·10802	60	·74769	·72305	·52449
1	·81085	·11086	·09896	1	·71751	·67791	·50315
2	·79221	·08948	·09089	2	·68632	·63120	·48065
3	·77367	·06796	·08350	3	·65404	·58284	·45688
4	·75514	·04630	·07649	4	·62056	·53267	·43173
15	·73662	·02449	·06963	65	·58576	·48060	·40507
6	·71800	·00251	·06243	6	·54952	·42646	·37678
7	·69925	5·98034	·05496	7	·51171	·37011	·34672
8	·68032	·95801	·04689	8	·47215	·31137	·31471
9	·66121	·93551	·03818	9	·43071	·25008	·28062
20	·64188	·91284	·02886	70	·38719	·18602	·24423
1	·62236	·89000	·01899	1	·34138	·11899	·20534
2	·60268	·86699	·00864	2	·29310	·04879	·16376
3	·58284	·84381	3·99799	3	·24208	3·97509	·11919
4	·56290	·82045	·98712	4	·18807	·89766	·07144
25	·54286	·79690	·97612	75	·13077	·81619	·02015
6	·52274	·77316	·96509	6	·06989	·73035	2·96506
7	·50257	·74923	·95411	7	·00505	·63978	·90579
8	·48235	·72508	·94314	8	2·93593	·54408	·84202
9	·46206	·70070	·93225	9	·86209	·44281	·77327
30	·44172	·67610	·92137	80	·78305	·33552	·69914
1	·42133	·65124	·91058	1	·69836	·22168	·61911
2	·40096	·62614	·89980	2	·60742	·10075	·53261
3	·38033	·60074	·88904	3	·50968	2·97206	·43911
4	·35972	·57506	·87828	4	·40451	·83499	·33796
35	·33902	·54909	·86753	85	·29119	·68875	·22845
6	·31823	·52278	·85674	6	·16890	·53254	·10978
7	·29734	·49614	·84591	7	·03675	·36549	1·98101
8	·27633	·46913	·83503	8	1·89394	·18659	·84142
9	·25521	·44176	·82409	9	·73930	1·99477	·68981
40	·23395	·41397	·81301	90	·57183	·78883	·52517
1	·21254	·38577	·80186	1	·38999	·56757	·34598
2	·19098	·35710	·79055	2	·19260	·32970	·15106
3	·16923	·32797	·77906	3	0·97938	·07295	0·94025
4	·14729	·29833	·76739	4	·74522	0·79704	·70816
45	·12514	·26816	·75549	95	·49242	·49956	·45737
6	·10276	·23742	·74335	6	·21606	·18027	·18273
7	·08012	·20607	·73093	7	$\bar{1}$·91597	$\bar{1}$·83914	$\bar{1}$·88413
8	·05721	·17412	·71820	8	·60948	·45266	·58017
9	·03399	·14145	·70511	9	·26794	$\bar{2}$·99228	·24160
				100	$\bar{2}$·89873	·27963	$\bar{2}$·87749
				1	·27963	...	·26260

TABLE NO. XXI.

Value of Annuities, and Single and Annual Premiums.

4 *per-cent.*

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	17·321	·29532	·01612	50	12·522	·47995	·03550
1	19·305	·21904	·01079	1	12·230	·49116	·03712
2	19·807	·19973	·00960	2	11·935	·50250	·03885
3	20·059	·19006	·00903	3	11·635	·51406	·04068
4	20·188	·18508	·00874	4	11·331	·52573	·04263
5	20·264	·18216	·00857	55	11·024	·53754	·04471
6	20·292	·18106	·00850	6	10·714	·54947	·04691
7	20·282	·18151	·00853	7	10·401	·56153	·04925
8	20·234	·18329	·00863	8	10·085	·57367	·05175
9	20·162	·18612	·00880	9	9·767	·58587	·05441
10	20·065	·18982	·00901	60	9·448	·59814	·05725
1	19·953	·19414	·00927	1	9·129	·61044	·06027
2	19·828	·19892	·00955	2	8·808	·62277	·06350
3	19·692	·20409	·00986	3	8·488	·63510	·06694
4	19·551	·20958	·01020	4	8·168	·64740	·07061
15	19·403	·21528	·01055	65	7·850	·65964	·07454
6	19·254	·22102	·01091	6	7·533	·67183	·07874
7	19·102	·22684	·01128	7	7·218	·68393	·08323
8	18·954	·23258	·01166	8	6·906	·69592	·08802
9	18·806	·23822	·01203	9	6·597	·70780	·09316
20	18·662	·24377	·01240	70	6·293	·71952	·09866
1	18·520	·24925	·01277	1	5·993	·73107	·10455
2	18·379	·25466	·01314	2	5·698	·74244	·11086
3	18·238	·26011	·01352	3	5·408	·75355	·11760
4	18·095	·26560	·01391	4	5·124	·76449	·12484
25	17·949	·27118	·01431	75	4·846	·77514	·13258
6	17·800	·27692	·01473	6	4·576	·78554	·14089
7	17·647	·28284	·01517	7	4·313	·79568	·14978
8	17·488	·28893	·01563	8	4·057	·80555	·15931
9	17·324	·29525	·01611	9	3·808	·81504	·16951
30	17·155	·30175	·01662	80	3·568	·82431	·18044
1	16·979	·30850	·01716	1	3·337	·83320	·19213
2	16·799	·31546	·01772	2	3·114	·84176	·20461
3	16·612	·32263	·01832	3	2·900	·85002	·21795
4	16·419	·33004	·01895	4	2·695	·85793	·23222
35	16·221	·33768	·01961	85	2·498	·86549	·24743
6	16·016	·34555	·02031	6	2·310	·87273	·26365
7	15·805	·35365	·02104	7	2·132	·87955	·28085
8	15·588	·36199	·02182	8	1·962	·88609	·29918
9	15·366	·37058	·02264	9	1·801	·89230	·31858
40	15·136	·37937	·02351	90	1·648	·89813	·33916
1	14·902	·38844	·02443	1	1·505	·90363	·36070
2	14·660	·39771	·02540	2	1·371	·90878	·38326
3	14·413	·40722	·02642	3	1·240	·91384	·40790
4	14·159	·41697	·02751	4	1·127	·91821	·43173
45	13·900	·42692	·02865	95	1·017	·92247	·45744
6	13·635	·43711	·02987	6	·921	·92613	·48214
7	13·364	·44752	·03115	7	·838	·92931	·50566
8	13·089	·45813	·03252	8	·697	·93474	·55085
9	12·807	·46894	·03396	9	·530	·94115	·61509
				100	·240	·95227	·76773
				1	...	·96154	·96154

TABLE No. XXII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.* 4 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
0	10·20954	11·32141	15·31431	16·30775	20·41908	21·29953	0
1	·08855	·29524	·14134	·27708	·19413	·26404	1
2	·04049	·27038	·07777	·24873	·11505	·23208	2
3	·00432	·24618	·03203	·22152	·05974	·20186	3
4	9·97377	·22235	14·99472	·19499	·01567	·17260	4
5	·94569	·19874	·96112	·16885	19·97655	·14401	5
6	·91972	·17528	·93068	·14301	·94164	·11588	6
7	·89545	·15186	·90279	·11727	·91013	·08796	7
8	·87255	·12840	·87696	·09156	·88137	·06009	8
9	·85072	·10486	·85273	·06577	·85474	·03214	9
10	·82967	·08117	·82967	·03977	·82967	·00402	10
1	·80907	·05732	·80729	·01355	·80551	20·97561	1
2	·78882	·03326	·78543	15·98709	·78204	·94686	2
3	·76877	·00898	·76387	·96031	·75897	·91774	3
4	·74875	10·98445	·74236	·93319	·73597	·88817	4
15	·72874	·95965	·72086	·90570	·71298	·85815	15
6	·70853	·93458	·69906	·87784	·68959	·82764	6
7	·68807	·90924	·67689	·84960	·66571	·79664	7
8	·66724	·88362	·65416	·82098	·64108	·76518	8
9	·64605	·85774	·63089	·79201	·61573	·73325	9
20	·62443	·83158	·60698	·76267	·58953	·70088	20
1	·60242	·80517	·58248	·73300	·56254	·66809	1
2	·58009	·77851	·55750	·70299	·53491	·63489	2
3	·55745	·75159	·53206	·67266	·50667	·60130	3
4	·53460	·72440	·50630	·64199	·47800	·56731	4
25	·51155	·69695	·48024	·61099	·44893	·53292	25
6	·48835	·66921	·45396	·57964	·41957	·49810	6
7	·46504	·64120	·42751	·54790	·38998	·46285	7
8	·44163	·61286	·40091	·51578	·36019	·42713	8
9	·41809	·58420	·37412	·48326	·33015	·39094	9
30	·39444	·55519	·34716	·45030	·29988	·35420	30
1	·37069	·52582	·32005	·41686	·26941	·31691	1
2	·34679	·49607	·29272	·38294	·23865	·27905	2
3	·32276	·46592	·26519	·34848	·20762	·24055	3
4	·29857	·43532	·23742	·31349	·17627	·20137	4
35	·27421	·40427	·20940	·27788	·14459	·16149	35
6	·24966	·37273	·18109	·24167	·11252	·12087	6
7	·22491	·34066	·15248	·20477	·08005	·07940	7
8	·19993	·30807	·12353	·16717	·04713	·03711	8
9	·17472	·27487	·09423	·12882	·01374	19·99385	9
40	·14923	·24107	·06451	·08966	18·97979	·94963	40
1	·12345	·20664	·03436	·04964	·94527	·90436	1
2	·09736	·17146	·00374	·00869	·91012	·85795	2
3	·07089	·13558	13·97255	14·96676	·87421	·81035	3
4	·04405	·09889	·94081	·92379	·83757	·76146	4
45	·01678	·06138	·90842	·87971	·80006	·71120	45
6	8·98905	·02296	·87534	·83443	·76163	·65946	6
7	·96001	9·98357	·84150	·78789	·72219	·60613	7
8	·93202	·94318	·80683	·73997	·68164	·55112	8
9	·90261	·90171	·77123	·69059	·63985	·49428	9

TABLE No. XXII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*
4 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
50	8·87253	9·85908	13·73463	14·63966	18·59673	19·43549	50
1	·84174	·81521	·69696	·58704	·55218	·37460	1
2	·81013	·77000	·65806	·53263	·50599	·31144	2
3	·77763	·72339	·61783	·47628	·45803	·24586	3
4	·74418	·67526	·57617	·41785	·40816	·17768	4
55	·70967	·62551	·53292	·35721	·35617	·10667	55
6	·67399	·57402	·48792	·29416	·30185	·03262	6
7	·63706	·52065	·44104	·22855	·24502	18·95532	7
8	·59875	·46528	·39209	·16014	·18543	·87446	8
9	·55889	·40776	·34082	·08874	·12275	·78980	9
60	·51738	·34792	·28707	·01410	·05676	·70101	60
1	·47405	·28558	·23059	13·93600	17·98713	·60774	1
2	·42871	·22058	·17110	·85412	·91349	·50965	2
3	·38118	·15266	·10832	·76816	·83546	·40632	3
4	·33125	·08164	·04194	·67781	·75263	·29730	4
65	·27869	·00728	12·97162	·58268	·66455	·18213	65
6	·22324	8·92925	·89696	·48239	·57068	·06028	6
7	·16465	·84730	·81759	·37650	·47053	17·93117	7
8	·10257	·76112	·73299	·26456	·36341	·79417	8
9	·03672	·67034	·64273	·14600	·24874	·64859	9
70	7·96671	·57459	·54623	·02028	·12575	·49368	70
1	·89213	·47346	·44288	12·88682	16·99363	·32862	1
2	·81260	·36646	·33210	·74488	·85160	·15247	2
3	·72759	·25314	·21310	·59374	·69861	16·96426	3
4	·63661	·13293	·08515	·43257	·53369	·76288	4
75	·53904	·00522	11·94731	·26050	·35558	·54716	75
6	·43431	7·86938	·79873	·07653	·16315	·31576	6
7	·32167	·72470	·63829	11·87960	15·95491	·06729	7
8	·20046	·57038	·46499	·66852	·72952	15·80013	8
9	·06981	·40555	·27753	·44196	·48525	·51251	9
80	6·92877	·22927	·07449	·19855	·22021	·20257	80
1	·77642	·04052	10·85448	10·93668	14·93254	14·86815	1
2	·61157	6·83819	·61572	·65472	·61987	·50707	2
3	·43313	·62103	·35658	·35077	·28003	·11681	3
4	·23982	·38766	·07513	·02276	13·91044	13·69458	4
85	·03021	·13656	9·76923	9·66834	·50825	·23724	85
6	5·80267	5·86610	·43644	·28506	·07021	12·74140	6
7	·55540	·57454	·07405	8·87034	12·59270	·20371	7
8	·28681	·25978	8·67968	·42096	·07255	11·61978	8
9	4·99457	4·91970	·24984	7·93374	11·50511	10·98537	9
90	·67666	·55172	7·78149	·40459	10·88632	·29486	90
1	·33001	·15357	·27003	6·83023	·21005	9·54381	1
2	3·95227	3·72287	6·71194	·20758	9·47161	8·72892	2
3	·54286	·25348	·10634	5·52558	8·66982	7·83350	3
4	·09157	2·74654	5·43792	4·78743	7·78427	6·86358	4
95	2·60301	·19590	4·71360	3·98145	6·82419	5·80161	95
6	·06732	1·60451	3·91858	·11106	5·76984	4·65005	6
7	1·48417	0·98832	·05237	2·21269	4·62057	3·46702	7
8	0·88823	·30183	2·16698	1·21211	3·44573	2·14672	8
9	·22218	$\bar{1}$·52613	1·17642	0·10931	2·13066	0·70654	9
100	$\bar{1}$·50079	$\bar{2}$·27963	0·10285	$\bar{2}$·27963	0·70491	$\bar{2}$·27963	100
1	$\bar{2}$·27963	...	$\bar{2}$·27963	...	$\bar{2}$·27963	...	1

TABLE NO. XXIII.

Values of Annuities. Joint Lives. Equal Ages.
4 per-cent.

x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}	x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}
0	12·938	9·850	7·594	50	9·695	8·036	6·899
1	16·095	13·669	11·747	1	9·407	7·764	6·644
2	16·978	14·824	13·093	2	9·117	7·492	6·389
3	17·453	15·470	13·871	3	8·826	7·219	6·135
4	17·725	15·859	14·353	4	8·533	6·945	5·882
5	17·908	16·134	14·705	55	8·238	6·673	5·630
6	18·012	16·305	14·936	6	7·944	6·401	5·380
7	18·047	16·386	15·060	7	7·649	6·131	5·132
8	18·024	16·391	15·091	8	7·354	5·862	4·887
9	17·953	16·332	15·045	9	7·061	5·597	4·646
10	17·844	16·222	14·940	60	6·769	5·334	4·408
1	17·711	16·079	14·794	1	6·479	5·075	4·175
2	17·557	15·910	14·616	2	6·193	4·820	3·946
3	17·386	15·720	14·414	3	5·909	4·569	3·723
4	17·207	15·518	14·197	4	5·629	4·324	3·505
15	17·018	15·305	13·969	65	5·353	4·084	3·293
6	16·829	15·093	13·742	6	5·082	3·850	3·088
7	16·641	14·884	13·519	7	4·816	3·622	2·888
8	16·458	14·683	13·308	8	4·556	3·401	2·696
9	16·281	14·492	13·108	9	4·302	3·186	2·511
20	16·112	14·312	12·923	70	4·054	2·979	2·333
1	15·950	14·142	12·751	1	3·814	2·779	2·163
2	15·791	13·979	12·589	2	3·580	2·587	1·999
3	15·637	13·823	12·435	3	3·354	2·402	1·844
4	15·481	13·668	12·283	4	3·136	2·226	1·695
25	15·325	13·513	12·134	75	2·925	2·057	1·555
6	15·166	13·356	11·982	6	2·723	1·896	1·421
7	15·002	13·194	11·827	7	2·530	1·743	1·295
8	14·833	13·028	11·666	8	2·344	1·598	1·177
9	14·659	12·857	11·502	9	2·166	1·460	1·065
30	14·479	12·681	11·332	80	1·998	1·331	·960
1	14·293	12·497	11·156	1	1·837	1·208	·862
2	14·102	12·309	10·975	2	1·685	1·094	·771
3	13·905	12·114	10·788	3	1·541	·987	·687
4	13·701	11·914	10·595	4	1·406	·886	·608
35	13·491	11·708	10·397	85	1·278	·793	·536
6	13·276	11·497	10·194	6	1·157	·706	·469
7	13·054	11·280	9·985	7	1·045	·626	·408
8	12·827	11·057	9·772	8	·940	·551	·353
9	12·594	10·829	9·552	9	·842	·483	·302
40	12·355	10·596	9·329	90	·750	·420	·256
1	12·111	10·358	9·101	1	·666	·363	·216
2	11·860	10·115	8·868	2	·590	·313	·181
3	11·606	9·868	8·633	3	·514	·263	·146
4	11·346	9·616	8·393	4	·452	·224	·120
45	11·082	9·360	8·150	95	·392	·185	·095
6	10·812	9·101	7·904	6	·345	·156	·076
7	10·538	8·839	7·655	7	·319	·145	·070
8	10·260	8·573	7·404	8	·259	·111	·050
9	9·979	8·305	7·152	9	·201	·086	·038
				100	·060	·015	·004

TABLES.

FOUR-AND-A-HALF PER CENT.

ONE LIFE.

TABLE NO. XXIV.—D, N, S, C, M, R.
" " XXV.—LOG D, LOG N, LOG M.
" " XXVI.—*a*, A, P.

TWO, THREE, AND FOUR LIVES.

TABLE NO. XXVII.—LOG D, LOG N.
" " XXVIII.—*a*.

Constants.

Constant.	Number.	Logarithm.
i	·045	$\bar{2}$·6532125
$(1+i)$	1·045	0·0191163
$(1+i)^{\frac{1}{2}}$	1·0222524	0·0095581
$(1+i)^{\frac{1}{4}}$	1·0110650	0·0047791
v	·9569378	$\bar{1}$·9808837
$v^{\frac{1}{2}}$	·9782320	$\bar{1}$·9904419
$v^{\frac{1}{4}}$	·9890561	$\bar{1}$·9952209
d	·0430622	$\bar{2}$·6340962
δ	·0440169	$\bar{2}$·6436193
$j_{(2)}$	·0445048	$\bar{2}$·6484072
$j_{(4)}$	·0442600	$\bar{2}$·6460110

TABLE NO. XXIV.

Commutation Table.

$4\frac{1}{2}$ per-cent.

x	D_x	N_x	S_x	C_x	M_x	R_x
0	127 280	2 016 855	36 266 604	13 739·	34 951·	490 091·
1	108 060	1 908 795	34 249 749	3 628·1	21 211·8	455 139·7
2	99 781	1 809 014	32 340 954	2 081·2	17 583·7	433 927·9
3	93 403	1 715 611	30 531 940	1 380·3	15 502·5	416 344·2
4	87 999	1 627 612	28 816 329	1 063·3	14 122·2	400 841·7
5	83 148	1 544 464	27 188 717	814·74	13 058·90	386 719·54
6	78 752	1 465 712	25 644 253	626·08	12 244·16	373 660·64
7	74 736	1 390 976	24 178 541	480·28	11 618·08	361 416·48
8	71 036	1 319 940	22 787 565	374·81	11 137·80	349 798·40
9	67 602	1 252 338	21 467 625	298·79	10 762·99	338 660·60
10	64 393	1 187 945	20 215 287	251·41	10 464·20	327 897·61
1	61 368	1 126 577	19 027 342	217·59	10 212·79	317 433·41
2	58 507	1 068 070	17 900 765	195·24	9 995·20	307 220·62
3	55 794	1 012 276	16 832 695	181·97	9 799·96	297 225·42
4	53 208	959 068	15 820 419	174·14	9 617·99	287 425·46
15	50 743	908 325	14 861 351	178·01	9 443·85	277 807·47
6	48 380	859 945	13 953 026	181·70	9 265·84	268 363·62
7	46 115	813 830	13 093 081	192·44	9 084·14	259 097·78
8	43 937	769 893	12 279 251	201·48	8 891·70	250 013·64
9	41 844	728 049	11 509 358	210·63	8 690·22	241 121·94
20	39 831	688 218	10 781 309	217·44	8 479·59	232 431·72
1	37 898	650 320	10 093 091	220·98	8 262·15	223 952·13
2	36 045	614 275	9 442 771	221·28	8 041·17	215 689·98
3	34 272	580 003	8 828 496	219·40	7 819·89	207 648·81
4	32 577	547 426	8 248 493	215·27	7 600·49	199 828·92
25	30 959	516 467	7 701 067	209·51	7 385·22	192 228·43
6	29 416	487 051	7 184 600	202·32	7 175·71	184 843·21
7	27 947	459 104	6 697 549	196·23	6 973·39	177 667·50
8	26 547	432 557	6 238 445	189·17	6 777·16	170 694·11
9	25 215	407 342	5 805 888	183·16	6 587·99	163 916·95
30	23 946	383 396	5 398 546	176·56	6 404·83	157 328·96
1	22 738	360 658	5 015 150	171·15	6 228·27	150 924·13
2	21 588	339 070	4 654 492	165·89	6 057·12	144 695·86
3	20 492	318 578	4 315 422	160·98	5 891·23	138 638·74
4	19 449	299 129	3 996 844	156·19	5 730·25	132 747·51
35	18 455	280 674	3 697 715	152·13	5 574·06	127 017·26
6	17 508	263 166	3 417 041	148·33	5 421·93	121 443·20
7	16 606	246 560	3 153 875	144·57	5 273·60	116 021·27
8	15 746	230 814	2 907 315	141·21	5 129·03	110 747·67
9	14 927	215 887	2 676 501	138·58	4 987·82	105 618·64
40	14 146	201 741	2 460 614	135·40	4 849·24	100 630·82
1	13 401	188 340	2 258 873	133·20	4 713·84	95 781·58
2	12 691	175 649	2 070 533	131·23	4 580·64	91 067·74
3	12 013	163 636	1 894 884	129·03	4 449·41	86 487·10
4	11 367	152 269	1 731 248	127·48	4 320·38	82 037·69
45	10 750	141 519	1 578 979	125·95	4 192·90	77 717·31
6	10 161	131 357·7	1 437 459·6	124·57	4 066·95	73 524·41
7	9 598·9	121 758·8	1 306 101·9	123·44	3 942·38	69 457·46
8	9 062·1	112 696·7	1 184 343·1	122·75	3 818·94	65 515·08
9	8 549·1	104 147·6	1 071 646·4	121·89	3 696·19	61 696·14

TABLE NO. XXIV.

Commutation Table.

4½ *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
50	8 059·2	96 088·4	967 498·8	121·20	3 574·30	57 999·95
1	7 590·8	88 497·6	871 410·4	120·95	3 453·10	54 425·65
2	7 143·0	81 354·6	782 912·8	120·59	3 332·15	50 972·55
3	6 714·9	74 639·7	701 558·2	120·32	3 211·56	47 640·40
4	6 305·4	68 334·3	626 918·5	120·20	3 091·24	44 428·84
55	5 913·6	62 420·7	558 584·2	120·21	2 971·04	41 337·60
6	5 538·9	56 881·8	496 163·5	119·99	2 850·83	38 366·56
7	5 180·2	51 701·6	439 281·7	119·97	2 730·84	35 515·73
8	4 837·3	46 864·3	387 580·1	120·09	2 610·87	32 784·89
9	4 508·9	42 355·4	340 715·8	119·91	2 490·78	30 174·02
60	4 194·8	38 160·6	298 360·4	119·73	2 370·87	27 683·24
1	3 894·5	34 266·1	260 199·8	119·46	2 251·14	25 312·37
2	3 607·3	30 658·8	225 933·7	119·07	2 131·68	23 061·23
3	3 332·8	27 326·0	195 274·9	118·54	2 012·61	20 929·55
4	3 070·8	24 255·2	167 948·9	117·79	1 894·07	18 916·94
65	2 820·8	21 434·4	143 693·7	116·76	1 776·28	17 022·87
6	2 582·5	18 851·9	122 259·3	115·46	1 659·52	15 246·59
7	2 355·9	16 496·0	103 407·4	113·94	1 544·06	13 587·07
8	2 140·5	14 355·5	86 911·4	111·96	1 430·12	12 043·01
9	1 936·4	12 419·1	72 555·9	109·62	1 318·16	10 612·89
70	1 743·3	10 675·8	60 136·8	106·92	1 208·54	9 294·73
1	1 561·3	9 114·5	49 461·0	103·75	1 101·62	8 086·19
2	1 390·4	7 724·1	40 346·5	100·16	997·87	6 984·57
3	1 230·3	6 493·8	32 622·4	96·079	897·707	5 986·699
4	1 081·3	5 412·49	26 128·57	91·614	801·628	5 088·992
75	943·08	4 469·41	20 716·08	86·680	710·014	4 287·364
6	815·79	3 653·62	16 246·67	81·364	623·334	3 577·350
7	699·31	2 954·31	12 593·05	75·631	541·970	2 954·016
8	593·55	2 360·76	9 638·74	69·656	466·339	2 412·046
9	498·34	1 862·42	7 277·98	63·435	396·683	1 945·707
80	413·45	1 448·97	5 415·56	57·082	333·248	1 549·024
1	338·56	1 110·41	3 966·59	50·699	276·166	1 215·776
2	273·28	837·13	2 856·18	44·346	225·467	939·610
3	217·17	619·96	2 019·05	38·172	181·121	714·143
4	169·64	450·32	1 399·09	32·283	142·949	533·022
85	130·06	320·260	948·768	26·784	110·666	390·073
6	97·672	222·588	628·508	21·765	83·882	279·407
7	71·702	150·886	405·920	17·252	62·117	195·525
8	51·362	99·524	255·034	13·347	44·865	133·408
9	35·803	63·721	155·510	10·031	31·518	88·543
90	24·230	39·491	91·789	7·3223	21·4866	57·0253
1	15·865	23·626	52·298	5·1593	14·1643	35·5387
2	10·022	13·604	28·672	3·4861	9·0050	21·3744
3	6·105	7·499	15·068	2·2984	5·5189	12·3694
4	3·543	3·956	7·569	1·4205	3·2205	6·8505
95	1·970	1·986	3·613	·8478	1·8000	3·6300
6	1·038	·948	1·627	·4755	·9522	1·8300
7	·518	·430	·679	·2409	·4767	·8778
8	·254	·176	·249	·1281	·2358	·4011
9	·115	·061	·073	·0613	·1077	·1653
100	·049	·012	·012	·0352	·0464	·0576
1	·012	...	...	·0112	·0112	·0112

Table No. XXV.

Logarithms of D_x, N_x, *and* M_x.
4½ *per-cent.*

x	log D_x	log N_x	log M_x	x	log D_x	log N_x	log M_x
0	5·10477	6·30468	4·54346	50	3·90629	4·98267	3·55319
1	·03367	·28076	·32658	1	·88029	·94693	·53821
2	4·99905	·25744	·24512	2	·85388	·91038	·52273
3	·97036	·23442	·19042	3	·82704	·87297	·50672
4	·94448	·21155	·14990	4	·79971	·83464	·49013
5	·91985	·18879	·11591	55	·77185	·79533	·47290
6	·89626	·16605	·08792	6	·74342	·75497	·45497
7	·87353	·14333	·06513	7	·71435	·71351	·43629
8	·85148	·12054	·04681	8	·68460	·67084	·41679
9	·82996	·09771	·03193	9	·65407	·62690	·39634
10	·80884	·07478	·01970	60	·62271	·58162	·37491
1	·78794	·05177	·00915	1	·59045	·53486	·35239
2	·76721	·02861	3·99979	2	·55718	·48656	·32873
3	·74659	·00531	·99123	3	·52281	·43658	·30376
4	·72598	5·98185	·98308	4	·48725	·38480	·27740
15	·70538	·95824	·97515	65	·45037	·33110	·24952
6	·68467	·93447	·96688	6	·41204	·27536	·21998
7	·66384	·91053	·95828	7	·37215	·21738	·18868
8	·64283	·88643	·94898	8	·33051	·15703	·15537
9	·62163	·86216	·93903	9	·28699	·09409	·11998
20	·60022	·83773	·92838	70	·24138	·02841	·08225
1	·57862	·81313	·91710	1	·19349	3·95973	·04102
2	·55685	·78837	·90532	2	·14313	·88785	2·99907
3	·53494	·76343	·89320	3	·09002	·81250	·95314
4	·51291	·73833	·88084	4	·03393	·73340	·90397
25	·49078	·71305	·86836	75	2·97455	·65025	·85126
6	·46859	·68757	·85586	6	·91158	·56272	·79472
7	·44633	·66191	·84344	7	·84467	·47045	·73398
8	·42402	·63605	·83105	8	·77346	·37306	·66870
9	·40166	·60996	·81875	9	·69753	·27007	·59844
30	·37923	·58365	·80651	80	·61642	·16107	·52277
1	·35676	·55710	·79437	1	·52964	·04548	·44118
2	·33421	·53029	·78226	2	·43661	2·92279	·35309
3	·31159	·50322	·77020	3	·33680	·79236	·25797
4	·28890	·47586	·75818	4	·22954	·65352	·15518
35	·26612	·44820	·74617	85	·11414	·50550	·04403
6	·24324	·42024	·73415	6	1·98977	·34751	1·92367
7	·22027	·39192	·72211	7	·85553	·17866	·79321
8	·19718	·36325	·71003	8	·71064	1·99793	·65191
9	·17397	·33423	·69791	9	·55392	·80428	·49856
40	·15063	·30479	·68567	90	·38436	·59650	·33218
1	·12714	·27494	·67337	1	·20044	·37339	·15119
2	·10350	·24465	·66092	2	·00097	·13367	0·95448
3	·07966	·21389	·64830	3	0·78567	0·87502	·74185
4	·05564	·18261	·63552	4	·54942	·59725	·50792
45	·03141	·15082	·62251	95	·29454	·29789	·25527
6	·00694	·11846	·60927	6	·01610	$\bar{1}$·97674	$\bar{1}$·97875
7	3·98222	·08550	·59576	7	$\bar{1}$·71392	·63380	·67824
8	·95723	·05192	·58194	8	·40535	·24559	·37249
9	·93192	·01766	·56776	9	·06173	$\bar{2}$·78358	·03218
				100	$\bar{2}$·69043	·06925	$\bar{2}$·66661
				1	·06925	...	·05014

TABLE No. XXVI.

Values of Annuities, and Single and Annual Premiums for Assurance of a Unit.

4½ per-cent.

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	15·846	·27459	·01630	50	11·923	·44351	·03432
1	17·664	·19630	·01052	1	11·658	·45490	·03594
2	18·130	·17623	·00921	2	11·389	·46650	·03765
3	18·368	·16598	·00857	3	11·116	·47828	·03948
4	18·496	·16048	·00823	4	10·838	·49025	·04142
5	18·575	·15706	·00802	55	10·556	·50240	·04348
6	18·612	·15547	·00793	6	10·270	·51470	·04567
7	18·612	·15545	·00793	7	9·981	·52716	·04801
8	18·581	·15679	·00801	8	9·688	·53975	·05050
9	18·525	·15921	·00815	9	9·394	·55242	·05315
10	18·448	·16250	·00836	60	9·097	·56520	·05598
1	18·358	·16642	·00860	1	8·799	·57802	·05899
2	18·256	·17084	·00887	2	8·499	·59095	·06221
3	18·143	·17565	·00918	3	8·199	·60388	·06564
4	18·025	·18076	·00950	4	7·899	·61681	·06931
15	17·900	·18611	·00985	65	7·599	·62972	·07324
6	17·775	·19152	·01020	6	7·300	·64260	·07742
7	17·648	·19699	·01056	7	7·002	·65544	·08191
8	17·523	·20237	·01093	8	6·707	·66813	·08669
9	17·399	·20768	·01129	9	6·414	·68075	·09182
20	17·279	·21289	·01165	70	6·124	·69322	·09731
1	17·160	·21801	·01201	1	5·838	·70393	·10295
2	17·042	·22308	·01236	2	5·555	·71770	·10948
3	16·923	·22817	·01273	3	5·278	·72966	·11622
4	16·804	·23331	·01310	4	5·006	·74138	·12344
25	16·683	·23855	·01349	75	4·739	·75285	·13118
6	16·557	·24393	·01389	6	4·479	·76408	·13947
7	16·428	·24952	·01432	7	4·225	·77501	·14834
8	16·294	·25529	·01476	8	3·977	·78567	·15785
9	16·155	·26127	·01523	9	3·737	·79599	·16803
30	16·011	·26747	·01572	80	3·505	·80603	·17894
1	15·861	·27391	·01625	1	3·280	·81572	·19059
2	15·707	·28058	·01679	2	3·063	·82505	·20305
3	15·546	·28748	·01737	3	2·855	·83401	·21636
4	15·380	·29463	·01799	4	2·654	·84264	·23058
35	15·208	·30203	·01863	85	2·462	·85092	·24576
6	15·031	·30968	·01932	6	2·279	·85882	·26192
7	14·847	·31757	·02004	7	2·104	·86632	·27906
8	14·658	·32572	·02080	8	1·938	·87351	·29734
9	14·463	·33415	·02161	9	1·780	·88032	·31669
40	14·261	·34280	·02246	90	1·630	·88679	·33721
1	14·054	·35175	·02337	1	1·489	·89279	·35867
2	13·840	·36093	·02432	2	1·357	·89848	·38114
3	13·622	·37037	·02533	3	1·228	·90402	·40568
4	13·396	·38008	·02640	4	1·116	·90887	·42946
45	13·165	·39003	·02754	95	1·008	·91355	·45500
6	12·928	·40025	·02874	6	·913	·91759	·47948
7	12·685	·41071	·03001	7	·832	·92113	·50284
8	12·436	·42142	·03136	8	·692	·92713	·54830
9	12·183	·43235	·03280	9	·527	·93422	·61189
				100	·239	·94663	·76082
				1	...	·95694	·95694

TABLE NO. XXVII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.* 4½ *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
0	10·20954	11·28697	15·31431	16·27642	20·41908	21·27073	0
1	·08646	·25871	·13925	·24353	·19204	·23287	1
2	·03633	·23185	·07361	·21312	·11089	·19879	2
3	9·99807	·20575	·02578	·18395	·05349	·16652	3
4	·96543	·18001	14·98638	·15549	·00733	·13532	4
5	·93528	·15458	·95071	·12749	19·96614	·10486	5
6	·90722	·12927	·91818	·09982	·92914	·07489	6
7	·88087	·10404	·88821	·07229	·89555	·04517	7
8	·85589	·07878	·86030	·04477	·86471	·01549	8
9	·83197	·05346	·83398	·01720	·83599	20·98578	9
10	·80884	·02800	·80884	15·98944	·80884	·95588	10
1	·78616	·00238	·78438	·96148	·78260	·92571	1
2	·76382	10·97656	·76043	·93324	·75704	·89522	2
3	·74169	·95050	·73679	·90470	·73189	·86433	3
4	·71959	·92420	·71320	·87581	·70681	·83301	4
15	·69750	·89764	·68962	·84657	·68174	·80122	15
6	·67520	·87080	·66573	·81694	·65626	·76894	6
7	·65266	·84369	·64148	·78693	·63030	·73617	7
8	·62975	·81631	·61667	·75654	·60359	·70293	8
9	·60647	·78866	·59131	·72580	·57615	·66922	9
20	·58277	·76075	·56532	·69469	·54787	·63507	20
1	·55868	·73259	·53874	·66326	·51880	·60050	1
2	·53426	·70417	·51167	·63149	·48908	·56553	2
3	·50955	·67549	·48416	·59940	·45877	·53016	3
4	·48461	·64657	·45631	·56698	·42801	·49441	4
25	·45947	·61739	·42816	·53423	·39685	·45824	25
6	·43420	·58792	·39981	·50113	·36542	·42169	6
7	·40880	·55818	·37127	·46767	·33374	·38468	7
8	·38330	·52813	·34258	·43382	·30186	·34721	8
9	·35769	·49776	·31372	·39957	·26975	·30927	9
30	·33195	·46705	·28467	·36489	·23739	·27082	30
1	·30612	·43599	·25548	·32974	·20484	·23180	1
2	·28014	·40454	·22607	·29411	·17200	·19220	2
3	·25402	·37269	·19645	·25797	·13888	·15201	3
4	·22775	·34040	·16660	·22126	·10545	·11113	4
35	·20131	·30769	·13650	·18398	·07169	·06952	35
6	·17467	·27446	·10610	·14607	·03753	·02719	6
7	·14784	·24075	·07541	·10752	·00298	19·98404	7
8	·12078	·20648	·04438	·06822	18·96798	·94003	8
9	·09348	·17164	·01299	·02821	·93250	·89510	9
40	·06591	·13618	13·98119	14·98736	·89647	·84919	40
1	·03805	·10009	·94896	·94567	·85987	·80223	1
2	·00988	·06330	·91626	·90306	·82264	·75414	2
3	8·98132	·02576	·88298	·85947	·78464	·70486	3
4	·95240	9·98742	·84916	·81485	·74592	·65429	4
45	·92305	·94826	·81469	·76910	·70633	·60235	45
6	·89323	·90821	·77952	·72218	·66581	·54893	6
7	·86291	·86720	·74360	·67397	·62429	·49393	7
8	·83204	·82519	·70685	·62441	·58166	·43724	8
9	·80054	·78208	·66916	·57337	·53778	·37872	9

TABLE NO. XXVII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*
4½ *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
50	8·76839	9·73782	13·63049	14·52078	18·49259	19·31825	50
1	·73551	·69232	·59073	·46651	·44595	·25568	1
2	·70181	·64549	·54974	·41044	·39767	·19086	2
3	·66724	·59725	·50744	·35243	·34764	·12359	3
4	·63170	·54749	·46369	·29234	.29568	·05373	4
55	·59510	·49610	·41835	·23004	·24160	18·98102	55
6	·55735	·44298	·37128	·16533	·18521	·90528	6
7	·51833	·38798	·32231	·09806	·12629	·82627	7
8	·47794	·33096	·27128	·02796	·06462	·74371	8
9	·43600	·27180	·21793	13·95488	17·99986	·65733	9
60	·39240	·21032	·16209	·87857	·93178	·56683	60
1	·34699	·14635	·10353	·79876	·86007	·47185	1
2	·29957	·07965	·04196	·71518	·78435	·37201	2
3	·24995	·01009	12·97709	·62753	·70423	·26694	3
4	·19794	8·93741	·90863	·53547	·61932	·15619	4
65	·14330	·86135	·83623	·43862	·52916	·03926	65
6	·08576	·78165	·75948	·33662	·43320	17·91566	6
7	·02509	·69802	·67803	·22899	·33097	·78477	7
8	7·96093	·61015	·59135	·11528	·22177	·64599	8
9	·89300	·51766	·49901	12·99499	·10502	·49861	9
70	·82090	·42020	·40042	·86752	16·97994	·34191	70
1	·74424	·31735	·29499	·73227	·84574	·17502	1
2	·66263	·20863	·18213	·58855	·70163	16·99705	2
3	·57553	·09356	·06104	·43562	·54655	·80701	3
4	·48247	7·97159	11·93101	·27265	·37955	·60379	4
75	·38282	·84212	·79109	·09875	·19936	·38621	75
6	·27600	·70451	·64042	11·91296	·00484	·15296	6
7	·16129	·55805	·47791	·71420	15·79453	15·90261	7
8	·03799	·40193	·30252	·50126	·56705	·63356	8
9	6·90525	·23530	·11297	·27286	·32069	·34406	9
80	·76214	·05721	10·90786	·02755	·05358	·03222	80
1	·60770	6·86665	·68576	10·76381	14·76382	14·69587	1
2	·44076	·66247	·44491	·47995	·44906	·33286	2
3	·26025	·44347	·18370	·17409	·10715	13·94067	3
4	·06485	·20825	9·90016	9·84417	13·73547	·51648	4
85	5·85316	5·95527	·59218	·48786	·33120	·05717	85
6	·62354	·68291	·25731	·10264	12·89108	12·55938	6
7	·37418	·38945	8·89283	8·68597	·41148	·01970	7
8	·10351	·07280	·49638	·23464	11·88925	11·43380	8
9	4·80919	4·73078	·06446	7·74544	·31973	10·79739	9
90	·48919	·36088	7·59402	·21434	10·69885	·10486	90
1	·14046	3·96080	·08048	6·63798	·02050	9·35182	1
2	3·76064	·52815	6·52031	·01334	9·27998	8·53491	2
3	·34915	·05679	5·91263	5·32932	8·47611	7·63746	3
4	2·89577	2·54787	·24212	4·58917	7·58847	6·66551	4
95	·40513	1·99523	4·51572	3·78117	6·62631	5·60148	95
6	1·86736	·40183	3·71862	2·90870	5·56988	4·44784	6
7	·28212	0·78369	2·85032	·00834	4·41852	3·26278	7
8	0·68410	·09527	1·96285	1·00574	3·24160	1·94043	8
9	·01597	$\bar{1}$·31773	0·97021	$\bar{1}$·90099	1·92445	0·49823	9
100	$\bar{1}$·29249	$\bar{2}$·06925	$\bar{1}$·89455	$\bar{2}$·06935	0·49661	$\bar{2}$·06925	100
1	$\bar{2}$·06925	...	$\bar{2}$·06925	...	$\bar{2}$·06925	...	1

TABLE No. XXVIII.

Values of Annuities. Joint Lives. Equal Ages.
$4\frac{1}{2}$ *per-cent.*

x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}	x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}
0	11·952	9·165	7·106	50	9·320	7·768	6·694
1	14·868	12·714	10·986	1	9·053	7·512	6·453
2	15·686	13·788	12·243	2	8·784	7·256	6·211
3	16·132	14·394	12·973	3	8·512	6·998	5·970
4	16·390	14·761	13·427	4	8·237	6·740	5·729
5	16·569	15·024	13·763	55	7·962	6·482	5·488
6	16·674	15·193	13·988	6	7·685	6·224	5·249
7	16·717	15·278	14·113	7	7·407	5·967	5·012
8	16·707	15·292	14·151	8	7·129	5·711	4·776
9	16·653	15·248	14·119	9	6·852	5·457	4·544
10	16·564	15·157	14·029	60	6·575	5·206	4·316
1	16·452	15·035	13·903	1	6·300	4·957	4·091
2	16·321	14·887	13·746	2	6·027	4·712	3·870
3	16·174	14·720	13·566	3	5·756	4·471	3·654
4	16·018	14·542	13·372	4	5·489	4·235	3·442
15	15·854	14·353	13·167	65	5·225	4·003	3·237
6	15·689	14·165	12·962	6	4·965	3·777	3·037
7	15·525	13·978	12·761	7	4·709	3·556	2·843
8	15·366	13·800	12·570	8	4·459	3·341	2·656
9	15·212	13·630	12·390	9	4·214	3·133	2·475
20	15·065	13·470	12·224	70	3·975	2·932	2·301
1	14·925	13·320	12·070	1	3·742	2·737	2·134
2	14·788	13·177	11·925	2	3·516	2·549	1·974
3	14·653	13·039	11·787	3	3·296	2·369	1·822
4	14·520	12·902	11·652	4	3·084	2·196	1·676
25	14·385	12·766	11·518	75	2·879	2·031	1·538
6	14·247	12·628	11·383	6	2·682	1·873	1·406
7	14·105	12·485	11·244	7	2·493	1·723	1·283
8	13·958	12·338	11·101	8	2·312	1·580	1·165
9	13·806	12·186	10·953	9	2·138	1·445	1·055
30	13·649	12·029	10·800	80	1·973	1·317	·952
1	13·486	11·865	10·640	1	1·815	1·197	·855
2	13·317	11·696	10·476	2	1·666	1·084	·765
3	13·142	11·522	10·307	3	1·525	·978	·682
4	12·961	11·341	10·132	4	1·391	·879	·604
35	12·776	11·155	9·950	85	1·265	·786	·532
6	12·583	10·964	9·765	6	1·146	·700	·466
7	12·385	10·767	9·573	7	1·036	·621	·406
8	12·181	10·564	9·377	8	·932	·547	·350
9	11·972	10·357	9·175	9	·835	·480	·300
40	11·756	10·143	8·969	90	·744	·417	·255
1	11·536	9·925	8·757	1	·661	·361	·214
2	11·309	9·701	8·541	2	·585	·311	·180
3	11·077	9·473	8·322	3	·510	·261	·145
4	10·840	9·240	8·098	4	·449	·222	·119
45	10·598	9·003	7·871	95	·389	·184	·094
6	10·351	8·763	7·640	6	·342	·155	·076
7	10·099	8·519	7·407	7	·317	·144	·070
8	9·844	8·271	7·171	8	·258	·110	·050
9	9·584	8·021	6·933	9	·200	·085	·037
				100	·060	·015	·004

TABLES.

FIVE PER CENT.

ONE LIFE.

TABLE NO. XXIX.—D, N, S, C, M, R.
„ „ XXX.—LOG D, LOG N, LOG M.
„ „ XXXI.—*a*, A, P.

TWO, THREE, AND FOUR LIVES.

TABLE NO. XXXII.—LOG D, LOG N.
„ „ XXXIII.—*a*.

Constants.

Constant.	Number.	Logarithm.
i	·05	$\bar{2}$·6989700
$(1+i)$	1·05	0·0211893
$(1+i)^{\frac{1}{2}}$	1·0246951	0·0105946
$(1+i)^{\frac{1}{4}}$	1·0122722	0·0052973
v	·9523810	$\bar{1}$·9788107
$v^{\frac{1}{2}}$	·9759001	$\bar{1}$·9894054
$v^{\frac{1}{4}}$	·9878766	$\bar{1}$·9947027
d	·0476190	$\bar{2}$·6777807
δ	·0487902	$\bar{2}$·6883323
$j_{(2)}$	·0493902	$\bar{2}$·6936404
$j_{(4)}$	·0490889	$\bar{2}$·6909835

TABLE NO. XXIX.

Commutation Table.

5 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
0	127 280	1 855 852	31 354 993	13 674·	32 848·	395 604·
1	107 550	1 748 302	29 499 141	3 593·6	19 173·8	362 756·4
2	98 833	1 649 469	27 750 839	2 051·6	15 580·2	343 582·6
3	92 075	1 557 394	26 101 370	1 354·2	13 528·6	328 002·4
4	86 336	1 471 058	24 543 976	1 038·2	12 174·4	314 473·8
5	81 186	1 389 872	23 072 918	791·74	11 136·20	302 299·39
6	76 528	1 313 344	21 683 046	605·49	10 344·46	291 163·19
7	72 279	1 241 065	20 369 702	462·29	9 738·97	280 818·73
8	68 375	1 172 690	19 128 637	359·05	9 276·68	271 079·76
9	64 760	1 107 930	17 955 947	284·86	8 917·63	261 803·08
10	61 392	1 046 538	16 848 017	238·55	8 632·77	252 885·45
1	58 229	988 309	15 801 479	205·48	8 394·22	244 252·68
2	55 251	933 058	14 813 170	183·49	8 188·74	235 858·46
3	52 437	880 621	13 880 112	170·21	8 005·25	227 669·72
4	49 769	830 852	12 999 491	162·10	7 835·04	219 664·47
15	47 237	783 615	12 168 639	164·92	7 672·94	211 829·43
6	44 823	738 792	11 385 024	167·54	7 508·02	204 156·49
7	42 521	696 271	10 646 232	176·60	7 340·48	196 648·47
8	40 319	655 952	9 949 961	184·01	7 163·88	189 307·99
9	38 216	617 736	9 294 009	191·46	6 979·87	182 144·11
20	36 204	581 532	8 676 273	196·70	6 788·41	175 164·24
1	34 283	547 249	8 094 741	198·96	6 591·71	168 375·83
2	32 453	514 796	7 547 492	198·28	6 392·75	161 784·12
3	30 709	484 087	7 032 696	195·65	6 194·47	155 391·37
4	29 051	455 036	6 548 609	191·06	5 998·82	149 196·90
25	27 476	427 560	6 093 573	185·06	5 807·76	143 198·08
6	25 983	401 577	5 666 013	177·85	5 622·70	137 390·32
7	24 567	377 010	5 264 436	171·68	5 444·85	131 767·62
8	23 226	353 784	4 887 426	164·72	5 273·17	126 322·77
9	21 955	331 829	4 533 642	158·72	5 108·45	121 049·60
30	20 751	311 078	4 201 813	152·27	4 949·73	115 941·15
1	19 611	291 467	3 890 735	146·91	4 797·46	110 991·42
2	18 530	272 937	3 599 268	141·71	4 650·55	106 193·96
3	17 506	255 431	3 326 331	136·86	4 508·84	101 543·41
4	16 535	238 896	3 070 900	132·16	4 371·98	97 034·57
35	15 616	223 280	2 832 004	128·11	4 239·82	92 662·59
6	14 744	208 536	2 608 724	124·31	4 111·71	88 422·77
7	13 918	194 618	2 400 188	120·59	3 987·40	84 311·06
8	13 134	181 484	2 205 570	117·23	3 866·81	80 323·66
9	12 392	169 092	2 024 086	114·49	3 749·58	76 456·85
40	11 687	157 405	1 854 994	111·34	3 635·09	72 707·27
1	11 019	146 386	1 697 589	109·00	3 523·75	69 072·18
2	10 386	136 000·4	1 551 202·6	106·88	3 414·75	65 548·43
3	9 784·1	126 216·3	1 415 202·2	104·59	3 307·87	62 133·68
4	9 213·6	117 002·7	1 288 985·9	102·84	3 203·28	58 825·81
45	8 672·0	108 330·7	1 171 983·2	101·12	3 100·44	55 622·53
6	8 157·9	100 172·8	1 063 652·5	99·536	2 999·318	52 522·087
7	7 669·9	92 502·9	963 479·7	98·161	2 899·782	49 522·769
8	7 206·4	85 296·5	870 976·8	97·149	2 801·621	46 622·987
9	6 766·1	78 530·4	785 680·3	96·013	2 704·472	43 821·366

TABLE NO. XXIX.

Commutation Table.

5 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
50	6 348·0	72 182·4	707 149·9	95·012	2 608·459	41 116·894
1	5 950·7	66 231·7	634 967·5	94·363	2 513·447	38 508·435
2	5 573·0	60 658·7	568 735·8	93·635	2 419·084	35 994·988
3	5 214·0	55 444·7	508 077·1	92·980	2 325·449	33 575·904
4	4 872·7	50 572·0	452 632·4	92·446	2 232·469	31 250·455
55	4 548·2	46 023·8	402 060·4	92·013	2 140·023	29 017·986
6	4 239·7	41 784·1	356 036·6	91·411	2 048·010	26 877·963
7	3 946·3	37 837·8	314 252·5	90·954	1 956·599	24 829·953
8	3 667·4	34 170·4	276 414·7	90·615	1 865·645	22 873·354
9	3 402·2	30 768·2	242 244·3	90·047	1 775·030	21 007·709
60	3 150·1	27 618·1	211 476·1	89·481	1 684·983	19 232·679
1	2 910·7	24 707·4	183 858·0	88·861	1 595·502	17 547·696
2	2 683·2	22 024·2	159 150·6	88·143	1 506·641	15 952·194
3	2 467·2	19 557·0	137 126·4	87·339	1 418·498	14 445·553
4	2 262·5	17 294·5	117 569·4	86·369	1 331·159	13 027·055
65	2 068·4	15 226·1	100 274·9	85·212	1 244·790	11 695·896
6	1 884·6	13 341·5	85 048·8	83·855	1 159·578	10 451·106
7	1 711·0	11 630·5	71 707·3	82·363	1 075·723	9 291·528
8	1 547·2	10 083·3	60 076·8	80·545	993·360	8 215·805
9	1 393·0	8 690·3	49 993·5	78·484	912·815	7 222·445
70	1 248·2	7 442·1	41 303·2	76·187	834·331	6 309·630
1	1 112·5	6 329·60	33 861·09	73·573	758·144	5 475·299
2	985·98	5 343·62	27 531·49	70·694	684·571	4 717·155
3	868·34	4 475·28	22 187·87	67·489	613·877	4 032·584
4	759·50	3 715·78	17 712·59	64·044	546·388	3 418·707
75	659·28	3 056·50	13 996·81	60·307	482·344	2 872·319
6	567·58	2 488·92	10 940·31	56·338	422·037	2 389·975
7	484·22	2 004·70	8 451·39	52·119	365·699	1 967·938
8	409·03	1 595·67	6 446·69	47·775	313·580	1 602·239
9	341·80	1 253·87	4 851·02	43·300	265·805	1 288·659
80	282·21	971·66	3 597·15	38·778	222·505	1 022·854
1	230·00	741·66	2 625·49	34·278	183·727	800·349
2	184·77	556·89	1 883·83	29·839	149·449	616·622
3	146·13	410·76	1 326·94	25·563	119·610	467·173
4	113·61	297·145	916·183	21·516	94·047	347·563
85	86·682	210·463	619·038	17·766	72·531	253·516
6	64·787	145·676	408·575	14·368	54·765	180·985
7	47·335	98·341	262·899	11·335	40·397	126·220
8	33·745	64·596	164·558	8·7271	29·0622	85·8226
9	23·411	41·185	99·962	6·5278	20·3351	56·7604
90	15·768	25·417	58·777	4·7424	13·8073	36·4253
1	10·275	15·142	33·360	3·3256	9·0649	22·6180
2	6·461	8·681	18·218	2·2364	5·7393	13·5531
3	3·915	4·766	9·537	1·4675	3·5029	7·8138
4	2·264	2·502	4·771	·9026	2·0354	4·3109
95	1·251	1·251	2·269	·5361	1·1328	2·2755
6	·656	·595	1·018	·2993	·5967	1·1427
7	·326	·269	·423	·1509	·2974	·5460
8	·160	·109	·154	·0799	·1465	·2486
9	·071	·038	·045	·0380	·0666	·1021
100	·031	·007	·007	·0217	·0286	·0355
1	·007	...	...	·0069	·0069	·0069

TABLE No. XXX.

Logarithms of D_x, N_x, *and* M_x.

5 *per-cent.*

x	log D_x	log N_x	log M_x	x	log D_x	log N_x	log M_x
0	5·10477	6·26855	4·51651	50	3·80264	4·85843	3·41639
1	·03160	·24262	·28271	1	·77457	·82107	·40026
2	4·99490	·21735	·19257	2	·74609	·78290	·38365
3	·96414	·19240	·13127	3	·71717	·74386	·36650
4	·93619	·16764	·08543	4	·68777	·70391	·34879
5	·90948	·14298	·04673	55	·65784	·66298	·33041
6	·88382	·11836	·01469	6	·62733	·62101	·31133
7	·85901	·09381	3·98851	7	·59619	·57793	·29150
8	·83490	·06919	·96739	8	·56436	·53365	·27082
9	·81131	·04450	·95025	9	·53176	·48810	·24920
10	·78811	·01974	·93615	60	·49833	·44119	·22660
1	·76514	5·99489	·92398	1	·46399	·39282	·20290
2	·74234	·96991	·91321	2	·42865	·34290	·17800
3	·71964	·94479	·90338	3	·39221	·29130	·15183
4	·69696	·91952	·89404	4	·35458	·23792	·12424
15	·67428	·89411	·88496	65	·31563	·18259	·09510
6	·65150	·86852	·87552	6	·27523	·12522	·06431
7	·62860	·84278	·86573	7	·23326	·06562	·03169
8	·60551	·81687	·85515	8	·18955	·00359	2·99711
9	·58224	·79081	·84385	9	·14395	3·93903	·96039
20	·55876	·76457	·83177	70	·09627	·87170	·92134
1	·53508	·73819	·81900	1	·04631	·80138	·87975
2	·51125	·71164	·80569	2	2·99387	·72783	·83542
3	·48726	·68493	·79201	3	·93869	·65082	·78808
4	·46316	·65805	·77806	4	·88053	·57005	·73750
25	·43896	·63100	·76401	75	·81907	·48522	·68335
6	·41469	·60377	·74994	6	·75403	·39601	·62535
7	·39036	·57635	·73599	7	·68504	·30205	·56313
8	·36598	·54873	·72207	8	·61176	·20295	·49635
9	·34154	·52092	·70829	9	·53377	·09826	·42457
30	·31704	·49287	·69458	80	·45058	2·98751	·34735
1	·29249	·46459	·68102	1	·36173	·87020	·26418
2	·26787	·43607	·66751	2	·26663	·74577	·17450
3	·24318	·40727	·65406	3	·16474	·61359	·07777
4	·21841	·37822	·64068	4	·05541	·47298	1·97334
35	·19356	·34885	·62735	85	1·93793	·32317	·86052
6	·16862	·31919	·61402	6	·81149	·16340	·73850
7	·14357	·28919	·60069	7	·67518	1·99273	·60635
8	·11841	·25883	·58735	8	·52821	·81021	·46333
9	·09313	·22812	·57398	9	·36942	·61474	·30824
40	·06771	·19703	·56052	90	·19779	·40512	·14010
1	·04215	·16551	·54701	1	·01179	·18018	0·95736
2	·01643	·13354	·53337	2	0·81025	0·93859	·75886
3	3·99052	·10113	·51955	3	·59288	·67808	·54443
4	·96443	·06819	·50560	4	·35456	·39843	·30865
45	·93812	·03475	·49142	95	·09761	·09719	·05415
6	·91158	·00074	·47702	6	$\bar{1}$·81709	$\bar{1}$·77417	$\bar{1}$·77576
7	·88479	4·96616	·46237	7	·51284	·42945	·47334
8	·85772	·93093	·44741	8	·20220	·03949	·16581
9	·83034	·89504	·43209	9	$\bar{2}$·85650	$\bar{2}$·57588	$\bar{2}$·82378
				100	·48313	$\bar{3}$·85988	·45675
				1	$\bar{3}$·85988	...	$\bar{3}$·83869

Table No. XXXI.

Values of Annuities, and Single and Annual Premiums for Assurance of a Unit.

5 *per-cent.*

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	14·581	·25807	·01656	50	11·371	·41091	·03322
1	16·256	·17828	·01033	1	11·130	·42237	·03482
2	16·690	·15764	·00891	2	10·885	·43407	·03652
3	16·915	·14694	·00820	3	10·634	·44600	·03834
4	17·039	·14101	·00782	4	10·379	·45816	·04027
5	17·120	·13717	·00757	55	10·119	·47051	·04232
6	17·161	·13517	·00744	6	9·856	·48306	·04450
7	17·171	·13474	·00742	7	9·588	·49580	·04683
8	17·151	·13567	·00747	8	9·317	·50870	·04930
9	17·108	·13770	·00760	9	9·044	·52172	·05195
10	17·046	·14062	·00779	60	8·767	·53490	·05476
1	16·973	·14416	·00802	1	8·489	·54816	·05777
2	16·888	·14821	·00829	2	8·208	·56150	·06098
3	16·794	·15267	·00858	3	7·927	·57494	·06441
4	16·694	·15743	·00890	4	7·644	·58838	·06807
15	16·589	·16244	·00924	65	7·361	·60182	·07197
6	16·482	·16750	·00958	6	7·079	·61529	·07616
7	16·375	·17264	·00994	7	6·798	·62868	·08063
8	16·269	·17768	·01029	8	6·517	·64204	·08541
9	16·165	·18265	·01064	9	6·239	·65530	·09053
20	16·062	·18750	·01099	70	5·963	·66845	·09601
1	15·963	·19227	·01134	1	5·689	·68146	·10187
2	15·863	·19699	·01168	2	5·420	·69430	·10815
3	15·764	·20172	·01203	3	5·154	·70695	·11488
4	15·664	·20649	·01239	4	4·892	·71940	·12209
25	15·561	·21137	·01276	75	4·636	·73161	·12981
6	15·455	·21640	·01315	6	4·385	·74357	·13808
7	15·346	·22163	·01356	7	4·140	·75525	·14693
8	15·232	·22703	·01399	8	3·901	·76664	·15642
9	15·114	·23268	·01444	9	3·669	·77768	·16658
30	14·991	·23853	·01492	80	3·443	·78844	·17746
1	14·863	·24464	·01542	1	3·225	·79882	·18909
2	14·730	·25098	·01596	2	3·014	·80885	·20151
3	14·591	·25756	·01652	3	2·811	·81852	·21478
4	14·448	·26441	·01712	4	2·616	·82781	·22895
35	14·298	·27151	·01775	85	2·428	·83674	·24408
6	14·144	·27887	·01841	6	2·249	·84530	·26021
7	13·984	·28650	·01912	7	2·078	·85343	·27730
8	13·817	·29440	·01987	8	1·914	·86123	·29553
9	13·646	·30259	·02066	9	1·759	·86860	·31480
40	13·469	·31104	·02150	90	1·612	·87561	·33524
1	13·285	·31979	·02239	1	1·474	·88221	·35665
2	13·095	·32881	·02333	2	1·344	·88840	·37904
3	12·901	·33809	·02432	3	1·217	·89444	·40350
4	12·699	·34767	·02538	4	1·106	·89968	·42714
45	12·492	·35752	·02650	95	·999	·90477	·45261
6	12·279	·36765	·02769	6	·906	·90922	·47706
7	12·061	·37808	·02895	7	·825	·91306	·50023
8	11·836	·38877	·03029	8	·688	·91962	·54495
9	11·606	·39971	·03171	9	·524	·92743	·60854
				100	·238	·94107	·76010
				1	...	·95238	·95238

TABLE No. XXXII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*
5 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
0	10·20954	11·25460	15·31431	16·24684	20·41908	21·24343	0
1	·08439	·22420	·13718	·21173	·18997	·20322	1
2	·03218	·19535	·06946	·17924	·10674	·16699	2
3	9·99185	·16729	·01956	·14808	·04727	·13271	3
4	·95714	·13966	14·97809	·11767	19·99904	·09954	4
5	·92491	·11234	·94034	·08782	·95577	·06718	5
6	·89478	·08518	·90574	·05827	·91670	·03531	6
7	·86635	·05812	·87369	·02890	·88103	·00376	7
8	·83931	·03104	·84372	15·99957	·84813	20·97229	8
9	·81332	·00389	·81533	·97019	·81734	·94078	9
10	·78811	10·97664	·78811	·94065	·78811	·90909	10
1	·76336	·94922	·76158	·91090	·75980	·87715	1
2	·73895	·92160	·73556	·88088	·73217	·84487	2
3	·71474	·89375	·70984	·85055	·70494	·81221	3
4	·69057	·86565	·68418	·81989	·67779	·77911	4
15	·66640	·83730	·65852	·78885	·65064	·74554	15
6	·64203	·80868	·63256	·75744	·62309	·71148	6
7	·61742	·77978	·60624	·72564	·59506	·67692	7
8	·59243	·75061	·57935	·69347	·56627	·64189	8
9	·56708	·72117	·55192	·66093	·53676	·60638	9
20	·54131	·69148	·52386	·62804	·50641	·57044	20
1	·51514	·66153	·49520	·59482	·47526	·53406	1
2	·48866	·63133	·46607	·56127	·44348	·49730	2
3	·46187	·60090	·43648	·52740	·41109	·46015	3
4	·43486	·57020	·40656	·49321	·37826	·42261	4
25	·40765	·53925	·37634	·45870	·34503	·38468	25
6	·38030	·50804	·34591	·42384	·31152	·34633	6
7	·35283	·47656	·31530	·38862	·27777	·30756	7
8	·32526	·44476	·28454	·35303	·24382	·26834	8
9	·29757	·41266	·25360	·31702	·20963	·22866	9
30	·26976	·38023	·22248	·28060	·17520	·18845	30
1	·24185	·34743	·19121	·24373	·14057	·14768	1
2	·21380	·31427	·15973	·20639	·10566	·10636	2
3	·18561	·28071	·12804	·16853	·07047	·06442	3
4	·15726	·24674	·09611	·13011	·03496	·02181	4
35	·12875	·21232	·06394	·09114	18·99913	19·97850	35
6	·10005	·17742	·03148	·05150	·96291	·93444	6
7	·07114	·14201	13·99871	·01123	·92628	·88958	7
8	·04201	·10605	·96561	14·97027	·88921	·84386	8
9	·01264	·06952	·93215	·92855	·85166	·79723	9
40	8·98299	·03242	·89827	·88603	·81355	·74962	40
1	·95306	9·99463	·86397	·84266	·77488	·70096	1
2	·92281	·95617	·82919	·79837	·73557	·65118	2
3	·89218	·91697	·79384	·75312	·69550	·60021	3
4	·86119	·87700	·75795	·70682	·65471	·54796	4
45	·82976	·83618	·72140	·65941	·61304	·49432	45
6	·79787	·79448	·68416	·61081	·57045	·43922	6
7	·76548	·75183	·64617	·56095	·52686	·38254	7
8	·73253	·70817	·60734	·50971	·48215	·32416	8
9	·69896	·66342	·56758	·45702	·43620	·26397	9

TABLE No. XXXII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*

5 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	$\log D_{xx}$	$\log N_{xx}$	$\log D_{xxx}$	$\log N_{xxx}$	$\log D_{xxxx}$	$\log N_{xxxx}$	
50	8·66474	9·61751	13·52684	14·40276	18·38894	19·20181	50
1	·62979	·57037	·48501	·34682	·34023	·13754	1
2	·59402	·52191	·44195	·28910	·28988	·07103	2
3	·55737	·47202	·39757	·22943	·23777	·00208	3
4	·51976	·42062	·35175	·16767	·18374	18·93051	4
55	·48109	·36760	·30434	·10370	·12759	·85612	55
6	·44126	·31283	·25519	·03731	·06912	·77869	6
7	·40017	·25619	·20415	13·96834	·00813	·69797	7
8	·35770	·19753	·15104	·89658	17·94438	·61372	8
9	·31369	·13672	·09562	·82182	·87755	·52561	9
60	·26802	·07357	·03771	·74382	·80740	·43339	60
1	·22053	·00792	12·97707	·66232	·73361	·33668	1
2	·17104	8·93960	·91343	·57704	·65582	·23512	2
3	·11935	·86837	·84649	·48769	·57363	·12833	3
4	·06527	·79401	·77596	·39391	·48665	·01582	4
65	·00856	·71628	·70149	·29535	·39442	17·89715	65
6	7·94895	·63490	·62267	·19162	·29639	·77178	6
7	·88620	·54958	·53914	·08225	·19208	·63912	7
8	·81997	·46000	·45039	12·96681	·08081	·49856	8
9	·74996	·36581	·35597	·84476	16·96198	·34939	9
70	·67579	·26663	·25531	·71553	·83483	·19089	70
1	·59706	·16206	·14781	·57851	·69856	·02218	1
2	·51337	·05162	·03287	·43302	·55237	16·84242	2
3	·42420	7·93481	11·90971	·27830	·39522	·65055	3
4	·32907	·81109	·77761	·11351	·22615	·44548	4
75	·22734	·67987	·63561	11·93781	·04388	·22606	75
6	·11845	·54049	·48287	·75019	15·84729	15·99094	6
7	·00166	·39226	·31828	·54960	·63490	·73873	7
8	6·87629	·23434	·14082	·33484	·40535	·46782	8
9	·74149	·06592	10·94921	·10459	·15693	·17641	9
80	·59630	6·88603	·74202	10·85743	14·88774	14·86267	80
1	·43979	·69363	·51785	·59180	·59591	·52442	1
2	·27078	·48763	·27493	·30606	·27908	·15951	2
3	·08819	·26677	·01164	9·99830	13·93509	13·76537	3
4	5·89072	·02967	9·72603	·66646	·56134	·33925	4
85	·67695	5·77485	·41597	·30822	·15499	12·87800	85
6	·44526	·50061	·07903	8·92108	12·71280	·37823	6
7	·19383	·20526	8·71248	·50248	·23113	11·83659	7
8	4·92108	4·88670	·31395	·04922	11·70682	·24871	8
9	·62469	·54279	7·87996	7·55806	·13523	10·61032	9
90	·30262	·17097	·40745	·02498	10·51228	9·91580	90
1	3·95181	3·76896	6·89183	6·44665	9·83185	·16074	1
2	·56992	·33437	·32959	5·82004	·08926	8·34183	2
3	·15636	2·86106	5·71984	·13402	8·28332	7·44235	3
4	2·70091	·35017	·04726	4·39187	7·39361	6·46838	4
95	·20820	1·79553	4·31879	3·58184	6·42938	5·40233	95
6	1·66835	·20011	3·51961	2·70733	5·37087	4·24662	6
7	·08104	0·58005	2·64924	1·80495	4·21744	3·05953	7
8	0·48095	$\bar{1}$·88968	1·75970	0·80035	3·03845	1·73513	8
9	$\bar{1}$·81074	·11029	0·76498	$\bar{1}$·69367	1·71922	0·29090	9
100	·08519	$\bar{3}$·85988	$\bar{1}$·68725	$\bar{3}$·85988	0·28931	$\bar{3}$·85988	100
1	$\bar{3}$·85988	...	$\bar{3}$·85988	...	$\bar{3}$·85988	...	1

TABLE No. XXXIII.

Values of Annuities. Joint Lives. Equal Ages.
5 *per-cent.*

x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}	x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}
0	11·093	8·561	6·673	50	8·970	7·515	6·499
1	13·798	11·873	10·310	1	8·721	7·275	6·271
2	14·560	12·876	11·488	2	8·470	7·033	6·042
3	14·978	13·444	12·174	3	8·216	6·790	5·812
4	15·224	13·791	12·604	4	7·959	6·545	5·582
5	15·397	14·044	12·924	55	7·700	6·300	5·352
6	15·502	14·208	13·140	6	7·440	6·055	5·124
7	15·551	14·296	13·266	7	7·178	5·810	4·896
8	15·550	14·317	13·309	8	6·916	5·566	4·670
9	15·509	14·284	13·287	9	6·653	5·324	4·447
10	15·436	14·208	13·212	60	6·391	5·083	4·227
1	15·341	14·103	13·102	1	6·129	4·845	4·009
2	15·228	13·974	12·963	2	5·869	4·609	3·796
3	15·101	13·826	12·802	3	5·611	4·377	3·587
4	14·965	13·668	12·628	4	5·355	4·149	3·382
15	14·822	13·500	12·442	65	5·102	3·925	3·182
6	14·677	13·332	12·257	6	4·852	3·706	2·988
7	14·533	13·164	12·074	7	4·607	3·492	2·799
8	14·394	13·005	11·902	8	4·366	3·284	2·617
9	14·259	12·853	11·739	9	4·129	3·082	2·440
20	14·131	12·711	11·589	70	3·898	2·886	2·270
1	14·008	12·578	11·450	1	3·673	2·696	2·107
2	13·889	12·451	11·319	2	3·453	2·513	1·950
3	13·773	12·329	11·196	3	3·241	2·337	1·800
4	13·657	12·208	11·075	4	3·034	2·167	1·657
25	13·539	12·088	10·956	75	2·835	2·005	1·521
6	13·420	11·965	10·835	6	2·643	1·851	1·392
7	13·296	11·839	10·710	7	2·458	1·703	1·270
8	13·167	11·708	10·581	8	2·281	1·563	1·155
9	13·034	11·572	10·448	9	2·111	1·430	1·046
30	12·896	11·432	10·310	80	1·949	1·304	·944
1	12·752	11·285	10·165	1	1·794	1·186	·848
2	12·603	11·134	10·016	2	1·648	1·074	·759
3	12·448	10·977	9·862	3	1·509	·970	·677
4	12·288	10·814	9·702	4	1·377	·872	·600
35	12·122	10·646	9·536	85	1·253	·780	·528
6	11·950	10·472	9·366	6	1·136	·695	·463
7	11·773	10·292	9·190	7	1·027	·617	·403
8	11·589	10·108	9·008	8	·924	·544	·348
9	11·399	9·917	8·822	9	·828	·477	·299
40	11·205	9·722	8·631	90	·739	·415	·253
1	11·004	9·521	8·435	1	·656	·359	·213
2	10·798	9·315	8·234	2	·581	·309	·179
3	10·587	9·105	8·030	3	·507	·260	·144
4	10·371	8·889	7·821	4	·446	·221	·119
45	10·149	8·670	7·608	95	·387	·183	·094
6	9·922	8·446	7·392	6	·340	·154	·075
7	9·691	8·218	7·173	7	·316	·143	·070
8	9·455	7·987	6·950	8	·256	·110	·050
9	9·214	7·753	6·726	9	·199	·085	·037
				100	·060	·015	·004

TABLES.

SIX PER CENT.

ONE LIFE.

Table No. XXXIV.—D, N, S, C, M, R.
" " XXXV.—Log D, Log N, Log M.
" " XXXVI.—*a*, A, P.

TWO, THREE, AND FOUR LIVES.

Table No. XXXVII.—Log D, Log N.
" " XXXVIII.—*a*.

Constants.

Constant.	Number.	Logarithm.
i	·06	$\bar{2}$·7781513
$(1+i)$	1·06	0·0253059
$(1+i)^{\frac{1}{2}}$	1·0295630	0·0126529
$(1+i)^{\frac{1}{4}}$	1·0146738	0·0063265
v	·9433962	$\bar{1}$·9746941
$v^{\frac{1}{2}}$	·9712860	$\bar{1}$·9873471
$v^{\frac{1}{4}}$	·9855384	$\bar{1}$·9936735
d	·0566038	$\bar{2}$·7528454
δ	·0582689	$\bar{2}$·7654369
$j_{(2)}$	·0591260	$\bar{2}$·7717785
$j_{(4)}$	·0586954	$\bar{2}$·7686041

Table No. XXXIV.

Commutation Table.

6 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
0	127 280	1 595 826	23 957 822	13 545·	29 748·	269 475·
1	106 530	1 489 296	22 361 996	3 526·1	16 203·1	239 727·4
2	96 977	1 392 319	20 872 700	1 994·1	12 677·0	223 524·3
3	89 493	1 302 826	19 480 381	1 303·8	10 682·9	210 847·3
4	83 125	1 219 701	18 177 555	990·13	9 379·12	200 164·36
5	77 428	1 142 273	16 957 854	747·96	8 388·99	190 785·24
6	72 297	1 069 976	15 815 581	566·63	7 641·03	182 396·25
7	67 639	1 002 337	14 745 605	428·52	7 074·40	174 755·22
8	63 381	938 956	13 743 268	329·69	6 645·88	167 680·82
9	59 465	879 491	12 804 312	259·10	6 316·19	161 034·94
10	55 839	823 652	11 924 821	214·93	6 057·09	154 718·75
1	52 464	771 188	11 101 169	183·38	5 842·16	148 661·66
2	49 311	721 877	10 329 981	162·22	5 658·78	142 819·50
3	46 358	675 519	9 608 104	149·06	5 496·56	137 160·72
4	43 584	631 935	8 932 585	140·62	5 347·50	131 664·16
15	40 976	590 959	8 300 650	141·71	5 206·88	126 316·66
6	38 516	552 443	7 709 691	142·60	5 065·17	121 109·78
7	36 193	516 250	7 157 248	148·89	4 922·57	116 044·61
8	33 995	482 255	6 640 998	153·69	4 773·68	111 122·04
9	31 918	450 337	6 158 743	158·39	4 619·99	106 348·36
20	29 952	420 385	5 708 406	161·20	4 461·60	101 728·37
1	28 096	392 289	5 288 021	161·51	4 300·40	97 266·77
2	26 344	365 945	4 895 732	159·44	4 138·89	92 966·37
3	24 693	341 252	4 529 787	155·84	3 979·45	88 827·48
4	23 140	318 112	4 188 535	150·75	3 823·61	84 848·03
25	21 679	296 433	3 870 423	144·64	3 672·86	81 024·42
6	20 308	276 125	3 573 990	137·69	3 528·22	77 351·56
7	19 020	257 105	3 297 865	131·66	3 390·53	73 823·34
8	17 812	239 293	3 040 760	125·13	3 258·87	70 432·81
9	16 679	222 614	2 801 467	119·44	3 133·74	67 173·94
30	15 615	206 999	2 578 853	113·50	3 014·30	64 040·20
1	14 618	192 381	2 371 854	108·47	2 900·80	61 025·90
2	13 682	178 699	2 179 473	103·65	2 792·33	58 125·10
3	12 804	165 895	2 000 774	99·159	2 688·678	55 332·774
4	11 980	153 915	1 834 879	94·846	2 589·519	52 644·096
35	11 207	142 708	1 680 964	91·073	2 494·673	50 054·577
6	10 481	132 227·4	1 538 255·6	87·539	2 403·600	47 559·904
7	9 800·5	122 426·9	1 406 028·2	84·114	2 316·061	45 156·304
8	9 161·8	113 265·1	1 283 601·3	81·001	2 231·947	42 840·243
9	8 562·1	104 703·0	1 170 336·2	78·363	2 150·946	40 608·296
40	7 999·3	96 703·7	1 065 633·2	75·485	2 072·583	38 457·350
1	7 470·9	89 232·8	968 929·5	73·202	1 997·098	36 384·767
2	6 974·8	82 258·0	879 696·7	71·100	1 923·896	34 387·669
3	6 508·9	75 749·1	797 438·7	68·922	1 852·796	32 463·773
4	6 071·6	69 677·5	721 689·6	67·129	1 783·874	30 610·977
45	5 660·8	64 016·7	652 012·1	65·385	1 716·745	28 827·103
6	5 275·0	58 741·7	587 995·4	63·753	1 651·360	27 110·358
7	4 912·6	53 829·1	529 253·7	62·280	1 587·607	25 458·998
8	4 572·3	49 256·8	475 424·6	61·056	1 525·327	23 871·391
9	4 252·4	45 004·4	426 167·8	59·772	1 464·271	22 346·064

TABLE No. XXXIV.

Commutation Table.

6 *per-cent.*

x	D_x	N_x	S_x	C_x	M_x	R_x
50	3 951·9	41 052·5	381 163·4	58·591	1 404·499	20 881·793
1	3 669·6	37 382·9	340 110·9	57·642	1 345·908	19 477·294
2	3 404·3	33 978·6	302 728·0	56·658	1 288·266	18 131·386
3	3 154·9	30 823·7	268 749·4	55·730	1 231·608	16 843·120
4	2 920·6	27 903·1	237 925·7	54·888	1 175·878	15 611·512
55	2 700·4	25 202·7	210 022·6	54·115	1 120·990	14 435·634
6	2 493·4	22 709·3	184 819·9	53·255	1 066·875	13 314·644
7	2 299·1	20 410·2	162 110·6	52·488	1 013·620	12 247·769
8	2 116·4	18 293·8	141 700·4	51·799	961·132	11 234·149
9	1 944·8	16 349·0	123 406·6	50·989	909·333	10 273·017
60	1 783·8	14 565·2	107 057·6	50·190	858·344	9 363·684
1	1 632·6	12 932·6	92 492·4	49·373	808·154	8 505·340
2	1 490·8	11 441·8	79 559·8	48·512	758·781	7 697·186
3	1 357·9	10 083·9	68 118·0	47·615	710·269	6 938·405
4	1 233·4	8 850·5	58 034·1	46·642	662·654	6 228·136
65	1 117·0	7 733·5	49 183·6	45·583	616·012	5 565·482
6	1 008·2	6 725·28	41 450·07	44·434	570·429	4 949·470
7	906·67	5 818·61	34 724·79	43·231	525·995	4 379·041
8	812·12	5 006·49	28 906·18	41·879	482·764	3 853·046
9	724·29	4 282·20	23 899·69	40·422	440·885	3 370·282
70	642·85	3 639·35	19 617·49	38·869	400·463	2 939·397
1	567·58	3 071·77	15 978·14	37·182	361·594	2 538·934
2	498·29	2 573·48	12 906·37	35·389	324·412	2 167·340
3	434·69	2 138·79	10 332·89	33·466	289·023	1 842·928
4	376·63	1 762·16	8 194·10	31·459	255·557	1 553·905
75	323·84	1 438·32	6 431·94	29·343	224·098	1 298·348
6	276·17	1 162·15	4 993·62	27·154	194·755	1 074·250
7	233·38	928·77	3 831·47	24·883	167·601	879·495
8	195·29	733·48	2 902·70	22·594	142·718	711·894
9	161·64	571·84	2 169·22	20·284	120·124	569·176
80	132·21	439·63	1 597·38	17·995	99·840	449·052
1	106·73	332·901	1 157·745	15·757	81·845	349·212
2	84·932	247·969	824·844	13·587	66·088	267·367
3	66·537	181·432	576·875	11·530	52·501	201·279
4	51·241	130·191	395·443	9·6130	40·9712	148·7777
85	38·728	91·463	265·252	7·8629	31·3582	107·8065
6	28·673	62·790	173·789	6·2988	23·4953	76·4483
7	20·751	42·039	110·999	4·9222	17·1965	52·9530
8	14·654	27·385	68·960	3·7540	12·2743	35·7565
9	10·070	17·315	41·575	2·7815	8·5203	23·4822
90	6·719	10·596	24·260	2·0017	5·7388	14·9619
1	4·337	6·259	13·664	1·3904	3·7371	9·2231
2	2·701	3·558	7·405	·9262	2·3467	5·4860
3	1·622	1·936	3·847	·6020	1·4205	3·1393
4	·928	1·008	1·911	·3668	·8185	1·7188
95	·508	·500	·903	·2158	·4517	·9003
6	·264	·236	·403	·1194	·2359	·4486
7	·130	·106	·167	·0596	·1165	·2127
8	·063	·043	·061	·0312	·0569	·0962
9	·028	·015	·018	·0147	·0257	·0393
100	·012	·003	·003	·0084	·0110	·0136
1	·003	...	...	·0026	·0026	·0026

TABLE NO. XXXV.

Logarithms of D_x, N_x, *and* M_x.

6 *per-cent.*

x	log D_x	log N_x	log M_x	x	log D_x	log N_x	log M_x
0	5·10477	6·20298	4·47346	50	3·59681	4·61334	3·14752
1	·02748	·17298	·20960	1	·56462	·57267	·12901
2	4·98667	·14373	·10302	2	·53203	·53121	·11002
3	·95179	·11488	·02869	3	·49899	·48889	·09047
4	·91973	·08625	3·97216	4	·46547	·44565	·07037
5	·88890	·05778	·92371	55	·43143	·40145	·04961
6	·85912	·02938	·88315	6	·39680	·35620	·02812
7	·83020	·00100	·84969	7	·36155	·30984	·00587
8	·80196	5·97265	·82255	8	·32560	·26231	2·98278
9	·77426	·94423	·80046	9	·28888	·21349	·95872
10	·74694	·91574	·78226	60	·25134	·16331	·93366
1	·71986	·88716	·76658	1	·21288	·11170	·90749
2	·69294	·85847	·75272	2	·17343	·05850	·88012
3	·66612	·82964	·74009	3	·13287	·00363	·85142
4	·63933	·80068	·72815	4	·09111	3·94697	·82128
15	·61253	·77156	·71658	65	·04805	·88838	·78959
6	·58564	·74229	·70460	6	·00353	·82771	·75620
7	·55862	·71286	·69219	7	2·95745	·76482	·72099
8	·53141	·68328	·67886	8	·90962	·69953	·68373
9	·50403	·65354	·66464	9	·85991	·63167	·64433
20	·47643	·62365	·64949	70	·80811	·56103	·60256
1	·44864	·59361	·63351	1	·75403	·48739	·55822
2	·42068	·56342	·61688	2	·69748	·41052	·51109
3	·39258	·53307	·59983	3	·63818	·33017	·46093
4	·36436	·50258	·58247	4	·57591	·24606	·40749
25	·33604	·47192	·56501	75	·51033	·15785	·35044
6	·30766	·44111	·54755	6	·44117	·06528	·28950
7	·27921	·41012	·53026	7	·36807	2·96791	·22427
8	·25072	·37892	·51307	8	·29067	·86539	·15448
9	·22216	·34754	·49606	9	·20856	·75727	·07962
30	·19354	·31597	·47919	80	·12125	·64309	1·99930
1	·16488	·28416	·46252	1	·02828	·52231	·91299
2	·13614	·25212	·44596	2	1·92907	·39440	·82012
3	·10734	·21985	·42954	3	·82306	·25871	·72017
4	·07845	·18730	·41322	4	·70962	·11458	·61248
35	·04948	·15445	·39702	85	·58802	1·96125	·49635
6	·02042	·12133	·38086	6	·45747	·79789	·37098
7	3·99125	·08789	·36476	7	·31704	·62365	·23545
8	·96198	·05411	·34867	8	·16595	·43751	·08899
9	·93258	·01995	·33262	9	·00305	·23842	0·93045
40	·90305	4·98544	·31652	90	0·82730	·02518	·75882
1	·87337	·95053	·30040	1	·63719	0·79654	·57253
2	·84353	·91518	·28418	2	·43153	·55127	·37046
3	·81351	·87938	·26783	3	·21003	·28704	·15244
4	·78330	·84310	·25137	4	$\bar{1}$·96760	·00363	$\bar{1}$·91302
45	·75288	·80630	·23469	95	·70653	$\bar{1}$·69867	·65486
6	·72222	·76895	·21785	6	·42190	·37195	·37273
7	·69131	·73102	·20074	7	·11353	·02366	·06651
8	·66013	·69247	·18336	8	$\bar{2}$·79878	$\bar{2}$·63028	$\bar{2}$·75543
9	·62863	·65325	·16563	9	·44896	·16343	·40993
				100	·07147	$\bar{3}$·44411	·03997
				1	$\bar{3}$·44411	...	$\bar{3}$·41880

TABLE NO. XXXVI.

Values of Annuities, and Single and Annual Premiums for Assurance of a Unit.

6 *per-cent.*

x	a_x	A_x	P_x	x	a_x	A_x	P_x
0	12·537	·23372	·01726	50	10·388	·35539	·03121
1	13·980	·15210	·01015	1	10·187	·36677	·03278
2	14·357	·13072	·00851	2	9·981	·37843	·03446
3	14·558	·11937	·00767	3	9·770	·39037	·03625
4	14·673	·11283	·00720	4	9·554	·40262	·03815
5	14·753	·10835	·00688	55	9·333	·41513	·04018
6	14·800	·10569	·00669	6	9·108	·42788	·04233
7	14·818	·10459	·00661	7	8·878	·44088	·04463
8	14·815	·10486	·00663	8	8·644	·45413	·04709
9	14·790	·10622	·00673	9	8·406	·46756	·04971
10	14·750	·10847	·00689	60	8·165	·48119	·05250
1	14·699	·11136	·00709	1	7·922	·49501	·05549
2	14·640	·11476	·00734	2	7·675	·50897	·05867
3	14·572	·11857	·00761	3	7·426	·52306	·06208
4	14·499	·12269	·00792	4	7·176	·53724	·06571
15	14·422	·12707	·00824	65	6·924	·55149	·06960
6	14·343	·13151	·00857	6	6·671	·56581	·07376
7	14·264	·13601	·00891	7	6·418	·58015	·07821
8	14·186	·14043	·00925	8	6·165	·59444	·08297
9	14·109	·14475	·00958	9	5·912	·60872	·08806
20	14·035	·14896	·00991	70	5·661	·62295	·09352
1	13·963	·15306	·01023	1	5·412	·63707	·09936
2	13·891	·15711	·01055	2	5·165	·65104	·10561
3	13·819	·16116	·01087	3	4·920	·66489	·11231
4	13·747	·16524	·01120	4	4·679	·67855	·11949
25	13·674	·16942	·01155	75	4·441	·69201	·12717
6	13·597	·17374	·01190	6	4·208	·70523	·13541
7	13·518	·17826	·01228	7	3·980	·71812	·14421
8	13·434	·18296	·01268	8	3·756	·73082	·15366
9	13·347	·18789	·01310	9	3·538	·74312	·16377
30	13·257	·19304	·01354	80	3·325	·75518	·17459
1	13·161	·19844	·01401	1	3·119	·76685	·18617
2	13·061	·20409	·01451	2	2·920	·77813	·19852
3	12·957	·20999	·01505	3	2·727	·78906	·21172
4	12·848	·21616	·01561	4	2·541	·79958	·22582
35	12·734	·22261	·01621	85	2·362	·80971	·24086
6	12·616	·22932	·01684	6	2·190	·81943	·25688
7	12·492	·23633	·01752	7	2·026	·82872	·27388
8	12·362	·24361	·01823	8	1·869	·83761	·29197
9	12·228	·25121	·01899	9	1·719	·84606	·31113
40	12·089	·25910	·01980	90	1·577	·85412	·33144
1	11·944	·26732	·02065	1	1·443	·86167	·35265
2	11·794	·27584	·02156	2	1·318	·86882	·37490
3	11·638	·28466	·02252	3	1·194	·87581	·39918
4	11·476	·29381	·02355	4	1·087	·88190	·42265
45	11·309	·30326	·02464	95	·982	·88783	·44795
6	11·136	·31306	·02580	6	·891	·89296	·47213
7	10·957	·32317	·02703	7	·813	·89739	·49495
8	10·773	·33360	·02834	8	·678	·90500	·53923
9	10·583	·34435	·02973	9	·518	·91405	·60207
				100	·236	·93004	·75256
				1	...	·94340	·94340

TABLE No. XXXVII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.*
6 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
0	10·20954	11·19532	15·31431	16·19240	20·41908	21·19301	0
1	·08027	·16065	·13306	·15275	·18585	·14805	1
2	·02395	·12772	·06123	·11608	·09851	·10745	2
3	9·97950	·09569	·00721	·08088	·03492	·06904	3
4	·94068	·06420	14·96163	·04657	19·98258	·03189	4
5	·90433	·03306	·91976	·01284	·93519	20·99568	5
6	·87008	·00212	·88104	15·97955	·89200	·96006	6
7	·83754	10·97134	·84488	·94647	·85222	·92478	7
8	·80637	·94057	·81078	·91347	·81519	·88965	8
9	·77627	·90975	·77828	·88044	·78029	·85451	9
10	·74694	·87882	·74694	·84726	·74694	·81921	10
1	·71808	·84774	·71630	·81389	·71452	·78367	1
2	·68955	·81647	·68616	·78026	·68277	·74780	2
3	·66122	·78498	·65632	·74632	·65142	·71155	3
4	·63294	·75325	·62655	·71204	·62016	·67485	4
15	·60465	·72125	·59677	·67739	·58889	·63767	15
6	·57617	·68898	·56670	·64236	·55723	·59999	6
7	·54744	·65644	·53626	·60693	·52508	·56182	7
8	·51833	·62364	·50525	·57114	·49217	·52315	8
9	·48887	·59056	·47371	·53496	·45855	·48402	9
20	·45898	·55724	·44153	·49845	·42408	·44444	20
1	·42870	·52367	·40876	·46160	·38882	·40444	1
2	·39809	·48987	·37550	·42444	·35291	·36404	2
3	·36719	·45582	·34180	·38696	·31641	·32327	3
4	·33606	·42154	·30776	·34918	·27946	·28212	4
25	·30473	·38702	·27342	·31108	·24211	·24057	25
6	·27327	·35224	·23888	·27265	·20449	·19866	6
7	·24168	·31721	·20415	·23388	·16662	·15631	7
8	·21000	·28187	·16928	·19474	·12856	·11354	8
9	·17819	·24625	·13422	·15521	·09025	·07030	9
30	·14626	·21029	·09898	·11528	·05170	·02657	30
1	·11424	·17400	·06360	·07489	·01296	19·98228	1
2	·08207	·13735	·02800	·03407	18·97393	·93744	2
3	·04977	·10034	13·99220	14·99271	·93463	·89199	3
4	·01730	·06288	·95615	·95083	·89500	·84590	4
35	8·98467	·02502	·91986	·90838	·85505	·79912	35
6	·95185	9·98668	·88328	·86533	·81471	·75159	6
7	·91882	·94784	·84639	·82162	·77396	·70327	7
8	·88558	·90849	·80918	·77723	·73278	·65412	8
9	·85209	·86856	·77160	·73209	·69111	·60404	9
40	·81833	·82804	·73361	·68616	·64889	·55301	40
1	·78428	·78689	·69519	·63940	·60610	·50092	1
2	·74991	·74505	·65629	·59172	·56267	·44773	2
3	·71517	·70249	·61683	·54307	·51849	·39335	3
4	·68006	·65916	·57682	·49340	·47358	·33770	4
45	·64452	·61499	·53616	·44262	·42780	·28067	45
6	·60851	·56995	·49480	·39067	·38109	·22217	6
7	·57200	·52397	·45269	·33744	·33338	·16209	7
8	·53494	·47696	·40975	·28285	·28456	·10034	8
9	·49725	·42890	·36587	·22681	·23449	03675	9

TABLE NO. XXXVII.

Logarithms of D *and* N. *Joint Lives. Equal Ages.* 6 *per-cent.*

x	TWO LIVES.		THREE LIVES.		FOUR LIVES.		x
	log D_{xx}	log N_{xx}	log D_{xxx}	log N_{xxx}	log D_{xxxx}	log N_{xxxx}	
50	8·45891	9·37967	13·32101	14·16920	18·18311	18·97120	50
1	·41984	·32921	·27506	·10992	·13028	·90356	1
2	·37996	·27743	·22789	·04883	·07582	·83365	2
3	·33919	·22425	·17939	13·98580	·01959	·76131	3
4	·29746	·16953	·12945	·92070	17·96144	·68635	4
55	·25468	·11321	·07793	·85336	·90118	·60856	55
6	·21073	·05511	·02466	·78362	·83859	·52772	6
7	·16553	8·99516	12·96951	·71129	·77349	·44359	7
8	·11894	·93319	·91228	·63616	·70562	·35591	8
9	·07081	·86906	·85274	·55802	·63467	·26439	9
60	·02103	·80260	·79072	·47664	·56041	·16873	60
1	7·96942	·73363	·72596	·39175	·48250	·06859	1
2	·91582	·66196	·65821	·30307	·40060	17·96356	2
3	·86001	·58739	·58715	·21029	·39429	·85328	3
4	·80180	·50967	·51249	·11311	·22318	·73730	4
65	·74098	·42859	·43391	·01111	·12684	·61512	65
6	·67725	·34384	·35097	12·90391	·02469	·48623	6
7	·61039	·25513	·26333	·79109	16·91627	·35005	7
8	·54004	·16215	·17046	·67216	·80088	·20593	8
9	·46592	·06457	·07193	·54662	·67794	·05323	9
70	·38763	7·96196	11·96715	·41387	·54667	16·89112	70
1	·30478	·85394	·85553	·27333	·40628	·71883	1
2	·21698	·74004	·73648	·12428	·25598	·53543	2
3	·12369	·61976	·60920	11·96599	·09471	·33993	3
4	·02445	·49255	·47299	·79763	15·92153	·13120	4
75	6·91860	·35782	·32687	·61832	·73514	15·90811	75
6	·80559	·21492	·17001	·42708	·53443	·66930	6
7	·68469	·06315	·00131	·22285	·31793	·41337	7
8	·55520	6·90165	10·81973	·00441	·08426	·13871	8
9	·41628	·72963	·62400	10·77046	14·83172	14·84354	9
80	·26697	·54614	·41269	·51960	·55841	·52602	80
1	·10634	·35011	·18440	·25025	·26246	·18398	1
2	5·93322	·14048	9·93737	9·96075	13·94152	13·81525	2
3	·74651	5·91594	·66996	·64924	·59341	·41729	3
4	·54493	·67516	·38024	·31361	·21555	12·98730	4
85	·32704	·41662	·06606	8·95157	12·80508	·52218	85
6	·09124	·13865	8·72501	·56059	·35878	·01853	6
7	4·83569	4·83954	·35434	·13814	11·87299	11·47298	7
8	·55882	·51721	7·95169	7·68099	·34456	10·88116	8
9	·25832	·16950	·51359	·18594	10·76886	·23882	9
90	3·93213	3·79386	·03696	6·64898	·14179	9·54035	90
1	·57721	·38801	6·51723	·06674	9·45725	8·78130	1
2	·19120	2·94956	5·95087	5·43618	8·71054	7·95841	2
3	2·77351	·47236	·33699	4·74617	7·90047	·05492	3
4	·31395	1·95758	4·66030	·00004	·00665	6·07693	4
95	1·81712	·39900	3·92771	3·18602	6·03830	5·00685	95
6	·27316	0·79960	·12442	2·30745	4·97568	3·84704	6
7	0·68173	·17563	2·24993	1·40109	3·81813	2·65591	7
8	·07753	$\bar{1}$·48143	1·35628	0·39249	2·63503	1·32744	8
9	$\bar{1}$·40320	$\bar{2}$·69841	0·35744	$\bar{1}$·28194	1·31168	$\bar{1}$·87925	9
100	$\bar{2}$·67353	$\bar{3}$·44411	$\bar{1}$·27559	$\bar{3}$·44411	$\bar{1}$·87765	$\bar{3}$·44411	100
1	$\bar{3}$·44411	...	$\bar{3}$·44411	...	$\bar{3}$·44411	...	1

TABLE NO. XXXVIII.

Values of Annuities. Joint Lives. Equal Ages.
6 *per-cent.*

x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Lives a_{xxxx}	x	Two Lives a_{xx}	Three Lives a_{xxx}	Four Liv a_{xxxx}
0	9·678	7·553	5·942	50	8·332	7·050	6·139
1	12·033	10·464	9·166	1	8·117	6·837	5·933
2	12·699	11·346	10·208	2	7·897	6·621	5·726
3	13·067	11·849	10·817	3	7·675	6·403	5·517
4	13·290	12·160	11·202	4	7·449	6·184	5·308
5	13·450	12·390	11·495	55	7·220	5·963	5·098
6	13·553	12·546	11·697	6	6·988	5·741	4·888
7	13·608	12·635	11·818	7	6·755	5·518	4·678
8	13·621	12·667	11·870	8	6·520	5·295	4·470
9	13·598	12·652	11·864	9	6·284	5·073	4·263
10	13·548	12·599	11·811	60	6·047	4·852	4·058
1	13·479	12·520	11·726	1	5·811	4·632	3·856
2	13·394	12·419	11·615	2	5·574	4·414	3·656
3	13·297	12·303	11·485	3	5·338	4·199	3·459
4	13·192	12·176	11·342	4	5·104	3·987	3·267
15	13·080	12·040	11·189	65	4·871	3·778	3·078
6	12·966	11·903	11·035	6	4·641	3·592	2·894
7	12·853	11·767	10·883	7	4·413	3·371	2·715
8	12·744	11·638	10·739	8	4·189	3·175	2·541
9	12·638	11·515	10·604	9	3·969	2·983	2·373
20	12·539	11·400	10·480	70	3·753	2·797	2·210
1	12·444	11·294	10·366	1	3·541	2·617	2·054
2	12·353	11·193	10·260	2	3·335	2·442	1·903
3	12·264	11·096	10·159	3	3·134	2·274	1·759
4	12·175	11·001	10·061	4	2·938	2·112	1·621
25	12·086	10·906	9·965	75	2·749	1·956	1·489
6	11·994	10·809	9·867	6	2·566	1·808	1·364
7	11·900	10·709	9·765	7	2·390	1·666	1·246
8	11·800	10·604	9·660	8	2·221	1·530	1·134
9	11·697	10·495	9·551	9	2·058	1·401	1·028
30	11·589	10·382	9·438	80	1·902	1·279	·928
1	11·475	10·263	9·318	1	1·753	1·164	·835
2	11·357	10·141	9·194	2	1·612	1·055	·748
3	11·235	10·012	9·065	3	1·477	·953	·667
4	11·107	9·878	8·931	4	1·350	·858	·591
35	10·974	9·739	8·792	85	1·229	·768	·521
6	10·835	9·595	8·647	6	1·115	·685	·457
7	10·691	9·446	8·498	7	1·009	·608	·398
8	10·542	9·291	8·343	8	·909	·536	·344
9	10·387	9·130	8·183	9	·815	·470	·295
40	10·226	8·965	8·019	90	·727	·409	·250
1	10·060	8·795	7·849	1	·647	·354	·211
2	9·889	8·619	7·675	2	·573	·306	·177
3	9·712	8·438	7·497	3	·500	·257	·143
4	9·530	8·252	7·313	4	·440	·219	·118
45	9·343	8·062	7·126	95	·382	·181	·093
6	9·150	7·868	6·936	6	·336	·152	·074
7	8·953	7·669	6·741	7	·312	·142	·069
8	8·750	7·466	6·543	8	·253	·109	·049
9	8·544	7·260	6·343	9	·197	·084	·037
				100	·059	·015	·004

CONVERSION TABLES.

Table No. XXXIX.—Single Premiums.

" " XL.—Annual Premiums.

TABLE NO. XXXIX.

SINGLE PREMIUM CONVERSION TABLE.

For finding by inspection the Value of A *from that of a.*

Value of a	VALUE OF A AT VARIOUS RATES OF INTEREST						
	$i=·025$	$i=·03$	$i=·035$	$i=·04$	$i=·045$	$i=·05$	$i=·06$
0	·97561	·97087	·96618	·96154	·95694	·95238	·94340
1	·95122	·94175	·93237	·92308	·91388	·90476	·88679
2	·92683	·91262	·89855	·88462	·87081	·85714	·83019
3	·90244	·88350	·86473	·84615	·82775	·80952	·77358
4	·87805	·85437	·83092	·80769	·78469	·76190	·71698
5	·85366	·82524	·79710	·76923	·74163	·71429	·66038
6	·82927	·79612	·76329	·73077	·69856	·66667	·60377
7	·80488	·76699	·72947	·69231	·65550	·61905	·54717
8	·78049	·73786	·69565	·65385	·61244	·57143	·49056
9	·75610	·70874	·66184	·61538	·56938	·52381	·43396
10	·73171	·67961	·62802	·57692	·52632	·47619	·37736
11	·70732	·65049	·59420	·53846	·48325	·42857	·32075
12	·68293	·62136	·56039	·50000	·44019	·38095	·26415
13	·65854	·59223	·52657	·46154	·39713	·33333	·20755
14	·63415	·56311	·49275	·42308	·35407	·28571	·15094
15	·60976	·53398	·45894	·38462	·31100	·23810	·09434
16	·58537	·50485	·42512	·34615	·26794	·19048	·03774
17	·56098	·47573	·39130	·30769	·22488	·14286	·00000
18	·53659	·44660	·35749	·26923	·18182	·09524	...
19	·51220	·41748	·32367	·23077	·13876	·04762	...
20	·48780	·38835	·28986	·19231	·09569	·00000	...
21	·46341	·35922	·25604	·15385	·05263	...	...
22	·43902	·33010	·22222	·11538	·00957	...	...
23	·41463	·30097	·18841	·07692	·00000	...	...
24	·39024	·27184	·15459	·03846	...	...	...
25	·36585	·24272	·12077	·00000	...	...	...
26	·34146	·21359	·08696	...	...	...	...
27	·31707	·18447	·05314	...	...	...	...
28	·29268	·15534	·01932	...	...	...	...
29	·26829	·12621	·00000	...	...	...	...

SINGLE PREMIUM CONVERSION TABLE.

Table of Differences.

Difference of a	DIFFERENCE OF A (SUBTRACTIVE)						
	$i=·025$	$i=·03$	$i=·035$	$i=·04$	$i=·045$	$i=·05$	$i=·06$
·1	·00244	·00291	·00338	·00385	·00431	·00476	·00566
·2	·00488	·00583	·00676	·00769	·00861	·00952	·01132
·3	·00732	·00874	·01015	·01154	·01292	·01429	·01698
·4	·00976	·01165	·01353	·01538	·01722	·01905	·02264
·5	·01220	·01456	·01691	·01923	·02153	·02381	·02830
·6	·01463	·01748	·02029	·02308	·02584	·02857	·03396
·7	·01707	·02039	·02367	·02692	·03014	·03333	·03962
·8	·01951	·02330	·02705	·03077	·03445	·03810	·04528
·9	·02195	·02621	·03044	·03462	·03876	·04286	·05094
·01	·00024	·00029	·00034	·00038	·00043	·00048	·00057
·02	·00049	·00058	·00068	·00077	·00086	·00095	·00113
·03	·00073	·00087	·00101	·00115	·00129	·00143	·00170
·04	·00098	·00117	·00135	·00154	·00172	·00190	·00226
·05	·00122	·00146	·00169	·00192	·00215	·00238	·00283
·06	·00146	·00175	·00203	·00231	·00258	·00286	·00340
·07	·00171	·00204	·00237	·00269	·00301	·00333	·00396
·08	·00195	·00233	·00271	·00308	·00344	·00381	·00453
·09	·00220	·00262	·00304	·00346	·00388	·00429	·00509
·001	·00002	·00003	·00003	·00004	·00004	·00005	·00006
·002	·00005	·00006	·00007	·00008	·00009	·00010	·00011
·003	·00007	·00009	·00010	·00012	·00013	·00014	·00017
·004	·00010	·00012	·00014	·00015	·00017	·00019	·00023
·005	·00012	·00015	·00017	·00019	·00022	·00024	·00028
·006	·00015	·00017	·00020	·00023	·00026	·00029	·00034
·007	·00017	·00020	·00024	·00027	·00030	·00033	·00040
·008	·00020	·00023	·00027	·00031	·00034	·00038	·00045

For finding by inspection the Value of P from that of a.

Value of a	VALUE OF P AT VARIOUS RATES OF INTEREST						
	$i=·025$	$i=·03$	$i=·035$	$i=·04$	$i=·045$	$i=·05$	$i=·06$
0	·97561	·97087	·96618	·96154	·95694	·95238	·94340
1	·47561	·47087	·46618	·46154	·45694	·45238	·44340
2	·30894	·30421	·29952	·29487	·29027	·28571	·27673
3	·22561	·22087	·21618	·21154	·20694	·20238	·19340
4	·17561	·17087	·16618	·16154	·15694	·15238	·14340
5	·14228	·13754	·13285	·12821	·12361	·11905	·11006
6	·11847	·11373	·10904	·10440	·09980	·09524	·08625
7	·10061	·09587	·09118	·08654	·08194	·07738	·06840
8	·08672	·08199	·07730	·07265	·06805	·06349	·05451
9	·07561	·07087	·06618	·06154	·05694	·05238	·04340
10	·06652	·06178	·05709	·05245	·04785	·04329	·03431
11	·05894	·05421	·04952	·04487	·04027	·03571	·02673
12	·05253	·04780	·04311	·03846	·03386	·02930	·02032
13	·04704	·04230	·03761	·03297	·02837	·02381	·01483
14	·04228	·03754	·03285	·02821	·02361	·01905	·01006
15	·03811	·03337	·02868	·02404	·01944	·01488	·00590
16	·03443	·02970	·02501	·02036	·01576	·01121	·00222
17	·03117	·02643	·02174	·01709	·01249	·00794	·00000
18	·02824	·02351	·01882	·01417	·00957	·00501	...
19	·02561	·02087	·01618	·01154	·00694	·00238	...
20	·02323	·01849	·01380	·00916	·00456	·00000	...
21	·02106	·01633	·01164	·00699	·00239	...	...
22	·01909	·01435	·00966	·00502	·00042	...	...
23	·01728	·01254	·00785	·00321	·00000	...	...
24	·01561	·01087	·00618	·00154	...	...	...
25	·01407	·00934	·00465	·00000	...	...	...
26	·01265	·00791	·00322	...	...	...	...
27	·01132	·00659	·00190	...	...	...	...
28	·01009	·00536	·00007	...	...	...	...
29	·00894	·00421	·00000	...	...	...	...
30	·00787	·00313	...	...	...	...	...

Table of Differences.

Value of a	DIFFERENCE OF P (SUBTRACTIVE)								
	$\Delta a=·1$	$\Delta a=·2$	$\Delta a=·3$	$\Delta a=·4$	$\Delta a=·5$	$\Delta a=·6$	$\Delta a=·7$	$\Delta a=·8$	$\Delta a=·9$
0	·09091	·16667	·22077	·28571	·33333	·37500	·41176	·44444	·47368
1	·02381	·04545	·06522	·08333	·10000	·11538	·12963	·14286	·15517
2	·01075	·02083	·03030	·03921	·04762	·05555	·06306	·07017	·07692
3	·00610	·01190	·01744	·02273	·02778	·03261	·03723	·04167	·04592
4	·00392	·00769	·01132	·01481	·01818	·02143	·02456	·02759	·03051
5	·00274	·00538	·00794	·01042	·01282	·01515	·01742	·01961	·02174
6	·00201	·00397	·00587	·00772	·00953	·01128	·01299	·01465	·01628
7	·00154	·00305	·00452	·00595	·00735	·00872	·01006	·01136	·01264
8	·00122	·00241	·00358	·00473	·00585	·00694	·00802	·00907	·01010
9	·00099	·00196	·00291	·00385	·00476	·00566	·00654	·00741	·00826
10	·00082	·00162	·00241	·00319	·00395	·00470	·00544	·00616	·00688
11	·00068	·00136	·00203	·00268	·00333	·00396	·00459	·00520	·00581
12	·00058	·00116	·00173	·00229	·00285	·00339	·00393	·00446	·00498
13	·00051	·00101	·00150	·00199	·00246	·00294	·00340	·00386	·00432
14	·00044	·00088	·00131	·00173	·00215	·00257	·00298	·00338	·00378
15	·00039	·00077	·00115	·00152	·00189	·00226	·00262	·00298	·00333
16	·00034	·00068	·00102	·00135	·00168	·00200	·00232	·00264	·00295
17	·00031	·00061	·00091	·00121	·00151	·00180	·00208	·00237	·00265
18	·00027	·00055	·00082	·00108	·00135	·00161	·00187	·00212	·00238
19	·00025	·00049	·00074	·00098	·00122	·00146	·00169	·00192	·00215
20	·00023	·00045	·00067	·00089	·00111	·00132	·00154	·00175	·00196
21	·00021	·00041	·00062	·00082	·00102	·00121	·00141	·00160	·00179
22	·00019	·00038	·00056	·00074	·00093	·00111	·00129	·00146	·00164
23	·00018	·00035	·00052	·00069	·00085	·00102	·00118	·00135	·00151
24	·00016	·00032	·00047	·00063	·00078	·00094	·00109	·00124	·00139
25	·00015	·00029	·00044	·00058	·00072	·00087	·00101	·00115	·00128
26	·00014	·00027	·00041	·00054	·00068	·00081	·00094	·00107	·00120
27	·00012	·00025	·00037	·00050	·00062	·00074	·00087	·00099	·00111
28	·00012	·00023	·00035	·00047	·00058	·00070	·00081	·00092	·00103
29	·00011	·00022	·00033	·00043	·00054	·00065	·00076	·00086	·00097
30	·00010	·00021	·00031	·00041	·00051	·00061	·00071	·00081	·00091

UNIFORM SENIORITY.

TABLE No. XLI.—TWO LIVES.

TABLE No. XLII.—THREE LIVES.

NOTE ON THE USE OF THE TABLES.

Two Lives.—Let x be the age of the younger life, and y that of the older. Enter Table XLI with the difference, $y-x$, and add the result to x. This will give u, the age of the two lives of equal ages, which may be substituted for the given lives of ages x and y.

Thus, if there be two lives, aged 32 and 40 respectively, the difference, $y-x$, is 8. Entering Table XLI with 8, the result is 4·72, which, added to 32, gives 36·72, the age of the lives to be substituted for those aged 32 and 40. See example, Chap. xii, Art. 39, p. 210.

Three Lives.—Let the ages of the given lives be x, y, and z, where $x<y<z$. Enter Table XLI with $y-x$ to find u, as above. Then enter Table XLII with $z-u$, and the result, added to u, will give w, the age of the three lives of equal ages, which may be substituted for the given lives of ages x, y, and z.

Usually u will include a fraction, and the fractional portion of $z-u$ must be multiplied into the difference in the column Δ in Table XLII.

Thus, if there be three lives, aged 30, 35, and 45, respectively, the difference, $y-x$, is 5. Entering Table XLI with 5, the result is 2·78, which, added to 30, gives $u=32{\cdot}78$. We now have $z-u=45-32{\cdot}78$ $=12{\cdot}22$. The result of entering Table XLII with 12 is 5·58, and for the difference we have $\cdot60\times\cdot22=\cdot13$, making together 5·71. Whence $w=32{\cdot}78+5{\cdot}71=38{\cdot}49$. See example, Chap. xii, Art. 42, p. 212.

The following are the calculations:—

$$\begin{array}{rrr}
x=30 & y=35 & z=45 \\
2{\cdot}78 & y-x=\;5 & \\
\hline
u=32{\cdot}78 & z-u=12{\cdot}22 & \\
5{\cdot}58 & & \\
\cdot22\times\cdot60=\;\;\cdot13 & & \\
\hline
w=38{\cdot}49 & & \\
\hline\hline
\end{array}$$

TABLE NO. XLI.

For finding the two Equal Ages, u, which may be substituted for the two Unequal Ages, x and y, as determined by the equation $2c^u = c^x + c^y$.

$y-x$	$u-x$	$y-x$	$u-x$
1	0·51	26	19·38
2	1·05	27	20·30
3	1·60	28	21·23
4	2·18	29	22·16
5	2·78	30	23·09
6	3·41	31	24·04
7	4·05	32	24·98
8	4·72	33	25·93
9	5·40	34	26·89
10	6·10	35	27·85
11	6·83	36	28·81
12	7·57	37	29·78
13	8·33	38	30·75
14	9·10	39	31·72
15	9·89	40	32·69
16	10·69	41	33·67
17	11·51	42	34·64
18	12·34	43	35·62
19	13·19	44	36·60
20	14·04	45	37·59
21	14·91	46	38·57
22	15·79	47	39·56
23	16·67	48	40·55
24	17·57	49	41·53
25	18·47	50	42·52

TABLE No. XLII.

For finding the three Equal Ages, w, which may be substituted for the three Unequal Ages, x, y, and z, as determined by the equation $3c^w = c^x + c^y + c^z$, or its equivalent, $3c^w = 2c^u + c^z$.

$z-u$	$w-u$	Δ	$z-u$	$w-u$	Δ
0	0·00	·35	...	...	...
1	0·35	·36	26	15·84	·85
2	0·71	·38	27	16·69	·86
3	1·09	·41	28	17·55	·87
4	1·50	·43	29	18·42	·88
5	1·93	·45	30	19·30	·89
6	2·38	·48	31	20·19	·90
7	2·86	·50	32	21·09	·91
8	3·36	·52	33	22·00	·91
9	3·88	·54	34	22·91	·92
10	4·42	·57	35	23·83	·93
11	4·99	·59	36	24·76	·93
12	5·58	·61	37	25·69	·94
13	6·19	·63	38	26·63	·94
14	6·82	·65	39	27·57	·95
15	7·47	·67	40	28·52	·96
16	8·14	·70	41	29·48	·95
17	8·84	·71	42	30·43	·96
18	9·55	·73	43	31·39	·97
19	10·28	·75	44	32·36	·96
20	11·03	·76	45	33·32	·97
21	11·79	·78	46	34·29	·97
22	12·57	·80	47	35·26	·98
23	13·37	·81	48	36·24	·98
24	14·18	·82	49	37·22	·97
25	15·00	·84	50	38·19	...

TABLE No. XLIII.

Endowment Assurances for two Joint Lives (equal ages). Annual Premiums for 100, *payable in n Years or at First Death* 3 *per-cent.*

Duration Ages	5	6	7	8	9	10	11	12
20 20	18·789	15·547	13·238	11·511	10·172	9·105	8·237	7·516
21 21	18·817	15·574	13·264	11·537	10·198	9·131	8·262	7·541
22 22	18·839	15·597	13·287	11·559	10·220	9·153	8·283	7·562
23 23	18·857	15·616	13·305	11·577	10·238	9·171	8·302	7·580
24 24	18·872	15·631	13·320	11·594	10·254	9·187	8·318	7·597
25 25	18·885	15·643	13·333	11·606	10·268	9·201	8·332	7·612
26 26	18·895	15·654	13·345	11·619	10·281	9·215	8·347	7·627
27 27	18·906	15·666	13·357	11·632	10·294	9·229	8·361	7·643
28 28	18·917	15·678	13·370	11·645	10·309	9·245	8·377	7·660
29 29	18·929	15·690	13·384	11·660	10·324	9·260	8·394	7·677
30 30	18·941	15·704	13·398	11·675	10·340	9·277	8·413	7·697
31 31	18·956	15·719	13·415	11·692	10·359	9·297	8·434	7·719
32 32	18·971	15·735	13·432	11·711	10·379	9·319	8·455	7·742
33 33	18·986	15·753	13·451	11·732	10·400	9·342	8·480	7·768
34 34	19·004	15·772	13·472	11·754	10·424	9·366	8·507	7·796
35 35	19·024	15·794	13·496	11·779	10·451	9·395	8·537	7·828
36 36	19·046	15·818	13·521	11·806	10·479	9·425	8·569	7·862
37 37	19·069	15·843	13·548	11·836	10·511	9·458	8·604	7·898
38 38	19·096	15·872	13·578	11·868	10·546	9·495	8·643	7·940
39 39	19·123	15·903	13·612	11·901	10·583	9·536	8·685	7·985
40 40	19·154	15·937	13·649	11·943	10·625	9·580	8·732	8·034
41 41	19·188	15·973	13·689	11·985	10·671	9·628	8·783	8·088
42 42	19·227	16·015	13·733	12·034	10·721	9·681	8·840	8·148
43 43	19·267	16·059	13·781	12·084	10·776	9·739	8·901	8·213
44 44	19·312	16·108	13·834	12·141	10·836	9·804	8·969	8·284
45 45	19·361	16·162	13·892	12·203	10·902	9·873	9·043	8·362
46 46	19·415	16·221	13·956	12·271	10·975	9·951	9·125	8·449
47 47	19·474	16·286	14·026	12·346	11·055	10·036	9·215	8·543
48 48	19·539	16·357	14·102	12·429	11·143	10·129	9·313	8·648
49 49	19·611	16·435	14·188	12·519	11·239	10·231	9·421	8·762
50 50	19·689	16·520	14·279	12·618	11·344	10·343	9·540	8·888
51 51	19·776	16·616	14·381	12·727	11·461	10·467	9·672	9·026
52 52	19·869	16·718	14·493	12·847	11·588	10·602	9·815	9·178
53 53	19·974	16·832	14·615	12·979	11·729	10·752	9·973	9·345
54 54	20·087	16·956	14·750	13·124	11·883	10·916	10·147	9·528
55 55	20·211	17·093	14·898	13·282	12·052	11·096	10·338	9·730
56 56	20·348	17·243	15·060	13·456	12·239	11·294	10·548	9·952
57 57	20·497	17·407	15·239	13·649	12·444	11·512	10·779	10·196
58 58	20·663	17·589	15·436	13·860	12·670	11·754	11·034	10·465
59 59	20·844	17·788	15·652	14·092	12·919	12·017	11·314	...
60 60	21·043	18·008	15·890	14·348	13·193	12·308	...	...

Table No. XLIII.—*Continued.*

Endowment Assurances for two Joint Lives (equal ages).

Annual Premiums for 100, *payable in n Years or at First Death.*
3 *per-cent.*

Duration	13	14	15	16	17	18	19	20
Ages								
20 20	6·909	6·392	5·947	5·559	5·220	4·920	4·655	4·418
21 21	6·934	6·416	5·970	5·583	5·244	4·945	4·680	4·444
22 22	6·955	6·438	5·992	5·605	5·266	4·967	4·703	4·467
23 23	6·973	6·457	6·011	5·625	5·286	4·988	4·724	4·488
24 24	6·991	6·474	6·029	5·643	5·305	5·008	4·744	4·510
25 25	7·006	6·490	6·046	5·661	5·323	5·026	4·763	4·530
26 26	7·022	6·506	6·063	5·678	5·342	5·046	4·784	4·551
27 27	7·038	6·524	6·081	5·697	5·361	5·066	4·805	4·573
28 28	7·056	6·542	6·100	5·718	5·383	5·089	4·829	4·598
29 29	7·075	6·562	6·121	5·740	5·406	5·113	4·854	4·625
30 30	7·095	6·584	6·144	5·763	5·431	5·140	4·882	4·654
31 31	7·118	6·608	6·170	5·790	5·459	5·169	4·913	4·686
32 32	7·143	6·634	6·197	5·819	5·489	5·200	4·946	4·721
33 33	7·170	6·662	6·227	5·851	5·523	5·235	4·982	4·759
34 34	7·200	6·694	6·260	5·885	5·559	5·273	5·022	4·801
35 35	7·233	6·728	6·296	5·923	5·599	5·316	5·066	4·847
36 36	7·268	6·766	6·336	5·965	5·643	5·361	5·115	4·897
37 37	7·307	6·807	6·379	6·011	5·691	5·411	5·167	4·953
38 38	7·351	6·853	6·427	6·061	5·744	5·467	5·225	5·013
39 39	7·399	6·903	6·480	6·116	5·801	5·528	5·289	5·080
40 40	7·450	6·957	6·537	6·176	5·865	5·594	5·358	5·152
41 41	7·507	7·017	6·601	6·243	5·934	5·667	5·435	5·232
42 42	7·570	7·084	6·670	6·316	6·011	5·747	5·519	5·320
43 43	7·638	7·156	6·745	6·395	6·094	5·834	5·610	5·416
44 44	7·714	7·236	6·829	6·483	6·186	5·931	5·711	5·521
45 45	7·797	7·322	6·921	6·579	6·287	6·036	5·821	5·636
46 46	7·888	7·418	7·021	6·685	6·398	6·152	5·942	5·762
47 47	7·988	7·523	7·132	6·801	6·519	6·279	6·075	5·901
48 48	8·097	7·638	7·253	6·928	6·652	6·419	6·220	6·052
49 49	8·218	7·765	7·387	7·068	6·798	6·572	6·380	6·219
50 50	8·350	7·905	7·533	7·221	6·959	6·739	6·554	6·400
51 51	8·496	8·058	7·694	7·390	7·136	6·923	6·746	...
52 52	8·656	8·226	7·870	7·575	7·329	7·125	...	...
53 53	8·832	8·412	8·064	7·778	7·541	...	...	...
54 54	9·026	8·615	8·278	8·001	...	...	...	...
55 55	9·238	8·838	8·512	...	...	...	...	...
56 56	9·472	9·084	...	...	...	...	...	...
57 57	9·729	...	...	...	...	...	...	...

Table No. XLIII.—*Continued.*

Endowment Assurances for two Joint Lives (equal ages).

Annual Premiums for 100, *payable in n Years or at First Death.*
3 *per-cent.*

Duration	21	22	23	24	25	26	27	28
Ages								
20 20	4·206	4·016	3·844	3·688	3·547	3·418	3·301	3·194
21 21	4·232	4·042	3·871	3·716	3·575	3·447	3·330	3·224
22 22	4·256	4·066	3·895	3·741	3·601	3·473	3·357	3·252
23 23	4·278	4·089	3·918	3·765	3·625	3·499	3·384	3·279
24 24	4·300	4·111	3·942	3·789	3·650	3·525	3·411	3·307
25 25	4·321	4·134	3·965	3·813	3·675	3·551	3·438	3·335
26 26	4·343	4·156	3·989	3·838	3·702	3·578	3·467	3·365
27 27	4·366	4·181	4·015	3·865	3·730	3·608	3·497	3·397
28 28	4·393	4·208	4·043	3·895	3·761	3·640	3·531	3·433
29 29	4·420	4·238	4·074	3·926	3·794	3·675	3·568	3·471
30 30	4·450	4·269	4·107	3·961	3·831	3·713	3·607	3·512
31 31	4·484	4·304	4·144	4·000	3·871	3·755	3·651	3·558
32 32	4·521	4·343	4·184	4·042	3·915	3·801	3·699	3·607
33 33	4·561	4·385	4·228	4·087	3·962	3·851	3·751	3·662
34 34	4·604	4·431	4·276	4·137	4·015	3·905	3·808	3·721
35 35	4·653	4·481	4·328	4·193	4·072	3·965	3·870	3·785
36 36	4·705	4·536	4·386	4·253	4·135	4·030	3·938	3·857
37 37	4·764	4·597	4·449	4·319	4·204	4·102	4·012	3·934
38 38	4·827	4·663	4·518	4·391	4·279	4·180	4·094	4·018
39 39	4·896	4·736	4·594	4·470	4·361	4·266	4·183	4·110
40 40	4·972	4·815	4·677	4·556	4·451	4·359	4·279	4·210
41 41	5·056	4·902	4·768	4·651	4·549	4·461	4·385	4·319
42 42	5·148	4·998	4·867	4·754	4·656	4·572	4·500	4·439
43 43	5·247	5·102	4·976	4·867	4·773	4·693	4·625	...
44 44	5·357	5·216	5·094	4·990	4·901	4·825	...	...
45 45	5·477	5·341	5·224	5·125	5·040	...	...	...
46 46	5·609	5·478	5·366	5·272	...	...	...	...
47 47	5·753	5·627	5·521	...	...	...	...	...
48 48	5·910	5·791	...	...	...	...	...	...
49 49	6·083	...	...	...	...	...	...	...

TABLE No. XLIII.—*Continued.*

Endowment Assurances for two Joint Lives (equal ages).
Annual Premiums for 100, *payable in n Years or at First Death.*
3 *per-cent.*

Duration Ages	29	30	31	32	33	34	35	36
20 20	3·097	3·007	2·926	2·851	2·782	2·719	2·662	2·609
21 21	3·127	3·038	2·957	2·883	2·815	2·754	2·697	2·646
22 22	3·156	3·068	2·988	2·915	2·848	2·788	2·732	2·682
23 23	3·184	3·098	3·018	2·946	2·881	2·821	2·767	2·717
24 24	3·213	3·127	3·049	2·978	2·914	2·856	2·802	2·754
25 25	3·242	3·158	3·081	3·012	2·948	2·891	2·840	2·793
26 26	3·274	3·191	3·115	3·047	2·985	2·929	2·879	2·834
27 27	3·307	3·225	3·152	3·085	3·025	2·970	2·922	2·878
28 28	3·344	3·264	3·191	3·126	3·068	3·015	2·968	2·926
29 29	3·384	3·305	3·234	3·171	3·114	3·063	3·018	2·978
30 30	3·427	3·350	3·281	3·220	3·165	3·116	3·073	3·034
31 31	3·474	3·399	3·333	3·273	3·220	3·173	3·132	3·096
32 32	3·526	3·453	3·389	3·331	3·280	3·236	3·195	3·162
33 33	3·583	3·512	3·450	3·394	3·346	3·303	3·266	3·234
34 34	3·644	3·576	3·516	3·463	3·417	3·377	3·341	3·312
35 35	3·712	3·646	3·589	3·538	3·495	3·457	3·424	...
36 36	3·785	3·722	3·667	3·620	3·578	3·543	...	...
37 37	3·865	3·805	3·753	3·708	3·670	...	...	...
38 38	3·952	3·895	3·846	3·804	...	...	...	...
39 39	4·048	3·994	3·948	...	...	...	...	...
40 40	4·151	4·100	...	...	...	...	...	...
41 41	4·264	...	...	...	...	...	...	...

Duration Ages	37	38	39	40	41	42	43	44
20 20	2·561	2·518	2·478	2·442	2·409	2·379	2·353	2·329
21 21	2·599	2·556	2·517	2·482	2·450	2·422	2·396	2·374
22 22	2·636	2·594	2·556	2·522	2·492	2·465	2·440	2·419
23 23	2·673	2·632	2·596	2·563	2·534	2·508	2·485	2·464
24 24	2·711	2·672	2·637	2·605	2·578	2·553	2·531	2·512
25 25	2·751	2·713	2·680	2·650	2·623	2·600	2·579	2·562
26 26	2·793	2·757	2·725	2·697	2·672	2·650	2·631	2·614
27 27	2·839	2·805	2·774	2·747	2·724	2·703	2·686	...
28 28	2·889	2·856	2·827	2·802	2·780	2·761	...	...
29 29	2·943	2·912	2·884	2·861	2·840	...	...	...
30 30	3·001	2·971	2·946	2·924	...	...	...	...
31 31	3·064	3·036	3·013	...	...	...	...	...
32 32	3·133	3·107	...	...	...	...	...	...
33 33	3·206	...	...	...	...	...	...	...

Duration Ages	45	46	47	48	49	50
20 20	2·307	2·289	2·272	2·257	2·244	2·233
21 21	2·353	2·336	2·320	2·306	2·295	...
22 22	2·400	2·383	2·368	2·356	...	...
23 23	2·447	2·431	2·418	...	...	...
24 24	2·496	2·481	...	...	...	...

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